

Original Research Article

ON SOME TOPOLOGICAL PROPERTIES OF $C_h(X)$ AND OF THE DUAL OPERATOR

Abstract

The hypo-topology on the algebra $C(X)$ of real-valued continuous functions defined on a Tychonoff space 2^X

$f \in C(X)$ with its hypograph, $hypof = \{(x, t) \in X \times \mathbb{R} : f(x) \geq t\}$. This topology is very useful in the calculus of variations and in optimization theory (e.g. Maximization problems). We denote $C(X)$ with the hypo-topology by $C_h(X)$.

Our study deals with fundamental properties of these function spaces, and with the linear operators on them and as well as the characterization of the topological properties of $C_h(X)$ in terms of topological properties of the base space X .

We are studying the linear operator between the functional algebras $C_h(X)$ and $C_h(Y)$. We are primarily concerned with the continuity of the evaluation functional, the general evaluation and the continuity of the characters of $C_h(X)$ before the investigation of the properties of the dual operator f^* of a continuous function

$f: X \rightarrow Y$. This operator is defined by $f^*: C_h(Y) \rightarrow C_h(X)$, where $f^*(g) = g \circ f$ for all $g \in C(Y)$. The continuity of f^* enables us to characterize the continuity of an algebra homomorphism of the type $\varphi: C_h(Y) \rightarrow C_h(X)$ for a real compact space Y . For such a space, we present a type of Riesz theorem with states that an algebra homomorphism $\varphi: C_h(Y) \rightarrow C_h(X)$ is continuous if and only if there exists a unique hypo-function $f: X \rightarrow Y$ such that $\varphi = f^*$.

There after, we give the equivalence between the properties of f and those of f^* . The study of the continuous linear functional on $C_h(X)$ helps us to compute the topological dual space of this algebra. We show here that this dual space is useful only when the set of isolated points of X is dense.

Keywords: Hypo-topology, hypograph, hypo-function, π –basis.

INTRODUCTION

For a given 2^X Tychonoff space X , we denote by the set of all closed subsets of X . The Fell topology on 2^X is defined by the open subbases of the form

$$U^- = \{A \in 2^X : A \cap U \neq \emptyset\} \text{ et } V^+ = \{A \in 2^X : A \subseteq V\}$$

where U is an open set of X and $X \setminus V$ is a compact set of X .

Let be $f: X \rightarrow \mathbb{R}$ a function. We define the hypograph f by

$$hypof = \{(x, t) \in X \times \mathbb{R} : f(x) \geq t\}.$$

If in addition, f is semicontinuous above (respectively below) $\text{hypof} \in 2^{X \times \mathbb{R}}$. We define the Fell topology on $C(X)$, called hypo-topology by identifying each $f \in C(X)$ with its hypograph. We denote $C(X)$ equipped with the hypo-topology by $C_h(X)$.

An element of the subbase of the open sets of $C_h(X)$ is of the form

$$U^- = \{f \in C(X) : (\text{hypof}) \cap U \neq \emptyset\} \text{ et}$$

$$V^+ = \{f \in C(X) : (\text{hypof}) \subseteq V\}$$

where U is an open subset of $X \times \mathbb{R}$ et $(X \times \mathbb{R}) \setminus V$ is a compact subset of $X \times \mathbb{R}$.

For a subset A of X and a subset V of \mathbb{R} , we set $[A, V] = \{f \in C(X) : f(A) \subseteq V\}$. With this notation, let U be an open subset of X , K a compact subset of X and ℓ an element of \mathbb{R} . We define

$$[U, \ell]^- = \{f \in C(X) : \exists x \in U \text{ with } f(x) > \ell\} \text{ and}$$

$$[K, \ell]^+ = \{f \in C(X) : f(x) < \ell \text{ for every } x \in K\}.$$

In such a way, $[U, \ell]^- = U \{[\{x\}, (\ell, +\infty)] : x \in U\} = (U \times (\ell, +\infty))^-$ and

$[K, \ell]^+ = ((X \times \mathbb{R}) \setminus (K \times \{\ell\}))^+ = [K, (-\infty, \ell)]$. Thus, the sets $[U, \ell]^-$ and $[K, \ell]^+$ are open in the hypo-topology. Here the intervals $(\ell, +\infty)$ et $(-\infty, \ell)$ represent the set of real numbers strictly greater than ℓ and strictly less than ℓ respectively.

