Robust Estimation of the Scale Parameter for Rayleigh Distribution under Type-I Hybrid Censoring

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ABSTRACT

Aims: This study aims to develop robust estimation techniques for the scale parameter of the Rayleigh distribution under Type-I hybrid censoring, addressing a gap in the existing reliability and survival literature.
Study design: A simulation-based study was conducted to compare the performance of maximum likelihood estimators (MLEs) and Bayesian estimators for the scale parameter.
Methodology: We derived likelihood functions and estimators for both MLE and Bayesian approaches. A comprehensive Monte Carlo simulation study was employed to evaluate the performance of these estimators, focusing on root mean squared errors (RMSEs) under various conditions.
Results: The results indicated that RMSEs decreased with increasing sample sizes and higher censoring parameters. Bayesian estimators consistently outperformed MLEs, particularly with well-chosen priors, demonstrating lower RMSEs across all scenarios.
Conclusion: The findings highlight the robustness and superiority of Bayesian methods in accurately estimating parameters under Type-I hybrid censoring, providing valuable insights for enhancing reliability and maintenance strategies in engineering systems. Future research may extend these methodologies to other distributions and real-world applications.

Keywords: Hybrid censoring, Maximum likelihood estimator, Conjugate prior, Scale-invariant loss, General entropy loss function, Bayes estimator.

1 Introduction

The Rayleigh distribution is a widely used model in reliability engineering and survival analysis, particularly for modeling the lifetimes of mechanical systems and electronic components. Its simplicity and relevance in practical applications make it a subject of significant interest. Bhattacharya and Tyagi (1990) used the Rayleigh distribution for modeling the survival time distribution for cancer patients in some specific clinical studies. Keeping in mind the concept of reliability for electrovaccuum devices, Polovko AM (1968) discussed the importance of this distribution. They consider the following distribution function of X which follows Rayleigh distribution. The cumulative distribution function of X is

$$F(x,\lambda) = 1 - e^{-\frac{x^2}{\lambda}} \quad x > 0, \lambda > 0 \tag{1}$$

and its probability distribution

$$f(x,\lambda) = \frac{2x}{\lambda} e^{-\frac{x^2}{\lambda}} \quad x > 0, \lambda > 0 \tag{2}$$

where λ is a scale parameter.

Mostert et al. (1998, 1999) used the Rayleigh model with a Bayesian approach to analyze survival data. However, real-world data often involve censoring, where the exact failure times of all items are not observed, posing challenges for parameter estimation.

Censoring can occur in various forms, and understanding these different types is crucial for effectively analyzing censored data. The most common types of censoring are right censoring, left censoring, and interval censoring. Right censoring occurs when the study ends before the event of interest (e.g., failure) happens for some subjects. The exact event time is unknown, but it is known to exceed a certain time. Left censoring occurs when the event of interest happens before the study begins. The exact event time is unknown, but it is known to be less than a certain time. Interval censoring occurs when the event of interest happens within a certain time interval. The exact event time is unknown, but it is known to fall between two observed times.

In addition to these basic forms of censoring, there are more specific schemes such as Type-I and Type-II censoring. Type-I censoring refers to time-based censoring where the study ends at a pre-specified time, regardless of how many events have occurred. Type-II censoring refers to failure-based censoring where the study ends after a pre-specified number of events have occurred. However, the Type-I censoring scheme has the advantage that the termination time of the experiment is insured, but the number of individuals to be observed is uncertain. On the other hand, in Type-II censoring the targeted individual is specified in advance, but the waiting time to terminate the experiment is a realized random variable. Indeed, none of these censoring schemes can control the total number of individuals to be observed and the termination time to complete the experiment simultaneously.

Hybrid censoring combines features of both Type-I and Type-II censoring. Type-I hybrid censoring, in particular, is a scheme where the study ends at a pre-specified time or after a pre-specified number of events, whichever comes first. This approach provides a flexible and realistic framework for analyzing life data and is especially useful in reliability testing and quality control.

Despite its practical relevance, the estimation of the scale parameter of the Rayleigh distribution under Type-I hybrid censoring has not been extensively studied, leaving a gap in the reliability and survival literature. Existing studies have explored parameter estimation for the Rayleigh distribution under complete and conventional censoring schemes. Methods such as maximum likelihood estimation (MLE) and Bayesian approaches have been developed, but their performance under Type-I hybrid censoring remains underexplored.

