A Note on Sum Formulae of Generalized Pentanacci Sequence

Abstract

During this paper, we first demonstrate the closed forms of sum formulae both $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{\lambda}$ and $\sum_{\lambda=1}^{\eta} \lambda z^{\lambda} A_{-\lambda}$ for the generalized Pentanacci numbers. Then, we grant summation formulae for the sequences such as Pentanacci, Pentanacci-Lucas and other fifth-order iteration sequences.

Keywords: Pentanacci numbers; Pentanacci-Lucas numbers; sum formulae; summing formulae

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1 Introduction

First of all, we revisit the Pentanacci sequences. The Pentanacci sequence is a higher-order generalization of the well-known Fibonacci sequence. Defined by a linear recurrence relation, each term in the Pentanacci sequence is the sum of the five preceding terms:

$$A_{\eta} = A_{\eta-1} + A_{\eta-2} + A_{\eta-3} + A_{\eta-4} + A_{\eta-5}, \eta \geq 5;$$

with initial conditions A_0, A_1, A_2, A_3, A_4 set to predefined values. The Pentanacci sequence has applications in theoretical mathematics, particularly in combinatorics and the analysis of recursive algorithms. Its study reveals deeper insights into the properties of linear recursions and their behavior in higher dimensions, offering attractive connections to both discrete mathematics and dynamical systems.

Now, we recall the generalized Pentanacci numbers. The generalized Pentanacci sequence $\{A_\eta(A_0, A_1, A_2, A_3, A_4; r, s, t, u, v)\}_{\eta\geq 0}$ (or $\{A_\eta\}_{\eta\geq 0}$) is defined in the following way:

$$\begin{aligned} A_\eta &= rA_{\eta-1} + sA_{\eta-2} + tA_{\eta-3} + uA_{\eta-4} + vA_{\eta-5}, \\ A_0 &= c_0, A_1 = c_1, A_2 = c_2, A_3 = c_3, A_4 = c_4, \eta \geq 5, \end{aligned} \quad (1.1)$$

where A_0, A_1, A_2, A_3, A_4 are arbitrary numbers (real or complex) and r, s, t, u, v are real numbers. The sequence $\{A_\eta\}_{\eta\geq 0}$ have negative subscripts by definition as

$$A_{-\eta} = -\frac{u}{v}A_{-\eta+1} - \frac{t}{v}A_{-\eta+2} - \frac{s}{v}A_{-\eta+3} - \frac{r}{v}A_{-\eta+4} + \frac{1}{v}A_{-\eta+5},$$

for $\eta = 1, 2, 3, ...$ where $v \neq 0$. Therefore, recurrence (1.1) satisfies for all integer η.

Over the last years, the Pentanacci sequence has been the subject of extensive research by various authors; see, for instance, [(8)], [(9)], [(11)], [(26)].

Table 1: Some special cases of the generalized Pentanacci sequences.

No	Sequences (Numbers)	Notation	Ref
1	Generalized Pentanacci	$\{V_\eta\} = \{A_\eta(A_0, A_1, A_2, A_3, A_4; 1, 1, 1, 1, 1)\}$	[(26)]
2	Generalized Fifth order Pell	$\{V_\eta\} = \{A_\eta(A_0, A_1, A_2, A_3, A_4; 2, 1, 1, 1, 1)\}$	[(27)]
3	Generalized Fifth order Jacobsthal	$\{V_\eta\} = \{A_\eta(A_0, A_1, A_2, A_3, A_4; 1, 1, 1, 1, 2)\}$	[(28)]
4	Generalized 5-primes	$\{V_\eta\} = \{A_\eta(A_0, A_1, A_2, A_3, A_4; 2, 3, 5, 7, 11)\}$	[(29)]

For some specific values of A_0, A_1, A_2, A_3, A_4 and r, s, t, u, v, it is worthwhile to present these special Pentanacci numbers in a table under a specific name. As an example, the literature employs the sequence names and symbols (refer to Table 2) for specific cases of r, s, t, u, v, along with their initial values.

The sequences and notations used in this study are as follows: Pentanacci $\{P_\eta\}$, Pentanacci-Lucas $\{Q_\eta\}$, fifth-order Pell $\{P_\eta^{(5)}\}$, fifth-order Pell-Lucas $\{Q_\eta^{(5)}\}$, modified fifth-order Pell $\{E_\eta^{(5)}\}$, fifth-order Jacobsthal $\{J_\eta^{(5)}\}$, fifth-order Jacobsthal-Lucas $\{j_\eta^{(5)}\}$, modified fifth-order Jacobsthal $\{K_\eta^{(5)}\}$, fifth-order Jacobsthal Perrin $\{Q_\eta^{(5)}\}$, adjusted fifth-order Jacobsthal $\{S_\eta^{(5)}\}$, modified fifth-order Jacobsthal-Lucas $\{R_\eta^{(5)}\}$, 5-primes $\{G_\eta\}$, Lucas 5-primes $\{H_\eta\}$, modified 5-primes $\{E_\eta\}$.

Table 2: Some members of generalized Pentanacci sequences.