I. RESULTS

This paper is an investigation of linear operators between algebras $C_h(X)$ et $C_h(Y)$. We are particularly interested in linear operators between these algebras. We study the continuity of some remarkable functions in relation to hypo-topology [2].

1. Properties of the dual operator. [8]; [9]; [10]; [11], [4]

a. Injection of the dual operator

A map $f: X \rightarrow Y$ is almost surjective if and only if $f(x)$ is a dense subset of Y .

b. Theorem

Let be $f: X \rightarrow Y$ a continuous function and $f^*: C_h(Y) \rightarrow C_h(X)$ its dual operator. Then f^* is injective if and only if f is almost surjective.

Evidence

Suppose f^* injective. To show that $f(x)$ is dense in Y , suppose the opposite, that is, $\overline{f(x)} \neq Y$. Let $y \in Y \setminus \overline{f(x)}$. Since Y is completely regular, it exists $g \in C(Y)$ such that $g(y) = 1$ et $g(\overline{f(x)}) = \{0\}$. It follows that, $g(f(x)) = \{0\}$ that is, $(g \circ f)(x) = \{0\}$. This means that, $g \circ f = 0$ that is, $f^*(g) = 0_X = f^*(0_Y)$.

From $f^*(g) = f^*(0_Y)$ we obtain $g = 0_Y$ because f^* is injective. $g = 0_Y$ contradicts the construction of g because $g(y) = 1 \neq 0$. In conclusion we must have $\overline{f(x)} = Y$ and f is almost surjective.

Conversely, let us assume f that is almost surjective. To see that f^* is injective, let $g, h \in C(Y)$ such that $f^*(g) = f^*(h)$. Let us show that $g = h$. Let therefore $y \in f(X)$. There exists $x \in X$ such that

$$y = f(x).$$

$$\begin{aligned} \text{From where, } g(y) &= g(f(x)) = (g \circ f)(x) = f^*(g)(x) = f^*(h)(x) = (h \circ f)(x) \\ &= h(f(x)) = h(y). \end{aligned}$$

So $g = h$ on $f(X)$, hence the equality $g = h$ on X .

This being true for everything $(g, h) \in C(Y) \times C(Y)$, we conclude that f^* is injective. ■

2) Surjection of the dual operator

A subset A of a Tychonoff space is c -immersed in X if and only if every continuous and bounded application $f: A \rightarrow \mathbb{R}$ admits a continuous extension $F: X \rightarrow \mathbb{R}$ (i.e. $f(a) = F(a)$ for all $a \in A$).

With this notion, we obtain the following result:

3. Theorem 1

Let be $f: X \rightarrow Y$ a continuous function and $f^*: C_h(Y) \rightarrow C_h(X)$ its dual operator.

Then f^* is surjective if and only if f is a homeomorphism from X to its image $f(X)$ et $f(X)$ is C -immé rgé in Y .

Evidence

Suppose f^* surjective. We first show that f is injective. To do this, let x et x' two elements of X be such that $f(x) = f(x')$.

Let us suppose for the sake of absurdity that $x \neq x'$. Since X is completely regular, there exists $g \in C(X)$ such that $g(x) = 1$ et $g(x') = 0$.

By surjectivity of f^* , there exists $h \in C(Y)$ such that $g = f^*(h)$. Now, $1 = g(x) = f^*(h)(x) = (h \circ f)(x) = h(f(x)) = h(f(x'))$

$= (h \circ f)(x) = f^*(h)(x') = g(x') = 0$ is a contradiction (because $1 \neq 0$). Hence $x = x'$ and f is injective.