This study aims to address this gap by developing robust estimation techniques for the scale parameter of the Rayleigh distribution in the context of Type-I hybrid censoring. The primary objective is to develop and evaluate new estimation methods, utilizing comprehensive simulations and analyzing real-world data to validate these methods. Understanding and accurately estimating the parameters of the Rayleigh distribution under hybrid censoring conditions is crucial for enhancing the reliability and maintenance strategies of engineering systems. This research will contribute to the field by offering new insights and methodologies, ultimately supporting better decision-making in reliability engineering and related disciplines.

Suppose the ordered lifetimes are denoted by $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. Type-I hybrid censoring scheme is described as follows. If n identical items are placed on test, and the experiment is terminated at the random time $T^* = min\{X_{R:n}, T)$ where R and T are fixed in advance, $0 \leq R \leq n$ and $T \in (0, \propto)$. In this research, the scale parameter λ will be estimated under the following sampling plans and we will get one of the following sampling plans:

Case 1: $\{x_{1:n} < x_{2:n} < \cdots < x_{R:n}\}$ if $x_{R:n} < T$ but $x_{R:n} > 0$ (3)

Case 2: $\{x_{1:n} < x_{2:n} < \cdots < x_{d:n}\}$ if $d < R \leq n$ and $x_{d:n} < T < x_{(d+1):n}$; $d > 0$

(4)

Both cases are presented in Figure 1.

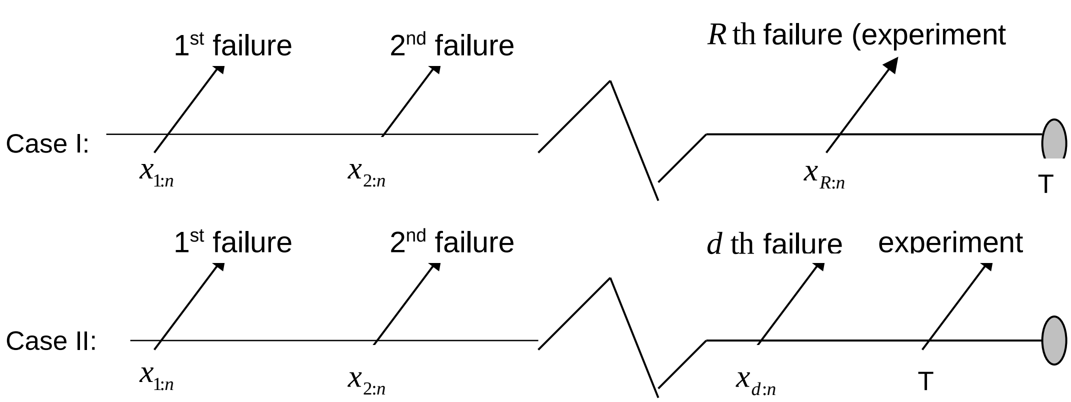

Figure 1 Type-I hybrid censoring scheme

The organization of this article is as follows. The likelihood functions for both cases and parameter estimations via maximum likelihood estimation are discussed in Section 2. The Bayes estimator for the scale parameter under different loss functions is derived in Section 3. The simulation study is presented in Section 4. The concluding remarks are presented in Section 5, followed by all proofs of the theorems in the Appendix and the Reference section.

2 Parameter Estimation

In this section, we present the estimation of the scale parameter using the maximum likelihood estimation (MLE) method. MLE is a widely used technique due to its desirable properties, such as consistency and efficiency. We derive the likelihood functions for both cases of Type-I hybrid censored data and obtain the MLEs for the scale parameter. The detailed steps and mathematical formulations are provided to ensure a thorough understanding of the estimation process.