Notation	OEIS [(12)]	Ref
$\{P_\eta\} = \{A_\eta(0, 1, 1, 2, 4; 1, 1, 1, 1, 1)\}$	A001591	[(26)]
$\{Q_\eta\} = \{A_\eta(5, 1, 3, 7, 15; 1, 1, 1, 1, 1)\}$	A074048	[(26)]
$\{P_\eta^{(5)}\} = \{A_\eta(0, 1, 2, 5, 13; 2, 1, 1, 1, 1)\}$	A141448	[(27)]
$\{Q_\eta^{(5)}\} = \{A_\eta(5, 2, 6, 17, 46; 2, 1, 1, 1, 1)\}$		[(27)]
$\{E_\eta^{(5)}\} = \{A_\eta(0, 1, 1, 3, 8; 2, 1, 1, 1, 1)\}$		[(27)]
$\{J_\eta^{(5)}\} = \{A_\eta(0, 1, 1, 1, 1; 1, 1, 1, 1, 2)\}$	A226310	[(28),(2)]
$\{j_\eta^{(5)}\} = \{A_\eta(2, 1, 5, 10, 20; 1, 1, 1, 1, 2)\}$	A226311	[(28),(2)]
$\{K_\eta^{(5)}\} = \{A_\eta(3, 1, 3, 10, 20; 1, 1, 1, 1, 2)\}$		[(28)]
$\{Q_\eta^{(5)}\} = \{A_\eta(3, 0, 2, 8, 16; 1, 1, 1, 1, 2)\}$		[(28)]
$\{S_\eta^{(5)}\} = \{A_\eta(0, 1, 1, 2, 4; 1, 1, 1, 1, 2)\}$		[(28)]
$\{R_\eta^{(5)}\} = \{A_\eta(5, 1, 3, 7, 15; 1, 1, 1, 1, 2)\}$		[(28)]
$\{G_\eta\} = \{A_\eta(0, 0, 0, 1, 2; 2, 3, 5, 7, 11)\}$		[(29)]
$\{H_\eta\} = \{A_\eta(5, 2, 10, 41, 150; 2, 3, 5, 7, 11)\}$		[(29)]
$\{E_\eta\} = \{A_\eta(0, 0, 0, 1, 1; 2, 3, 5, 7, 11)\}$		[(29)]

To simplify notation, we henceforth omit the superscripts in these sequences. For instance, we use P_η instead of $P_\eta^{(5)}$.

Table 3: Some special studies of sum formulas.

Sequences	Papers dealing with sum formulae
Pell and Pell-Lucas	[(1),(4),(32),(6),(7)]
Generalized Fibonacci	[(5),(13),(14),(15),(16),(17),(19)]
Generalized Tribonacci	[(3),(10),(18)]
Generalized Tetranacci	[(20),(25),(33)]
Generalized Pentanacci	[(21),(22)]
Generalized Hexanacci	[(23),(24)]

Theorem 1.1. *Let z be a real (or complex) number. For $\eta \geq 0$, we have the following formulas:*

(a) *If $rz + sz^2 + tz^3 + uz^4 + vz^5 - 1 \neq 0$, then*

$$\sum_{\lambda=0}^{\eta} z^{\lambda} A_{\lambda} = \frac{\Theta_1(z)}{rz + sz^2 + tz^3 + uz^4 + vz^5 - 1} = \frac{\Theta_1(z)}{\Theta(z)},$$

where,

$\Theta_1(z) = z^{\eta+4}A_{\eta+4} - (rz-1)z^{\eta+3}A_{\eta+3} - (sz^2+rz-1)z^{\eta+2}A_{\eta+2} - (sz^2+tz^3+rz-1)z^{\eta+1}A_{\eta+1} + vz^{\eta+5}A_{\eta} - z^4A_4 + z^3(rz-1)A_3 + z^2(sz^2+rz-1)A_2 + z(sz^2+tz^3+rz-1)A_1 + (sz^2+tz^3+uz^4+rz-1)A_0.$

(b) *If $r^2z + 2uz^2 - s^2z^2 + t^2z^3 - u^2z^4 + v^2z^5 + 2sz + 2rtz^2 + 2rvz^3 - 2suz^3 + 2tvz^4 - 1 \neq 0$, then*

$$\sum_{\lambda=0}^{\eta} z^{\lambda} A_{2\lambda} = \frac{\Theta_2(z)}{r^2z + 2uz^2 - s^2z^2 + t^2z^3 - u^2z^4 + v^2z^5 + 2sz + 2rtz^2 + 2rvz^3 - 2suz^3 + 2tvz^4 - 1},$$

where,

$\Theta_2(z) = -(uz^2+sz-1)z^{\eta+1}A_{2\eta+2} + (t+rs+vz+ruz)z^{\eta+2}A_{2\eta+1} + (u+t^2z-u^2z^2+v^2z^3+rt+2tvz^2+rvz-suz)z^{\eta+2}A_{2\eta} + (v+ru-svz+tuz)z^{\eta+2}A_{2\eta-1} + v(r+vz^2+tz)z^{\eta+2}A_{2\eta-2} + z^2(uz^2+sz-1)A_4 - z^3(t+rs+vz+ruz)A_3 + z(r^2z+uz^2-s^2z^2+2sz+rtz^2+rvz^3-suz^3-1)A_2 - z^3(v+ru-svz+tuz)A_1 + (r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+2sz+2rtz^2+rvz^3-2suz^3+tvz^4-1)A_0.$