Next, to show that $f: X \rightarrow f(X)$ is a homeomorphism, let us show that $f^{-1}: f(X) \rightarrow X$ is continuous. Let $y_0 \in f(X)$. Let us show that f^{-1} is continuous at y_0 .

To do this, let V be a neighborhood of $f^{-1}(y_0)$ in X . We must find a neighborhood W of y_0 in $f(X)$ tel que $f^{-1}(W) \subset V$. As $y_0 \in f(X)$, there exists $x_0 \in X$ such that $y_0 = f(x_0)$. Also, $f^{-1}(y_0) = f^{-1}(f(x_0)) = x_0$ and therefore $x_0 \in V$. As V is a neighborhood of x_0 in X and X is completely regular, there exists $g \in C(X)$ such that $g(x_0) = 0$ and $g(X \setminus V) = \{1\}$. By the surjectivity of f^* , there exists $h \in C(Y)$ such that $g = f^*(h)$.

NOW, $0 = g(x_0) = f^*(h)(x_0) = (hof)(x_0) = h(f(x_0)) = h(y_0)$.

By posing $W = h^{-1}(0,1) \cap f(X)$, we have that W is a neighborhood of y_0 in $f(X)$. We want to show that $f^{-1}(W) \subset V$. By calculating $f^{-1}(W)$, we have:

$$\begin{aligned} f^{-1}(W) &= f^{-1}[f(X) \cap h^{-1}(0,1)] \\ &= f^{-1}(f(X)) \cap f^{-1}(h^{-1}(0,1)) \\ &= X \cap (hof)^{-1}[0,1] \\ &= (hof)^{-1}[0,1] \\ &= (f^*(h))^{-1}[(0,1)] \end{aligned}$$

To see that $f^{-1}(W) \subset V$, either $x \in f^{-1}(W) = (f^*(h))^{-1}[0,1]$. So $f^*(h)(x) \in (0,1)$

Let us assume by absurdity that $x \notin V$. Then $x \in X \setminus V$ and so $g(x) = 1$ by the construction of g .

then turns out that $1 = g(x)f^*(h)(x) \in (0,1)$ is a contradiction because $1 \notin (0,1)$. Thus $x \in V$. This being true for all $x \in f^{-1}(W)$, we conclude that $f^{-1}(W) \subset V$ and f^{-1} is continuous in y_0 as desired.

But then y_0 being arbitrary in $f(X)$, f^{-1} is continuous on $f(X)$. Therefore f is a homeomorphism from X to $f(X)$.

Finally, it remains to prove that $f(x)$ is C -immersed in Y .

Let be $g: f(x) \rightarrow \mathbb{R}$ a bounded and continuous function. We must construct a continuous function $g^*: Y \rightarrow \mathbb{R}$ such that $\hat{g} = g$ on $f(x)$.

We have the composite: $X \rightarrow f(x) \rightarrow \mathbb{R}$, that is to say $g \circ f: X \rightarrow \mathbb{R}$ which is continuous being the composite of 2 continuous functions.

So $g \circ f \in C(X)$. By the surjectivity of f^* , there exists $g^* \in C(Y)$ such that $f^*(g^*) = g \circ f$, that is $g^* \circ f = g \circ f$. To see that g^* is the desired extension, let $y \in f(x)$. Then there exists $x \in X$ tel que $y = f(x)$.

Hence $g(y) = g(f(x)) = (g \circ f)(x) = (g^* \circ f)(x) = g^*(f(x)) = g^*(y)$, for everything $y \in f(x)$.

This means that $g = g^* \circ f$ et $f(x)$ is C -immersed in Y .

Conversely, suppose that $f: X \rightarrow f(X)$ is a homeomorphism such that $f(X)$ is C -immersed in Y . Let us show that f^* is surjective.