2.1 Maximum Likelihood Estimation

Suppose $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ are ordered of a random sample draw of the density given Equation (1). Based on hybrid censored data, the likelihood function is

$$L(\lambda) = \frac{n!}{(n-D^*)}\left[\prod_{i=1}^{D^*} f\left(x_{i:n}, \lambda\right)\right]\left[1-F\left(T^*, \lambda\right)\right]^{n-D^*} \tag{5}$$

where $T^* = \min\{X_{R:n}, T\}$ and D^* which takes either R and d denotes the number of observed lifetimes before time T^*.The maximum likelihood estimator of λ satisfies the following equations:

$$\frac{\partial}{\partial\lambda}\ln L(\lambda) = \sum_{i=1}^{D^*}\frac{\frac{\partial}{\partial\lambda} f\left(x_{i:n};\lambda\right)}{f(x_{i:n};\lambda)} - (n-D^*)\frac{\frac{\partial}{\partial\lambda}F(T^*;\lambda)}{1-F(T^*;\lambda)} = 0 \tag{6}$$

We may get the following relations

$$\frac{f'(x;\lambda)}{f(x;\lambda)} = -\frac{1}{\lambda}\left\{1+\ln\left[1-F\left(x;\lambda\right)\right]\right\} \tag{7}$$

$$\frac{F'(x,\lambda)}{1-F(x,\lambda)} = -\frac{1}{2\lambda}\left[\frac{x\, f\left(x;\lambda\right)}{1-F(x;\lambda)}\right] \tag{8}$$

Substituting these results in equation (6) finally we get

$$-\frac{D^*}{\lambda}+\frac{1}{\lambda^2}\left\{\sum_{i-1}^{D^*} x_{i:n}{}^2+(n-D^*)T^{*2}\right\}=0 \tag{9}$$

and hence the MLE of λ for case 1

$$\hat{\lambda}_1=\frac{\left\{\sum_{i-1}^{R} x_{i:n}{}^2+(n-R)\,x_{R:n}{}^2\right\}}{R} \tag{10}$$

For case 2

$$\hat{\lambda}_2=\frac{\left\{\sum_{i-1}^{d} x_{i:n}{}^2+(n-d)T^2\right\}}{d}. \tag{11}$$

3 Bayes Estimation

In this section, we focus on the estimation of the scale parameter using Bayesian methods. Unlike the maximum likelihood estimation (MLE) approach discussed in the previous section, Bayesian estimation incorporates prior information about the parameter in conjunction with the observed data. We derive the Bayes estimator for the scale parameter under various loss functions, providing a comprehensive comparison with the MLE approach.

Using different priors as well as loss functions, Bayesian estimation criteria have been taken into account here. Let us consider the following prior

$$g_1(\lambda)\propto\lambda^{-3} \tag{12}$$

Based on this prior, the joint density function of λ and data is

$$l(data,\lambda)\propto\lambda^{-(D^*+3)}\prod_{i=1}^{D^*}x_{i:n}\,e^{-\frac{s}{\lambda}};\quad \text{where } s=\sum_{i-1}^{D^*}x_{i:n}{}^2+(n-D^*)T^{*2} \tag{13}$$

Theorem 3.1. The posterior distribution of λ under improper prior $g_1(\lambda)$ and the censored sampling as specified in section 1 is

$$\pi(\lambda \mid data) = \frac{s^{(D^*+2)}\,\lambda^{-(D^*+3)}\,e^{-\frac{s}{\lambda}}}{\Gamma(D^*+2)} \tag{14}$$

which is nothing but the pdf of an Inverted Gamma distribution.

3.1 Using Symmetric Loss

If $\hat{\delta}^{\pi}$ be a Bayes estimator of λ, considering the scale-invariant squared-error loss function (SILF) of the form

$$L(\lambda,\ \delta^{\pi}) = \left(\frac{\lambda - \hat{\delta}^{\pi}}{\lambda}\right)^2 \tag{15}$$

Then the Bayes estimator of λ using this loss function is

$$\hat{\delta}_1^{\pi} = \frac{E(\omega(\lambda)\gamma(\lambda) \mid x)}{E(\omega(\lambda) \mid x)}$$; where $\omega(\lambda) = \dfrac{1}{\lambda^2}$ and $\gamma(\lambda) = \lambda$

(16)

$$= \frac{\int_0^\infty \lambda^{-(D^*+4)}\,e^{-\frac{s}{\lambda}}\,d\lambda}{\int_0^\infty \lambda^{-(D^*+5)}\,e^{-\frac{s}{\lambda}}\,d\lambda}$$

$$= \frac{s}{(D^*+3)} \tag{17}$$

Using inverted gamma, $g_2(\lambda) \propto \lambda^{-(\alpha+1)} e^{-\frac{1}{\beta\lambda}};\ \alpha, \beta > 0$ as a prior, the joint density function of λ and data are