(c) *If $r^2z + 2uz^2 - s^2z^2 + t^2z^3 - u^2z^4 + v^2z^5 + 2sz + 2rtz^2 + 2rvz^3 - 2suz^3 + 2tvz^4 - 1 = 0$, then*

$$\sum_{\lambda=0}^{\eta} z^{\lambda} A_{2\lambda+1} = \frac{\Theta_3(z)}{r^2z + 2uz^2 - s^2z^2 + t^2z^3 - u^2z^4 + v^2z^5 + 2sz + 2rtz^2 + 2rvz^3 - 2suz^3 + 2tvz^4 - 1},$$

where,

$\Theta_3(z) = (r+vz^2+tz)z^{\eta+1}A_{2\eta+2} + (s-s^2z+t^2z^2-u^2z^3+v^2z^4+uz+rvz^2-2suz^2+2tvz^3+rtz)z^{\eta+1}A_{2\eta+1} + (t+vz-svz^2+ruz-stz)z^{\eta+1}A_{2\eta} + (u-u^2z^2+v^2z^3+tvz^2+rvz-suz)z^{\eta+1}A_{2\eta-1} - v\left(uz^2+sz-1\right)z^{\eta+1}A_{2\eta-2} - z^2(r+vz^2+tz)A_4 + z(r^2z+uz^2+sz+rtz^2+rvz^3-1)A_3 - z^2(t+vz-svz^2+ruz-stz)A_2 + (r^2z+uz^2-s^2z^2+t^2z^3+2sz+2rtz^2+rvz^3-suz^3+tvz^4-1)A_1 + vz^2(uz^2+sz-1)A_0$

Proof. See Soykan [(30)], theorem 2.1, for the proof. □

Theorem 1.2. *For $\eta \geq 1$, we have the following formulas:If $v + rz^4 + sz^3 + tz^2 + uz - z^5 \neq 0$, then*

$$\sum_{\lambda=1}^{\eta} z^{\lambda} A_{-\lambda} = \frac{\Theta_4(z)}{v + rz^4 + sz^3 + tz^2 + uz - z^5},$$

where,
$\Theta_4(z) = -z^{\eta+1}A_{4-\eta} + (r-z)z^{\eta+1}A_{-\eta+3} + (s+rz-z^2)z^{\eta+1}A_{-\eta+2} + (t+rz^2+sz-z^3)z^{\eta+1}A_{-\eta+1} + (u + rz^3 + sz^2 + tz - z^4)z^{\eta+1}A_{-\eta} + zA_4 - z(r-z)A_3 + z(-s-rz+z^2)A_2 + z(-t-rz^2-sz+z^3)A_1 + z(-u-rz^3-sz^2-tz+z^4)A_0.$

Proof. See Soykan [(30)], theorem 4.1, for the proof. □

The paper is structured into 6 distinct sections. In section 1, we initially revisited the definition of generalized Pentanacci numbers, laying out the formal expressions and recurrence relations. We also drew from several important publications in the literature to build a foundation for our paper. For a better understanding, we present three detailed tables. They include notations, some members of sequences and related works. Moreover, we highlight two crucial theorems conducted by Soykan in [(30)], which have proven invaluable in this research. In Section 2, we compute the following sums for the generalized Pentanacci sequences $\{A_\eta\}_{\eta\geq 0}$ with positive subscripts: $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{\lambda}$, $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{2\lambda}$ and $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{2\lambda}$. In section 3, we provide the closed-form solutions (identities) for sums $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{\lambda}$, $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{2\lambda}$ and $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{2\lambda}$ for particular cases of $z = -1$ and $z = i$, specifically considering the sequence $\{A_\eta\}_{\eta\geq 0}$. In section 4, we compute the following sum for the generalized Pentanacci sequences $\{A_\eta\}_{\eta\geq 0}$ with negative subscripts: $\sum_{\lambda=1}^{\eta} \lambda z^{\lambda} A_{-\lambda}$. In section 5, we provide the closed-form solutions (identities) for sums $\sum_{\lambda=1}^{\eta} \lambda z^{\lambda} A_{-\lambda}$, for particular cases of $z = -1$ and $z = i$, specifically considering the sequence $A_\eta\}_{\eta\geq 0}$. Finally, section 6 is the conclusion. It summarises the findings of the article.

2 Sums $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{\lambda}$, $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{2\lambda}$ and $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{2\lambda+1}$ with non-negative subscripts

During this section, we compute the following sums for the generalized Pentanacci sequences $\{A_\eta\}_{\eta\geq 0}$ with positive subscripts: $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{\lambda}$, $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{2\lambda}$ and $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{2\lambda+1}$. The following theorem presents three significant summation formulas for generalized Pentanacci numbers with positive subscripts.

Theorem 2.1. *For $\eta \geq 0$ and let z be a real (or complex) number. Then, we get the following formulae:*

(a) *If $vz^5 + uz^4 + tz^3 + sz^2 + rz - 1 \neq 0$, then*

$$\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{\lambda} = \frac{\Omega_1}{(vz^5 + uz^4 + tz^3 + sz^2 + rz - 1)^2},$$