Let $g \in C(f(X))$. us consider the composite: $f(X) \rightarrow f(X) \rightarrow \mathbb{R}$ clearly, $g \circ f^{-1}$

$g \circ f^{-1} \in C(X)$. Since X is C -immersed in Y , there exists

$h \in C(Y)$ such as $h = g \circ f^{-1}$ on $f(X)$. We show that $g = \hat{f}(h)$. Indeed, if $x \in X$, then $f(x) \in f(X)$. Whence $h(f(x)) = (g \circ f^{-1})(f(x))$
 $= g[f^{-1}(f(x))] = g(x)$, that's to say $g(x) = h(f(x)) = (hof)(x)$
 $= f^*(h)(x)$ for everything $x \in X$.

So, we have $g = f^*(h)$. This being true for everything $g \in C(X)$, f^* is surjective. ■

Theorem 2

f^* is bijective if and only if f is a homeomorphism from X onto $f(X)$ and $f(X)$ is dense and C -immersed in Y .

3. The almost surjection of the dual operator. [8]; [16]

A function $f: X \rightarrow Y$ is a hypofunction if and only if for any open set V and any compact set K of X we have $f(K) \subset V \Rightarrow K \subset U$.

Any injective application is a hypo-function.

Theorem

Let be $f: X \rightarrow Y$ a function and $f^*: C_h(Y) \rightarrow C_h(X)$ its dual operator. Then f^* is almost surjective if and only if f is a hypo-function.

Evidence

Suppose that f^* is a hypofunction. Let us show that $f^*(C_h(Y))$ is dense in $C_h(X)$. To do this, it suffices to establish that any open set with a non-empty basis in $C_h(X)$ intersects $f^*(C_h(Y))$. So let

$B = [U_1, s_1]^- \cap \dots \cap [U_m, s_m]^- \cap [K_1, t_1]^+ \cap \dots \cap [K_n, t_n]^+$ an open set with a nonempty base in $C_h(X)$.

Either $I = \{1, \dots, m\}$ et $J = \{1, \dots, n\}$. Let $i \in I$. Let us define J_i, p_i, q_i, K_i, x_i et g_i

in the following manner. First either $J_i = \{j \in J : t_j \leq s_i\}$.

Choisissons p_i, q_i , in \mathbb{R} such that $s_i < p_i$ et $q_i < p_i$, and also such that $p_i < \min\{t_j : j \in J \setminus J_i\}$ if $J \setminus J_i \neq \emptyset$ et $q_i < \min\{t_j : j \in J_i\}$ si $J_i \neq \emptyset$.

Let's ask $K'_i = \begin{cases} \cup \{K_j : j \in J_i\} & \text{si } J_i \neq \emptyset \\ \emptyset & \text{si } J_i = \emptyset \end{cases}$

Since B is not empty, then $U_i \not\subset K'_i$. Since f is a hypo-function, there exists $x_i \in U_i \setminus K'_i$ such that $f(x_i) \notin f(K'_i)$. Finally, let be $g_i: Y \rightarrow [q_i, p_i]$ a continuous function such that $g_i(f(x_i)) = q_i$ et $g_i(Y) = p_i$ for all $y \in f(K'_i)$. We then define $g: Y \rightarrow \mathbb{R}$ by $g(y) = \max\{g_i(y) : i \in I\}$ for all $y \in Y$. Clearly $g \in C_h(Y)$. We will then show that $f^*(g) \in B \cap f^*(C_h(Y))$.

Let us show that $f^*(g) \in B$. For all $i \in I$, we know that $(gof)(x_i) = g(f(x_i)) \leq g_i(f(x_i)) = q_i > s_i$. Hence $gof \in [U_i, s_i]^-$ for all $i \in I$.

Now let's take $i \in I, j \in J$ et $x \in K_j$. If $j \in J_i$, then $x \in K'_i$; hence $g_i(f(x)) \in g_i(f(K'_i))$ and thus $g_i(f(x)) = p_i < t_j$; such that $g_i(f(x)) < q_i < t_j$; which means that $gof \in [K_j, t_j]^+$ for all $j \in J$. Hence $f^*(g) = gof \in B$.