For case 1: $l_1(data,\lambda) \propto \lambda^{-(R+\alpha+1)} \prod_{i=1}^{R} x_{i:n}\, e^{-\frac{1}{\lambda}(s_1+\frac{1}{\beta})}$; where $s_1 = \sum_{i-1}^{R} x_{i:n}{}^2 + (n-R)\,x_{R:n}{}^2$ (18)

For case 2: $l_2(data,\lambda) \propto \lambda^{-(d+\alpha+1)} \prod_{i=1}^{d} x_{i:n}\, e^{-\frac{1}{\lambda}(s_2+\frac{1}{\beta})}$; where $s_2 = \sum_{i-1}^{R} x_{i:n}{}^2 + (n-d)T^2$ (19)

Theorem 3.2. The posterior distribution of λ for given data is

For case 1: $\pi_1(\lambda \mid data) \sim Ig\left(\alpha + R, (\sum_{i-1}^{R} x_{i:n}{}^2 + (n-R)\, x_{R:n}{}^2 + \frac{1}{\beta})^{-1} \right)$ (20)

For case 2: $\pi_2(\lambda \mid data) \sim Ig\left(\alpha + d, (\sum_{i-1}^{d} x_{i:n}{}^2 + (n-d)T^2 + \frac{1}{\beta})^{-1} \right) \quad if \;\; d > 0$ (21)

$$\pi_2(\lambda \mid data) \sim Ig\left(\alpha, (\sum_{i-1}^{d} x_{i:n}{}^2 + nT^2 + \frac{1}{\beta})^{-1} \right) \quad if \;\; d = 0$$

where Ig for Inverted Gamma distribution.

Thus, based on (15) and (20), the Bayes estimator of λ for case 1 is

$$\hat{\delta}_{21}^{\pi} = \frac{E(\omega(\lambda)\gamma(\lambda) \mid data)}{E(\omega(\lambda) \mid data)} \quad , \omega(\lambda) = \frac{1}{\lambda^2}, \quad \gamma(\lambda) = \lambda$$

and as $E[\omega(\lambda)\gamma(\lambda) \mid x] = E\left[\frac{1}{\lambda} \middle| x \right]$

$$= \int_0^{\infty} \frac{\left(s_1 + \frac{1}{\beta}\right)^{R+\alpha} \lambda^{-(R+\alpha+2)} \; e^{-\frac{1}{\lambda}\left(s_1 + \frac{1}{\beta}\right)}}{\Gamma(R+\alpha)} \, d\lambda$$

$$= \frac{\Gamma(R+\alpha+1)}{\Gamma(R+\alpha)} \left(s_1 + \frac{1}{\beta}\right)^{-1}$$

and $E[\omega(\lambda) \mid x] = E\left[\frac{1}{\lambda^2} \middle| x \right]$

$$= \int_0^\infty \frac{\left(s_1 + \frac{1}{\beta}\right)^{R+\alpha} \lambda^{-(R+\alpha+3)}\ e^{-\frac{1}{\lambda}\left(s_1+\frac{1}{\beta}\right)}}{\Gamma(R+\alpha)}\, d\lambda$$

$$= \frac{\Gamma(R+\alpha+2)}{\Gamma(R+\alpha)}\left(s_1 + \frac{1}{\beta}\right)^{-2}$$

Therefore, $\hat{\delta}_{21}^{\pi} = \frac{\Gamma(R+\alpha+1)}{\Gamma(R+\alpha+2)}\left(s_1 + \frac{1}{\beta}\right)$

$$= k_1(s_1 + \frac{1}{\beta}) \quad \text{where } k_1 = \frac{\Gamma(R+\alpha+1)}{\Gamma(R+\alpha+2)} \qquad (22)$$

Similarly, we can derive estimator for case 2 using equation (21) as follows:

$$\hat{\delta}_{22}^{\pi} = \frac{\Gamma(d+\alpha+1)}{\Gamma(d+\alpha+2)}\left(s_2 + c_1 + \frac{1}{\beta}\right)$$

(23)

$$= k_2(s_2 + c_1 + \frac{1}{\beta})$$

$$= k_2(s_2' + \frac{1}{\beta})$$

where $k_2 = \frac{\Gamma(d+\alpha+1)}{\Gamma(d+\alpha+2)}$, $s_2 = \sum_{i=1}^{d} x_{i:n}^2$ $s_2' = s_2 + c_1$ and $c_1 = (n-d)T^2$