where,
$\Omega_1 = z^{\eta+4}(\eta(sz^2 + tz^3 + uz^4 + vz^5 + rz - 1) - 4 + 2sz^2 + tz^3 - vz^5 + 3rz)A_{\eta+4} - z^{\eta+3}(\eta(rz - 1)(sz^2 + tz^3 + uz^4 + vz^5 + rz - 1) + 3 - sz^2 + uz^4 + 2vz^5 + 3r^2z^2 - 6rz + 2rsz^3 + rtz^4 - rvz^6) A_{\eta+3} - z^{\eta+2}(\eta(sz^2 + rz - 1)(sz^2 + tz^3 + uz^4 + vz^5 + rz - 1) + 2 - 4sz^2 + tz^3 + 2uz^4 + 3vz^5 + 2r^2z^2 + 2s^2z^4 - 4rz + 4rsz^3 - ruz^5 + stz^5 - 2rvz^6 - svz^7)A_{\eta+2} - z^{\eta+1}(\eta(sz^2 + tz^3 + rz - 1)(sz^2 + tz^3 + uz^4 + vz^5 + rz - 1) + 1 + 2rsz^3 + 2rtz^4 - 2ruz^5 + 2stz^5 - 3rvz^6 - suz^6 - 2svz^7 - tvz^8 - 2sz^2 - 2tz^3 + 3uz^4 + 4vz^5 + r^2z^2 + s^2z^4 + t^2z^6 - 2rz)A_{\eta+1} + v$

$$z^{\eta+5}(\eta(sz^2+tz^3+uz^4+vz^5+rz-1)+3sz^2+2tz^3+uz^4+4rz-5)A_\eta - z^4(2sz^2+tz^3-
vz^5+3rz-4)A_4+z^3(-sz^2+uz^4+2vz^5+3r^2z^2-6rz+2rsz^3+rtz^4-rvz^6+3)A_3+
z^2(-4sz^2+tz^3+2uz^4+3vz^5+2r^2z^2+2s^2z^4-4rz+4rsz^3-ruz^5+stz^5-2rvz^6-svz^7+2)
A_2+z(-2sz^2-2tz^3+3uz^4+4vz^5+r^2z^2+s^2z^4+t^2z^6-2rz+2rsz^3+2rtz^4-2ruz^5+
2stz^5-3rvz^6-suz^6-2svz^7-tvz^8+1)A_1-vz^5(3sz^2+2tz^3+uz^4+4rz-5)A_0.$$

(b) *If* $r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+v^2z^5+2sz+2rtz^2+2rvz^3-2suz^3+2tvz^4-1\neq 0$*, then*

$$\sum_{\lambda=0}^{\eta}\lambda z^{\lambda}A_{2\lambda}
=\frac{\Omega_2}{(r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+v^2z^5+2sz+2rtz^2+2rvz^3-2suz^3+2tvz^4-1)^2},$$