So $B \cap f^*(C_h(Y)) \neq \emptyset$ and so $f^*(C_h(Y))$ is dense in $C_h(X)$.

Conversely, suppose that f is not a hypofunction. We will show that $f^*(C_h(Y))$ is not dense in $C_h(X)$. Since f is not a hypofunction, there exists an open set U of X and a compact subset K of X such that $U \not\subset K$ mais

$\phi(U) \subset \phi(K)$. Let us define $B = [U, 0]^- \cap [K, 0]^+$ which is an open set of $C_h(X)$. Let $x_0 \in U \setminus K$, and be $g: x \rightarrow [-1, 1]$ a continuous function such that $g(x_0) = -1$ and $g(x) = 1$ for all $x \in K$. Then $g \in B$, and thus $B \neq \emptyset$.

Let us show that $f^*(K) \in B$ for all $K \in C(Y)$ that is to say that $K^*(C_h(Y)) \cap B = \emptyset$. Let us suppose for absurdity that this is not true. Let then be $k \in K^*(C_h(Y))$ such that $kof \in B$.

As $kof \in [U, 0]^-$, there exists $x_1 \in U$ such that $k(f(x_1)) > 0$. As $f(x_1) \in f(U) \subseteq f(K)$, there exists such $x_2 \in K$ that $f(x_1) = f(x_2)$. Hence $k(f(x_2)) < 0$ because $kof \in [K, 0]^+$; which contradicts the inequality

$$k(f(x)) > 0.$$

So $K^*(C_h(Y)) \cap B = \emptyset$ and so $K^*(C_h(Y))$ is not dense in $C_h(X)$. ■

4. Embedding the dual operator. [8];[10];[6];[7];[1]

A continuous application $f: X \rightarrow Y$ is a k -function if and only if every compact set of Y is an image of f a compact set of X . (That is, $\forall K$ compact set of Y , there exists C compact set of X such that $K = f(C)$).

Theorem

Let be $f: X \rightarrow Y$ a continuous function and $f^*: C_h(Y) \rightarrow C_h(X)$ its dual operator. Then f^* is an embedding of $C_h(Y)$ in $C_h(X)$ if and only if f is a weakly open k -function.

Evidence

Let $\mathcal{R} = f^*(C(Y))$. First suppose that $f^*: C_h(Y) \rightarrow \mathcal{R}$ is a homeomorphism. By the continuity of f^* , f is already weakly open. It remains to show that f is a k -function. To do this, let A be a compact of X . Since $W = [A, (-\infty, 1)]$ is an open neighborhood of 0_Y , then $f^*(W)$ is an open neighborhood of 0_X in \mathcal{R} . There exist compacts K_1, K_2, \dots, K_n in X and nonempty open sets U_1, U_2, \dots, U_n in X and reals $t_1, t_2, \dots, t_n, s_1, s_2, \dots, s_n$ such that

$$0_X \in [K_i, t_i]^+ \cap \dots \cap [K_n, t_n]^+ \cap [U_i, s_i] \cap \dots \cap [U_m, s_m] \cap \mathcal{R} \subset f^*(w).$$

For each $1 \leq i \leq n$, be $x_i \in U_i$.

assume $k = \{x_1, x_2, \dots, x_n\} \cup K_1 \cup \dots \cup K_m$ that is a compact of X . We first show that $A \subset f(K)$. Indeed, assuming the opposite, there would exist $a \in A \setminus f(K)$. Since Y is a Tychonoff space, there would exist $g: Y \rightarrow [0, 1]$ a continuous such that

$g(a) = 1$ et $g(f(K)) = \{0\}$. So $f^*(g) \in f^*(W)$ and so it would exist

$h \in W = [A, (-\infty, 1)]$ such that $f^*(g) = f^*(h)$. By the injectivity of f^* , we will have $g = h$. From which $g(a) = h(a) < 1$ on the one hand and, on the other hand by $g(a) = 1$ the construction of g . With this contradiction, we must have $A \subset f(K)$. Now, by setting $C = K \cap f^{-1}(A)$, we have a compact of X such that $A = f(C)$. Therefore f is a k -function.