3.2 Using Asymmetric Loss

A widely used asymmetric loss function generalized by Zellner (1986) is linear-exponential (LINEX) loss function. However, it does not sound well for scale parameter (see for example Basu and Ebrahimi 1991). A modified linear exponential (MLINEX) loss function may be defined as follows:

$$L\left(\lambda, \hat{\delta}^{\pi}\right) \propto \left[\left(\frac{\hat{\delta}^{\pi}}{\lambda}\right)^{c} - c\ln\left(\frac{\hat{\delta}^{\pi}}{\lambda}\right) - 1\right] \quad c \neq 0$$

(24)

where $\hat{\delta}^{\pi}$ is the estimator of λ and c is the parameters of loss function.

The Bayes estimator under MLINEX (or general entropy loss (GE)) loss function is

$$\hat{\delta}^{\pi} = \left[E\left(\lambda^{-c}\right)\right]^{-\frac{1}{c}}$$

(25)

provided that $E\left(\lambda^{-c}\right)$ exists and is finite.

Hence, Bayes estimator using $g_1(\lambda)$ prior is

$$\hat{\delta}_3^{\pi} = \left[E\left(\lambda^{-c}\right)\right]^{-\frac{1}{c}} \qquad (26)$$

Now,

$$E\left(\lambda^{-c}\right) = \int_0^{\infty} \frac{s^{\left(D^*+2\right)} \lambda^{-\left(D^*+c+3\right)} e^{-\frac{s}{\lambda}}}{\Gamma\left(D^*+2\right)} d\lambda$$

$$= \frac{\Gamma(D^*+c+2)}{\Gamma(D^*+2)} s^{-c}$$

Therefore, $\hat{\delta}_3^{\pi} = \left[E\left(\lambda^{-c}\right)\right]^{-\frac{1}{c}}$

$$= \left[\frac{\Gamma(D^*+2)}{\Gamma D^*+c+2}\right]^{\frac{1}{c}} s \qquad (27)$$

Again $E[\lambda^{-c}]$ using prior $g_2(\lambda)$ is

$$E(\lambda^{-c}) = \int_0^{\infty} \frac{\left(s_1+\frac{1}{\beta}\right)^{R+\alpha} \alpha^{-(R+\alpha+c+1)} e^{-\frac{1}{\lambda}\left(s_1+\frac{1}{\beta}\right)}}{\Gamma(R+\alpha)} d\lambda$$

$$= \frac{\Gamma\left(R+\alpha+c\right)}{\Gamma(R+\alpha)}\left(s_1+\frac{1}{\beta}\right)^{-c}$$

Therefore, Bayes estimator using inverted gamma prior for case 1 is

$$\hat{\delta}_{41}^{\pi} = \left[\frac{\Gamma\left(R+\alpha\right)}{\Gamma\left(R+\alpha+c\right)}\right]^{\frac{1}{c}}\left(s_1+\frac{1}{\beta}\right)$$

(28)

Similarly for case 2 the estimator is

$$\hat{\delta}_{42}^{\pi} = \left[\frac{\Gamma\left(d+\alpha\right)}{\Gamma\left(d+\alpha+c\right)}\right]^{\frac{1}{c}}\left(s_2'+\frac{1}{\beta}\right) \qquad (29)$$

In the next section, we conduct a simulation study to compare the performance of the maximum likelihood estimator and Bayes estimator under different loss functions.

4 Simulation Study

Conducting an analytical comparison of the performance of different methods can be quite challenging. Therefore, we have carried out a Monte Carlo simulation study to facilitate this comparison. We employed the method of selecting ordered uniform random variates, as proposed by Balakrishnan and Aggarwala (2000). With some modifications to their approach, we followed these steps to generate hybrid censored samples:

i. Generating n independent $Uniform(0,1)$ random variates $W_1, W_2, \dots, W_n$.
ii. Setting $V_i = W_i^{\frac{1}{i}}$ for $i = 1,2,\dots,n$.
iii. Setting $U_i = 1 - (V_n V_{\text{n-1}} \dots V_{\text{n-i+1}})$ for $i = 1,2,\dots,n$ so that $U_1 < U_2 < \cdots < U_n$ is a set of ordered sample of size $n$from $Uniform(0,1)$ distribution.
iv. Using inverse transformation method let $X_i = \sqrt{-\lambda\, ln(1 - U_i)}$. Then $X_1 < X_2 < \cdots < X_n$ are observation obtained from Rayleigh distribution.
v. Selecting Type I hybrid censoring sample for T and R.