where,

$$\Omega_2=-z^{\eta+1}(\eta(uz^2+sz-1)(r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+v^2z^5+2sz+2rtz^2+2rvz^3-2suz^3+
2tvz^4-1)+1+s^2z^2+2t^2z^3-u^2z^4+u^3z^6+4v^2z^5-2sz+2rtz^2+4rvz^3+6tvz^4+2r^2uz^3-st^2z^4+
s^2uz^4+2su^2z^5-3sv^2z^6-2uv^2z^7-2rsvz^4+2rtuz^4-4stvz^5-2tuvz^6+r^2sz^2-uz^2)A_{2\eta+2}+
z^{\eta+2}(\eta(t+rs+vz+ruz)(r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+v^2z^5+2sz+2rtz^2+2rvz^3-2suz^3+
2tvz^4-1)+2rs^2z-t^3z^3-2v^3z^6-2rs-3vz-2t+r^3sz+r^2tz+4svz^2+2uvz^3+2ru^2z^3+2r^3uz^2+
2r^2vz^2+ru^3z^5-s^2vz^3+2tu^2z^4-4t^2vz^4-5tv^2z^5+u^2vz^5+2stuz^3-rst^2z^3+rs^2uz^3+2rsu^2
z^4-2r^2svz^3+2r^2tuz^3-3rsv^2z^5-2ruv^2z^6-3ruz+2stz+4rsuz^2-4rstvz^4-2rtuvz^5)A_{2\eta+1}
+z^{\eta+2}(\eta(u+t^2z-u^2z^2+v^2z^3+rt+2tvz^2+rvz-suz)(r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+v^2z^5+
2sz+2rtz^2+2rvz^3-2suz^3+2tvz^4-1)+4u^2z^2-3t^2z-2u-2u^3z^4-5v^2z^3-2rt+2r^2t^2z^2-
3r^2u^2z^3-s^2t^2z^3+4r^2v^2z^4+2s^2u^2z^4-3s^2v^2z^5+r^3tz+r^2uz-8tvz^2+rt^3z^3+4st^2z^2-4s^2
uz^2-6su^2z^3+2r^3vz^2+s^3uz^3+t^2uz^3+su^3z^5+8sv^2z^4+2rv^3z^6+3uv^2z^5+4rsvz^2+12stvz^3-
2r^2suz^2-rs^2vz^3-2rtu^2z^4+6r^2tvz^3+4rt^2vz^4+5rtv^2z^5-4s^2tvz^4-ru^2vz^5-2suv^2z^6+2
rstz-2stuvz^5-3rvz+5suz+4tuvz^4)A_{2\eta}+z^{\eta+2}(\eta(v+ru-svz+tuz)(r^2z+2uz^2-s^2z^2+
t^2z^3-u^2z^4+v^2z^5+2sz+2rtz^2+2rvz^3-2suz^3+2tvz^4-1)+r^3uz-3v^3z^5-2ru-2v+
r^2vz+2ru^3z^4-2rv^2z^3-4s^2vz^2+2tu^2z^3+s^3vz^3-t^2vz^3+tu^3z^5-4tv^2z^4+2sv^3z^6+2u^2
vz^4+4stuz^2+2rsu^2z^3-2r^2svz^2+2r^2tuz^2+rt^2uz^3-s^2tuz^3-2r^2uvz^3-3ruv^2z^5+2
stv^2z^5-su^2vz^5-2t^2uvz^5-2tuv^2z^6-2rstvz^3-4rtuvz^4+5svz-3tuz+2rsuz)A_{2\eta-1}+
vz^{\eta+2}(\eta(r+vz^2+tz)(r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+v^2z^5+2sz+2rtz^2+2rvz^3-2suz^3+2tvz^4-
1)+r^3z-2r-4vz^2-v^3z^7-3tz+4stz^2+6svz^3+2tuz^3+4uvz^4+2r^2tz^2+rt^2z^3-s^2tz^3+2ru^2
z^4+r^2vz^3-rv^2z^5-2s^2vz^4+tu^2z^5-t^2vz^5-2tv^2z^6+2rsz+2rsuz^3-2suvz^5)A_{2\eta-2}+
z^2(-r^2z-4uz^2+4s^2z^2-s^3z^3+t^2z^3+2u^2z^4+3v^2z^5-5sz+2rvz^3+6suz^3+4tvz^4+2r^2sz^2+
3r^2uz^3-2s^2uz^4-su^2z^5+t^2uz^5-2sv^2z^6-uv^2z^7+2rstz^3+4rtuz^4+2ruvz^5-2stvz^5+2)A_4+
z^3(3t+v^3z^6+3rs+4vz-4rs^2z-2r^3sz-2r^2tz-6svz^2-2tuz^2-4uvz^3+rs^3z^2-2rt^2z^2+s^2tz^2-
4ru^2z^3-3r^3uz^2-3r^2vz^2-2rv^2z^4+2s^2vz^3-tu^2z^4+t^2vz^4+2tv^2z^5+4ruz-4stz-8rsuz^2-
4rtvz^3+2suvz^4-2r^2stz^2+2rs^2uz^3+rsu^2z^4-4r^2tuz^3-rt^2uz^4+2rsv^2z^5-2r^2uvz^4+ruv^2z^6+
2rstvz^4)A_3+z(r^4z^2+2r^3tz^3+r^3vz^4-2r^2s^2z^3-r^2suz^4+4r^2sz^2+r^2t^2z^4+2r^2u^2z^5+2r^2uz^3-
r^2v^2z^6-2r^2z-3rs^2tz^4-2rs^2vz^5-4rstuz^5+4rstz^3-4rsuvz^6+2rsvz^4-rt^2vz^6+rtu^2z^6+
4rtuz^4-2rtv^2z^7-rtz^2+4ruvz^5-rv^3z^8+s^4z^4+2s^3uz^5-4s^3z^3+2s^2tvz^6+s^2u^2z^6-5s^2uz^4+
2s^2v^2z^7+6s^2z^2-st^2uz^6-2st^2z^4-8stvz^5+suv^2z^8+4suz^3-6sv^2z^6-4sz+2t^2z^3-2tuvz^6+6
tvz^4+u^3z^6-u^2z^4-2uv^2z^7-uz^2+4v^2z^5+1)A_2+z^3(3v+2v^3z^5+3ru-2r^3uz-2r^2vz-
2uvz^2-2ru^2z^2-ru^3z^4+7s^2vz^2-4tu^2z^3-2s^3vz^3-t^3uz^4+2tv^2z^4-sv^3z^6-u^2vz^4-8svz+
4tuz-2rtvz^2-6stuz^2+4suvz^3+rs^2uz^2+3r^2svz^2-5r^2tuz^2-4rt^2uz^3+2rsv^2z^4+2s^2tuz^3+
2stu^2z^4+st^2vz^4+2ruv^2z^5-2s^2uvz^4+tuv^2z^6-4rsuz+4rstvz^3)A_1+vz^3(3r-2r^3z+5vz^2-
t^3z^4+4tz-2ruz^2-6stz^2-8svz^3-4tuz^3-6uvz^4+rs^2z^2-5r^2tz^2-4rt^2z^3+2s^2tz^3-
ru^2z^4-4r^2vz^3-2rv^2z^5+3s^2vz^4-2t^2vz^5-tv^2z^6+u^2vz^6-4rsz-6rtvz^4+2stuz^4+4suvz^5)A_0.$$

(c) *If* $r^2z + 2uz^2 - s^2z^2 + t^2z^3 - u^2z^4 + v^2z^5 + 2sz + 2rtz^2 + 2rvz^3 - 2suz^3 + 2tvz^4 - 1 \neq 0$, *then*

$$\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{2\lambda+1}$$
$$= \frac{\Omega_3}{(r^2z + 2uz^2 - s^2z^2 + t^2z^3 - u^2z^4 + v^2z^5 + 2sz + 2rtz^2 + 2rvz^3 - 2suz^3 + 2tvz^4 - 1)^2},$$