For the converse, suppose that $f: X \rightarrow Y$ is a weakly open k -function. We already know that $f^*: C_h(Y) \rightarrow C_h(X)$ is continuous by the previous theorem. Also, since f is surjective, f^* is injective. By setting $\mathcal{R} = f^*(C(Y))$, we have $f^*: C_h(Y) \rightarrow \mathcal{R}$ bijective and continuous. To have the desired homeomorphism, we show that $(f^*)^{-1}: \mathcal{R} \rightarrow C_h(Y)$ is continuous. For this, let $W_1 = [A_1, (-\infty, t)]$ an open set of the subbase of $C_h(Y)$, where A is a compact set of Y and $t \in \mathbb{R}$. Let us show that $((f^*)^{-1})^{-1}(W_1)$ is an open set of \mathcal{R} . But $((f^*)^{-1})^{-1}(W_1) = f^*(W_1)$.

Let $g \in f^*(W_1)$. Then there exists $h \in W_1$ such that $g = f^*(h) = h \circ f$. Since f is a k -function and A_1 is a compact of Y , there exists C compact of X such that $A_1 = f(C)$. Then $g \in \mathcal{R} \cap [C, (-\infty, t)] \subset f^*(W_1)$ show that $f^*(W_1)$ is a neighborhood of g in \mathcal{R} . From where $f^*(W_1)$ is opened from \mathcal{R} .

Similarly, let be $W_2 = [U_2, S]$ another open number of the subbase of $C_h(Y)$ where U_2 is a non-empty open number of Y and S a real number.

Either $g \in ((f^*)^{-1})^{-1}(W_2) = f^*(W_2) = f^*(U[Y, (S, +\infty)] : y \in U_2)$

$= U f^*(U[Y, (S, +\infty)])$. There exists $y_0 \in U_2$ such that $g \in f^*([y_0, (S, +\infty)])$. Let $h \in [y_0, (S, +\infty)]$ such that $g = f^*(h) = h \circ f$. Also, as f is surjective and $y_0 \in U_2 \subset Y$, there exists $x_0 \in X$ such that $f(x_0) = y_0 \in U_2$. Then $x_0 \in f^{-1}(U_2)$. Let us set $W_3 = [f^{-1}(U_2), 1]$. Then as

$g(x_0) = h(f(x_0)) = h(y_0) > S$. SO $g \in [x_0, (S, +\infty)] \cap \mathcal{R} \subset [f^{-1}(U_2), S]$

$\cap \mathcal{R} = W_3 \cap \mathcal{R} \subset f^*(W_2)$. For the last inclusion, if $\ell \in W_3 \cap \mathcal{R}$, there exists $K \in C(Y)$ such that $\ell = f^*(h) = h \circ f$. $\ell \in W_3 = [f^{-1}(U_2), S]$ shows that there exists $z \in f^{-1}(U_2)$ such that $\ell(z) > S$. Now $f(z) \in U_2$ et

$k = (f(z)) > S$ implies that $k \in [U_2, S] = W_2$.

So, $f^*(k) \in f^*(W_2)$ that is to say that $\ell \in f^*(W)$. We obtain the implication $W_3 \cap \mathcal{R} \subset f^*(W_2)$. From $g \in f^*(W_2)$, we draw

$g \in W_3 \cap \mathcal{R} \subset f^*(W_2)$; which means that $f^*(W_2)$ is a neighborhood of g in \mathcal{R} for all $g \in f^*(W_2)$. From which $f^*(W_2)$ is open of \mathcal{R} .