Considering data as derived by using the above six steps, we have computed the values of MLEs $\hat{\lambda}_1$and $\hat{\lambda}_2$ specified in equation (10) and (11). We have also computed the values of Bayes estimators $\hat{\delta}_i^{\pi};\ i = 1,2,3,4$ using (17), (22), (23), (27), (28) and (29). This process will be repeated for M =10000 times and finally the summarized results are presented in the following tables.

Table 1 For particular value of $\lambda = 2$, the root mean squared errors of λ considering $g_1(\lambda)$as a prior.

T	n	R	$\hat{\lambda}$	$\hat{\delta}_1^{\pi}$	$\hat{\delta}_3^{\pi}(c=2)$	$\hat{\delta}_3^{\pi}(c=-2)$
2.0	10	8	0.650250	0.588981	0.525250	0.491019
		5	0.673721	0.657137	0.600728	0.572910
	20	15	0.606781	0.526661	0.485212	0.457626
		10	0.621620	0.568486	0.503121	0.474548
	30	25	0.560429	0.445666	0.434098	0.404926
		15	0.583516	0.496122	0.454696	0.428335
2.5	10	8	0.650445	0.588821	0.525250	0.491000
		5	0.673822	0.657076	0.600728	0.572711
	20	15	0.606898	0.526567	0.485212	0.457552
		10	0.621743	0.568362	0.503121	0.474414
	30	25	0.560467	0.445550	0.434098	0.404732
		15	0.583592	0.496082	0.454696	0.428213
3.0	10	8	0.650487	0.588622	0.525122	0.489801
		5	0.673865	0.657000	0.600523	0.572645
	20	15	0.606911	0.526454	0.485110	0.457478
		10	0.621786	0.568126	0.503082	0.474311
	30	25	0.560479	0.445445	0.433970	0.404633
		15	0.583601	0.496024	0.454519	0.428121

Based on the provided table, several observations can be made regarding the root mean squared errors (RMSEs) of the parameter estimates under different scenarios for maximum likelihood estimator $\hat{\lambda}$ and Bayesian estimators $\hat{\delta}_1^{\pi}$, $\hat{\delta}_3^{\pi}(c=2)$, $\hat{\delta}_3^{\pi}(c=-2)$. As T increases from 2.0 to 3.0, there is a general trend of decreasing RMSEs across all estimators. This indicates that higher values of T tend to yield more accurate parameter estimates. For each fixed value of T and R, the RMSEs decrease as the sample size (n) increases. For example, for T=2.0 and R=10, the RMSEs decrease from 0.65025 (for n=8) to 0.560429 (for n=25). This trend is consistent across different values of T, indicating that larger sample sizes improve the accuracy of parameter estimates.

Bayesian estimators $\hat{\delta}_1^{\pi}$, $\hat{\delta}_3^{\pi}(c=2)$, $\hat{\delta}_3^{\pi}(c=-2)$ consistently show lower RMSEs compared to MLE. For instance, for T=2.0, n=10, and R=8, the RMSEs for $\hat{\delta}_1^{\pi}$, $\hat{\delta}_3^{\pi}(c=2)$ and $\hat{\delta}_3^{\pi}(c=-2)$ are 0.588981, 0.52525, and 0.491019, respectively, all of which are lower than the RMSE for MLE (0.65025).

Among the Bayesian estimators, $\hat{\delta}_3^{\pi}(c=-2)$ generally has the lowest RMSEs, suggesting that it might be the most effective in reducing estimation errors. The observed trends of decreasing RMSEs with increasing T and n, as well as the superior performance of Bayesian estimators, are consistent across all combinations of T, n, and R. This consistency indicates robust performance improvements using Bayesian methods over MLE.

Table 2 For particular value of $\lambda = 2$, the root mean squared errors of λconsidering inverted gamma prior assuming $\alpha = 2, \beta = 1$.