where,

$$\begin{aligned}
&\Omega_3 = z^{\eta+1}(\eta(r + vz^2 + tz)(r^2z + 2uz^2 - s^2z^2 + t^2z^3 - u^2z^4 + v^2z^5 + 2sz + 2rtz^2 + 2rvz^3 - \\
&2suz^3 + 2tvz^4 - 1) - 3vz^2 - t^3z^4 - 2v^3z^7 - 2tz - r - 2ruz^2 + 2stz^2 + 4svz^3 + 2uvz^4 + rs^2z^2 - \\
&r^2tz^2 - 2rt^2z^3 + 3ru^2z^4 - 2r^2vz^3 - 4rv^2z^5 - s^2vz^4 + 2tu^2z^5 - 4t^2vz^5 - 5tv^2z^6 + u^2vz^6 + 4rsu \\
&z^3 - 6rtvz^4 + 2stuz^4)A_{2\eta+2} + z^{\eta+1}(\eta(s - s^2z + t^2z^2 - u^2z^3 + v^2z^4 + uz + rvz^2 - 2suz^2 + 2tvz^3 + rtz) \\
&(r^2z + 2uz^2 - s^2z^2 + t^2z^3 - u^2z^4 + v^2z^5 + 2sz + 2rtz^2 + 2rvz^3 - 2suz^3 + 2tvz^4 - 1) + 2s^2z - s - \\
&s^3z^2 - 3t^2z^2 + 4u^2z^3 - 2u^3z^5 - 5v^2z^4 - r^2s^2z^2 + 2r^2t^2z^3 - 3r^2u^2z^4 + 4r^2v^2z^5 - 3rvz^2 + 6suz^2 - \\
&8tvz^3 + r^3tz^2 + r^2uz^2 + rt^3z^4 + 2st^2z^3 - 4s^2uz^3 - 5su^2z^4 + 2r^3vz^3 + t^2uz^4 + 4sv^2z^5 + 2rv^3 \\
&z^7 + 3uv^2z^6 + 6stvz^4 + 4tuvz^5 + rs^2vz^4 - 2rtu^2z^5 + 6r^2tvz^4 + 4rt^2vz^5 + 5rtv^2z^6 - ru^2vz^6 - 4r^2 \\
&suz^3 - 2rstuz^4 - 2rtz - 2uz)A_{2\eta+1} + z^{\eta+1}(\eta(t + vz - svz^2 + ruz - stz)(r^2z + 2uz^2 - s^2z^2 + t^2z^3 - \\
&u^2z^4 + v^2z^5 + 2sz + 2rtz^2 + 2rvz^3 - 2suz^3 + 2tvz^4 - 1) + 5svz^2 - 2t^3z^3 - 3v^3z^6 - 2vz - t - 2tu \\
&z^2 - 2rt^2z^2 - s^2tz^2 + r^3uz^2 + r^2vz^2 + st^3z^4 + 2ru^3z^5 - 2rv^2z^4 - 4s^2vz^3 + 3tu^2z^4 + s^3vz^4 - 7t^2vz^4 - \\
&8tv^2z^5 + 2sv^3z^7 + 2u^2vz^5 - 2ruz + 2stz + 2rsuz^2 - 4rtvz^3 + 4stuz^3 - r^2stz^2 - 2r^2svz^3 - rt^2 \\
&uz^4 - 2s^2tuz^4 - 2stu^2z^5 - 2r^2uvz^4 + 4st^2vz^5 - 3ruv^2z^6 + 5stv^2z^6 - su^2vz^6 + 2rsu^2z^4 - \\
&4rtuvz^5)A_{2\eta} + z^{\eta+1}(\eta(u - u^2z^2 + v^2z^3 + tvz^2 + rvz - suz)(r^2z + 2uz^2 - s^2z^2 + t^2z^3 - u^2z^4 + \\
&v^2z^5 + 2sz + 2rtz^2 + 2rvz^3 - 2suz^3 + 2tvz^4 - 1) + u^2z^2 - u + u^3z^4 - 4v^2z^3 - u^4z^6 - v^4z^8 - \\
&2r^2u^2z^3 + r^2v^2z^4 - s^2u^2z^4 - 2s^2v^2z^5 - t^2v^2z^6 + 2u^2v^2z^7 - 3tvz^2 - s^2uz^2 + r^3vz^2 - 2t^2uz^3 - \\
&2su^3z^5 + 6sv^2z^4 - rv^3z^6 - 2tv^3z^7 - 2rvz + 2suz + 2rsvz^2 - 2rtuz^2 - 4ruvz^3 + 4stvz^3 - 4tuvz^4 - r^2 \\
&suz^2 - 2rtu^2z^4 + 2r^2tvz^3 + rt^2vz^4 + st^2uz^4 - s^2tvz^4 + 2ru^2vz^5 + suv^2z^6 + 3tu^2vz^6 + 4rsuvz^4 + \\
&4stuvz^5)A_{2\eta-1} - vz^{\eta+1}(\eta(uz^2 + sz - 1)(r^2z + 2uz^2 - s^2z^2 + t^2z^3 - u^2z^4 + v^2z^5 + 2sz + 2rtz^2 + \\
&2rvz^3 - 2suz^3 + 2tvz^4 - 1) + 1 + s^2z^2 + 2t^2z^3 - uz^2 - u^2z^4 + u^3z^6 + 4v^2z^5 - 2sz + 2rtz^2 + 4rvz^3 + 6 \\
&tvz^4 + r^2sz^2 + 2r^2uz^3 - st^2z^4 + s^2uz^4 + 2su^2z^5 - 3sv^2z^6 - 2uv^2z^7 - 2rsvz^4 + 2rtuz^4 - 4stvz^5 - \\
&2tuvz^6)A_{2\eta-2} + z^2(2r - r^3z + 4vz^2 + v^3z^7 + 3tz - 4stz^2 - 6svz^3 - 2tuz^3 - 4uvz^4 - 2r^2tz^2 - rt^2z^3 + \\
&s^2tz^3 - 2ru^2z^4 - r^2vz^3 + rv^2z^5 + 2s^2vz^4 - tu^2z^5 + t^2vz^5 + 2tv^2z^6 - 2rsz - 2rsuz^3 + 2suvz^5) \\
&A_4 + z(r^4z^2 + 2r^3tz^3 + r^3vz^4 + 2r^2suz^4 + 3r^2sz^2 + r^2t^2z^4 + 2r^2u^2z^5 + 2r^2uz^3 - r^2v^2z^6 - \\
&2r^2z - rs^2tz^4 - 2rs^2vz^5 + 4rstz^3 - 2rsuvz^6 + 4rsvz^4 - rt^2vz^6 + rtu^2z^6 + 4rtuz^4 - 2rtv^2z^7 - \\
&rtz^2 + 4ruvz^5 - rv^3z^8 + s^2uz^4 + s^2z^2 - st^2z^4 - 4stvz^5 + 2su^2z^5 - 3sv^2z^6 - 2sz + 2t^2z^3 - \\
&2tuvz^6 + 6tvz^4 + u^3z^6 - u^2z^4 - 2uv^2z^7 - uz^2 + 4v^2z^5 + 1)A_3 + z^2(-2r^3uz^2 + 2r^2stz^2 + 3r^2svz^3 - \\
&2r^2tuz^3 - r^2tz - 2r^2vz^2 + rs^2uz^3 + 2rst^2z^3 + 4rstvz^4 - 4rsuz^2 + 2rsv^2z^5 + 2rtuvz^5 - ru^3z^5 - \\
&2ru^2z^3 + 2ruv^2z^6 + 3ruz - s^3tz^3 - 2s^3vz^4 + 4s^2tz^2 - 2s^2uvz^5 + 7s^2vz^3 - st^2vz^5 + stu^2z^5 - \\
&2stv^2z^6 - 5stz + 4suvz^4 - sv^3z^7 - 8svz^2 + t^3z^3 + 4t^2vz^4 - 2tu^2z^4 + 5tv^2z^5 + 2t - u^2vz^5 - \\
&2uvz^3 + 2v^3z^6 + 3vz)A_2 + z^2(-2r^3vz^2 + 2r^2suz^2 - 5r^2tvz^3 + 3r^2u^2z^3 - r^2uz - 4r^2v^2z^4 + \\
&rs^2vz^3 + 2rstuz^3 - 4rsvz^2 - 4rt^2vz^4 + 4rtu^2z^4 - 6rtv^2z^5 + ru^2vz^5 - 2rv^3z^6 + 3rvz - s^3uz^3 + \\
&2s^2tvz^4 - 2s^2u^2z^4 + 4s^2uz^2 + 3s^2v^2z^5 - 6stvz^3 - su^3z^5 + 6su^2z^3 + 2suv^2z^6 - 5suz - 8sv^2z^4 - t^3 \\
&vz^5 + t^2u^2z^5 + t^2uz^3 - 2t^2v^2z^6 - tv^3z^7 + 4tvz^2 + 2u^3z^4 - 4u^2z^2 - 3uv^2z^5 + 2u + 5v^2z^3)A_1 + \\
&vz^2(-r^2z - 4uz^2 + 4s^2z^2 - s^3z^3 + t^2z^3 + 2u^2z^4 + 3v^2z^5 - 5sz + 2rvz^3 + 6suz^3 + 4tvz^4 + 2r^2sz^2 + \\
&3r^2uz^3 - 2s^2uz^4 - su^2z^5 + t^2uz^5 - 2sv^2z^6 - uv^2z^7 + 2rstz^3 + 4rtuz^4 + 2ruvz^5 - 2stvz^5 + 2)A_0.
\end{aligned}$$