In conclusion, $(f^*)^{-1}: \mathcal{R} \rightarrow C_h(Y)$ is continuous and thus $f^*: C_h(Y) \rightarrow C_h(X)$ is an extension of $C_h(Y)$ in $C_h(X)$. ■

It is known (see [16]) that $f^*: C_k(Y) \rightarrow C_k(X)$ is an embedding if and only if f is a k -function.

5. Homomorphism of algebras. [5];[8];[3];[14];[15]

5.1. Continuity of homomorphisms of algebras

1. Definition

a) An application $\lambda: C(Y) \rightarrow C(X)$ is a homomorphism of \mathbb{R} -algebras if and only if $\lambda(1_Y) = 1_X$, $\lambda(\alpha f + \beta g) = \alpha \lambda(f) + \beta \lambda(g)$ et

$\lambda(fg) = \lambda(f)\lambda(g)$ for everything $f, g \in C(Y)$ et $\alpha, \beta \in \mathbb{R}$.

Here 1_Y et 1_X are the unit elements of the algebras $C(Y)$ et $C(X)$ respectively.

Tychonoff space X in which every character of $C(X)$ is of the form e_x where $x \in X$ is called a full or real compact space. Among the full spaces, we can cite the compact spaces.

We equip $C(Y)$ and $C(X)$ with the hypo-topology and we obtain the following theorem:

2) Theorem

Let Y be a full space. An algebra homomorphism $\lambda: C_h(Y) \rightarrow C_h(X)$ is continuous if and only if there exists a unique weakly open continuous function $f: X \rightarrow Y$ such that $\lambda = f^*$.

Evidence

If f is continuous, weakly open such that $\lambda = f^*$, by Theorem 2, $\lambda = f^*$ is continuous.

For the converse, let us assume $\lambda: C_h(Y) \rightarrow C_h(X)$ continuous homomorphism of algebras. Let $x \in X$. Since $e_x \circ \lambda: C_h(Y) \rightarrow \mathbb{R}$ is a character of $C_h(Y)$ and Y is a full space, there exists $Y_x \in Y$ such that $e_x \circ \lambda = e_{Y_x}$. We then define $f: X \rightarrow Y$ by $f(x) = Y_x$ for all $x \in X$.

Now, let $g \in C(Y)$. Then $\lambda(g) \in C(X)$. From where for all $x \in X$, we have: $e_x(\lambda(g)) = (e_x \circ \lambda)(g) = e_{Y_x}(g) = e_{f(x)}(g) = g(f(x)) = (g \circ f)(x)$.

So $\lambda(g)(x) = e_x(\lambda(g)) = (g \circ f)(x)$ that is to say that $\lambda(g) = g \circ f = f^*(g)$ for all $g \in C(Y)$. This shows that $\lambda = f^*$.

Moreover, since Y is completely regular and $g \circ f = \lambda(g) \in C(X)$ for all $g \in C(Y)$, then f becomes continuous.

Finally, the uniqueness of f follows from the fact that $C(Y)$ separates the points of Y . Also, by the continuity of $\lambda = f^*$, f is weakly open. ■

6. TOPOLOGICAL DUAL OF $C_h(X)$. [19];[18];[17];[8];[12];[13]

6.1. The weak hypo-topology of $C_h(X)$

Although $C_h(X)$ is not in general a topological vector space, we can however speak of its topological dual $C'_h(X)$ by considering $C'_h(X) = \{\lambda: C_h(X) \rightarrow \mathbb{R}/\lambda \text{ is linear and continuous}\}$. The smallest (in the sense of inclusion) topology on $C(X)$ that makes every element of continuous $C'_h(X)$ is denoted by $C_S(X)$ and is called the weak hypo-topology. This topology is useful only for spaces X having dense isolated points.