T	n	R	$\hat{\lambda}$	$\hat{\delta}_2^\pi$	$\hat{\delta}_4^\pi(c=2)$	$\hat{\delta}_4^\pi(c=-2)$
2.0	10	8	0.650250	0.502302	0.491491	0.464686
		5	0.673721	0.580486	0.553286	0.517214
	20	15	0.606781	0.488592	0.479760	0.448673
		10	0.621620	0.537854	0.483452	0.465459
	30	25	0.560429	0.430220	0.403633	0.385526
		15	0.583516	0.472132	0.449651	0.416259
2.5	10	8	0.650445	0.502211	0.491233	0.464418
		5	0.673822	0.580324	0.553161	0.517141
	20	15	0.606898	0.488456	0.479564	0.448417
		10	0.621743	0.537655	0.483327	0.465291
	30	25	0.560467	0.430137	0.403473	0.385322
		15	0.583592	0.472085	0.449566	0.416155
3.0	10	8	0.650487	0.502054	0.491130	0.464211
		5	0.673865	0.580132	0.553062	0.517044
	20	15	0.606911	0.488287	0.479422	0.448171
		10	0.621786	0.537432	0.483175	0.465015
	30	25	0.560479	0.430011	0.403230	0.385156
		15	0.583601	0.471702	0.449561	0.416023

From Table 2, we observed that the RMSEs decrease as T increases from 2 to 3 across all estimators. This trend suggests that higher values of T lead to more accurate parameter estimates. For a fixed value of T and R, the RMSEs decrease as the sample size (n) increases. For example, for T=2 and R=10, the RMSEs decrease from 0.65025 (for n=8) to 0.560429 (for n=25) under MLE. This trend is consistent across different values of T, indicating that larger sample sizes improve the accuracy of parameter estimates.

Bayesian estimators ($\hat{\delta}_2^\pi$, $\hat{\delta}_4^\pi(c=2)$, $\hat{\delta}_4^\pi(c=-2)$) consistently show lower RMSEs compared to MLE. For instance, for T=2, n=10, and R=8, the RMSEs for $\hat{\delta}_2^\pi$, $\hat{\delta}_4^\pi(c=2)$, and $\hat{\delta}_4^\pi(c=-2)$are 0.502302, 0.491491, and 0.464686 respectively, all of which are lower than the RMSE for MLE (0.65025). Among the Bayesian estimators, $\hat{\delta}_4^{\pi}(c = -2)$ generally has the lowest RMSEs, suggesting it provides the most accurate parameter estimates.

The trends of decreasing RMSEs with increasing T and n, as well as the superior performance of Bayesian estimators, hold true across different values of R. For example, for T=3 and n=30, the RMSEs for R=15 are 0.5836 (MLE), 0.4717 ($\hat{\delta}_2^{\pi}$), 0.4496 ($\hat{\delta}_4^{\pi}(c = 2)$), and 0.4160 ($\hat{\delta}_4^{\pi}(c = -2)$), showing consistent performance improvements using Bayesian methods over MLE.

5 Concluding Remarks

In this study, we have addressed the estimation of the scale parameter of the Rayleigh distribution under Type-I hybrid censoring, a topic that has not been extensively explored in existing reliability and survival literature. Through the development and evaluation of robust estimation techniques, our research has aimed to fill this gap by leveraging both maximum likelihood estimation (MLE) and Bayesian approaches. We compared the performance of MLE and Bayesian estimators for the scale parameter through a simulation study.

Using simulated data, the results from Tables 1 and 2 reveal several key insights into the performance of different estimation methods under various conditions. Across all scenarios, the RMSEs decrease with increasing values of T and sample size n, indicating that higher T and larger n consistently lead to more accurate parameter estimates. This trend is evident for both MLE and Bayesian estimators. Notably, Bayesian estimators outperform the MLE in all cases, with $\hat{\delta}_4^{\pi}(c = -2)$ generally providing the lowest RMSEs, thereby demonstrating superior accuracy. Furthermore, while the variations in R influence the RMSEs, the overall pattern of Bayesian estimators exhibiting lower RMSEs than MLE remains consistent. These findings highlight the robustness of Bayesian methods, particularly with well-chosen priors, in enhancing the accuracy of parameter estimation in hybrid censored data scenarios. Thus, employing Bayesian approaches, especially $\hat{\delta}_4^{\pi}(c = -2)$, is recommended for more precise parameter estimation.