Proof. **(a)** Using the following recurrence relation

$$A_\eta = rA_{\eta-1} + sA_{\eta-2} + tA_{\eta-3} + uA_{\eta-4} + vA_{\eta-5},$$

i.e.

$$vA_{\eta-5} = A_\eta - rA_{\eta-1} - sA_{\eta-2} - tA_{\eta-3} - uA_{\eta-4}.$$

We have

$$
\begin{aligned}
v\times 0\times z^{0}A_{0} &= 0\times z^{0}A_{5}-r\times 0\times z^{0}A_{4}-s\times 0\times z^{0}A_{3}-t\times 0\times z^{0}A_{2}\\
&\quad -u\times 0\times z^{0}A_{1},\\
v\times 1\times z^{1}A_{1} &= 1\times z^{1}A_{6}-r\times 1\times z^{1}A_{5}-s\times 1\times z^{1}A_{4}-t\times 1\times z^{1}A_{3}\\
&\quad -u\times 1\times z^{1}A_{2},\\
v\times 2\times z^{2}A_{2} &= 2\times z^{2}A_{7}-r\times 2\times z^{2}A_{6}-s\times 2\times z^{2}A_{5}-t\times 2\times z^{2}A_{4}\\
&\quad -u\times 2\times z^{2}A_{3},\\
&\ \ \vdots\\
v(\eta-2)z^{\eta-2}A_{\eta-2} &= (\eta-2)z^{\eta-2}A_{\eta+3}-r(\eta-2)z^{\eta-2}A_{\eta+2}-s(\eta-2)z^{\eta-2}A_{\eta+1}\\
&\quad -t(\eta-2)z^{\eta-2}A_{\eta}-u(\eta-2)z^{\eta-2}A_{\eta-1},\\
v(\eta-1)z^{\eta-1}A_{\eta-1} &= (\eta-1)z^{\eta-1}A_{\eta+4}-r(\eta-1)z^{\eta-1}A_{\eta+3}-s(\eta-1)z^{\eta-1}A_{\eta+2}\\
&\quad -t(\eta-1)z^{\eta-1}A_{\eta+1}-u(\eta-1)z^{\eta-1}A_{\eta},\\
v\times\eta\times z^{\eta}A_{\eta} &= \eta\times z^{\eta}A_{\eta+5}-r\times\eta\times z^{\eta}A_{\eta+4}-s\times\eta\times z^{\eta}A_{\eta+3}\\
&\quad -t\times\eta\times z^{\eta}A_{\eta+2}-u\times\eta\times z^{\eta}A_{\eta+1}.
\end{aligned}
$$