We now take topological spaces X whose set of isolated points I_X is dense in X . For $x \in X$, we consider the multiplicative linear form $e_x: C(X) \rightarrow \mathbb{R}$ defined by $e_x(f) = f(x)$ for all $f \in C(X)$. Then the weak hypo-topology is generated by the e_x or $x \in I_X$.

An element of the subbase of open sets is of the form $[A, V]$ where A is a finite subset of I_X and V is an open set of \mathbb{R} . It is clear that $C_S(X)$ is the topology of simple convergence on isolated points of X .

We can also consider $C_S(X)$ as a locally convex vector space whose topology is generated by the semi-norm $p_A: C(X) \rightarrow \mathbb{R}$ defined by $p_A(f) = \sup\{|f(x)|: x \in A\}$ where A is a finite subset of I_X . A basic neighborhood of f in $C_S(X)$ is of the form $\langle f, A, \varepsilon \rangle = \{g \in C(X): |g(x) - f(x)| < \varepsilon, \forall x \in A\}$ where $A \subset I_X$, A is finite and $\varepsilon > 0$ is a real number.

6.2. Topological dual of $C_h(X)$

The weak hypo-topology is both less fine than hypo-topology and the topology of simple convergence on X . To compute the topological dual $C'_S(X)$ of $C_S(X)$, it is clear that $C'_h(X) \subset C'_S(X)$. Our goal is to show that there is equality between these two dual spaces.

1) Theorem

Let be $\lambda: C_S(X) \rightarrow \mathbb{R}$ a non-zero and continuous linear form. Then there exists x_1, x_2, \dots, x_n in I_X such that λ is a linear combination of e_{x_i} .

Evidence

As the open interval $(-1, 1)$ is a neighborhood of $\lambda(O_x) = 0$ in \mathbb{R} , by the continuity of λ at the point O_x , there exists a finite subset $A = \{x_1, x_2, \dots, x_n\}$ in I_X and $\varepsilon > 0$ real such that $\lambda(\langle O_x, A, \varepsilon \rangle) \subset (-1, 1)$.

We consider the $(n + 1)$ - linear forms $\lambda, e_{x_1}, e_{x_2}, \dots, e_{x_n}$ on $C(X)$. By a result of linear algebra either λ is a linear combination of e_{x_i} or then there exists $g \in C(X)$ such that $\lambda(g) = 1$ et $g \in \bigcap_{i=1}^{n+1} \text{Ker } e_{x_i}$. If there exists the same g then $g \in \langle O_x, A, \varepsilon \rangle$ which would imply $1 = \lambda(g) \in \lambda(\langle O_x, A, \varepsilon \rangle) \subset (-1, 1)$

With this contradiction, we conclude that λ must be a linear combination of e_{x_i} .

We have just shown that any continuous linear form on $C_S(X)$ is also continuous on $C_h(X)$. From which we have the following corollary: ■

2) Corollary

$$C'_S(X) = C'_h(X) = \left\{ \sum_{i=1}^n \alpha_i e_{x_i} : n \in \mathbb{N}, \alpha_i \in \mathbb{R} \text{ et } x_i \in I_x \right\} = \text{Eng}(e_{I_x})$$

Or $e_{I_x} = \{e_x : x \in I_x\}$.

CONCLUSION

We have now reached the end of our article, the aim of which was to study some topological properties of $C_h(X)$ and of the dual operator as a function of the topological properties of the Tychonoff space X .

At the operator level, we exploited the continuity of some special functions, before tackling the dual operator which allowed us to characterize the continuity of a homomorphism of algebras of type $\varphi: C_h(Y) \rightarrow C_h(X)$ for a replete space Y . For such a space, $\varphi: C_h(Y) \rightarrow C_h(X)$ is continuous if and only if there exists a unique hypo-function $f: X \rightarrow Y$ such that $\varphi = f^*$. For a continuous $f: X \rightarrow Y$ function, we define its dual application $f: C_h(Y) \rightarrow C_h(X)$ by $f^*(g) = g \circ f$, for all g in $C(Y)$.

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- 2.
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