To sum up, our findings indicate that Bayesian estimators generally outperform MLE in terms of root mean squared errors (RMSEs), particularly with well-chosen priors, thus providing more accurate parameter estimates. The flexibility and practical relevance of Type-I hybrid censoring make it an effective framework for analyzing life data, and our methodologies offer valuable insights and tools for enhancing reliability and maintenance strategies in engineering systems. This study not only contributes to the theoretical understanding of parameter estimation under hybrid censoring but also supports improved decision-making in reliability engineering and related disciplines.

Future research could explore extensions to different distributions or incorporate real-world datasets to validate these findings further.

Declarations

Ethics Statement

We have conducted ourselves with integrity, fidelity, and honesty. We have not intentionally engaged in or participated in malicious harm to another person or animal.

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Appendix

Proof of Theorem 2.2.1

The posterior density function of $\boldsymbol{\lambda}$ given data is

$$\boldsymbol{\pi}(\boldsymbol{\lambda}|\boldsymbol{data}) = \frac{\boldsymbol{l}(\boldsymbol{data},\boldsymbol{\lambda})}{\int_{\mathbf{0}}^{\infty}\boldsymbol{l}(\boldsymbol{data},\boldsymbol{\lambda})\boldsymbol{d\lambda}}$$

$$= \frac{\boldsymbol{\lambda}^{-(\boldsymbol{D}^{*}+\mathbf{3})}\prod_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{D}^{*}}\boldsymbol{x}_{\boldsymbol{i}:\boldsymbol{n}}\boldsymbol{e}^{-\frac{\boldsymbol{s}}{\boldsymbol{\lambda}}}}{\int_{\mathbf{0}}^{\infty}\boldsymbol{\lambda}^{-(\boldsymbol{D}^{*}+\mathbf{3})}\prod_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{D}^{*}}\boldsymbol{x}_{\boldsymbol{i}:\boldsymbol{n}}\boldsymbol{e}^{-\frac{\boldsymbol{s}}{\boldsymbol{\lambda}}}\boldsymbol{d\lambda}}$$

$$= \frac{\boldsymbol{s}^{(\boldsymbol{D}^{*}+\mathbf{2})}\boldsymbol{\lambda}^{-(\boldsymbol{D}^{*}+\mathbf{3})}\boldsymbol{e}^{-\frac{\boldsymbol{s}}{\boldsymbol{\lambda}}}}{\boldsymbol{\Gamma}(\boldsymbol{D}^{*}\rightleftarrows+\mathbf{2})}$$

Proof of Theorem 2.2.2 Considering case 1, the posterior density function of $\boldsymbol{\lambda}$, for given data is

$$\boldsymbol{\pi}_{\mathbf{1}}(\boldsymbol{\lambda}|\boldsymbol{data}) = \frac{\boldsymbol{l}_{\mathbf{1}}(\boldsymbol{data},\boldsymbol{\lambda})}{\int_{\mathbf{0}}^{\infty}\boldsymbol{l}_{\mathbf{1}}(\boldsymbol{data},\boldsymbol{\lambda})\boldsymbol{d\lambda}}$$

$$= \frac{\boldsymbol{\lambda}^{-(\boldsymbol{R}+\boldsymbol{\alpha}+\mathbf{1})}\prod_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{R}}\boldsymbol{x}_{\boldsymbol{i}:\boldsymbol{n}}\boldsymbol{e}^{-\frac{\mathbf{1}}{\boldsymbol{\lambda}}\left(\boldsymbol{s}_{\mathbf{1}}+\frac{\mathbf{1}}{\boldsymbol{\beta}}\right)}}{\int_{\mathbf{0}}^{\infty}\boldsymbol{\lambda}^{-(\boldsymbol{R}+\boldsymbol{\alpha}+\mathbf{1})}\prod_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{R}}\boldsymbol{x}_{\boldsymbol{i}:\boldsymbol{n}}\boldsymbol{e}^{-\frac{\mathbf{1}}{\boldsymbol{\lambda}}\left(\boldsymbol{s}_{\mathbf{1}}+\frac{\mathbf{1}}{\boldsymbol{\beta}}\right)}\boldsymbol{d\lambda}}$$

$$= \frac{\left(s_1+\frac{1}{\beta}\right)^{R+\alpha}}{\Gamma(R+\alpha)} \lambda^{-(R+\alpha+1)} e^{-\frac{1}{\lambda}\left(s_1+\frac{1}{\beta}\right)}$$

which is the density function of Inverted Gamma with parameters specified in equation (20).

This completes the proof for case 1. Likewise, case 2 can be proved.