By adding the equalities side by side and applying theorem 1.1 (a), we obtain (a).

(b) and (c) Using the following recurrence relation

$$A_{\eta}=rA_{\eta-1}+sA_{\eta-2}+tA_{\eta-3}+uA_{\eta-4}+vA_{\eta-5},$$

i.e.

$$rA_{\eta-1}=A_{\eta}-sA_{\eta-2}-tA_{\eta-3}-uA_{\eta-4}-vA_{\eta-5}.$$

We get

$$
\begin{aligned}
r\times 1\times x^{1}A_{3} &= 1\times x^{1}A_{4}-s\times 1\times x^{1}A_{2}-t\times 1\times x^{1}A_{1}\\
&\quad -u\times 1\times x^{1}A_{0}-v\times 1\times x^{1}A_{-1},\\
r\times 2\times x^{2}A_{5} &= 2\times x^{2}A_{6}-s\times 2\times x^{2}A_{4}-t\times 2\times x^{2}A_{3}\\
&\quad -u\times 2\times x^{2}A_{2}-v\times 2\times x^{2}A_{1},\\
&\ \ \vdots\\
r\times(\eta-1)\times x^{\eta-1}A_{2\eta-1} &= (\eta-1)\times x^{\eta-1}A_{2\eta}-s\times(\eta-1)\times x^{\eta-1}A_{2\eta-2}\\
&\quad -t\times(\eta-1)\times x^{\eta-1}A_{2\eta-3}-u\times(\eta-1)\times x^{\eta-1}A_{2\eta-4}\\
&\quad -v\times(\eta-1)\times x^{\eta-1}A_{2\eta-5},\\
r\times\eta\times x^{\eta}A_{2\eta+1} &= \eta\times x^{\eta}A_{2\eta+2}-s\times\eta\times x^{\eta}A_{2\eta}-t\times\eta\times x^{\eta}A_{2\eta-1}\\
&\quad -u\times\eta\times x^{\eta}A_{2\eta-2}-v\times\eta\times x^{\eta}A_{2\eta-3}.
\end{aligned}
$$

By adding the equalities side by side the above equalities, we have that

$$r(-0\times x^0A_1+\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda+1})=(\eta\times x^{\eta}A_{2\eta+2}-0\times x^0A_2-(-1)\times x^{-1}A_0$$
$$+\sum_{\lambda=0}^{\eta}(\lambda-1)x^{\lambda-1}A_{2\lambda})-s(-0\times x^0A_0+\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda})-t(-(\eta+1)x^{\eta+1}A_{2\eta+1}$$
$$+\sum_{\lambda=0}^{\eta}(\lambda+1)x^{\lambda+1}A_{2\lambda+1})-u(-(\eta+1)x^{\eta+1}A_{2\eta}+\sum_{\lambda=0}^{\eta}(\lambda+1)x^{\lambda+1}A_{2\lambda})$$
$$-v(-(\eta+2)x^{\eta+2}A_{2\eta+1}-(\eta+1)x^{\eta+1}A_{2\eta-1}+1\times x^1A_{-1}$$
$$+\sum_{\lambda=0}^{\eta}(\lambda+2)x^{\lambda+2}A_{2\lambda+1}).$$

Since

$$A_{-1}=-\frac{u}{v}A_0-\frac{t}{v}A_1-\frac{s}{v}A_2-\frac{r}{v}A_3+\frac{1}{v}A_4,$$

we obtain

$$r(-0\times x^0A_1+\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda+1})=(\eta\times x^{\eta}A_{2\eta+2}-0\times x^0A_2-(-1)\times x^{-1}A_0 \tag{2.1}$$
$$+x^{-1}\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda}-x^{-1}\sum_{\lambda=0}^{\eta}x^{\lambda}A_{2\lambda})-s(-0\times x^0A_0+\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda})-t(-(\eta+1)x^{\eta+1}A_{2\eta+1}$$
$$+x^1\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda+1}+x^1\sum_{\lambda=0}^{\eta}x^{\lambda}A_{2\lambda+1})-u(-(\eta+1)x^{\eta+1}A_{2\eta}+x^1\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda}$$
$$+x^1\sum_{\lambda=0}^{\eta}x^{\lambda}A_{2\lambda})-v(-(\eta+2)x^{\eta+2}A_{2\eta+1}-(\eta+1)x^{\eta+1}A_{2\eta-1}+1\times x^1(-\frac{u}{v}A_0-\frac{t}{v}A_1$$
$$-\frac{s}{v}A_2-\frac{r}{v}A_3+\frac{1}{v}A_4)+x^2\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda+1}+2x^2\sum_{\lambda=0}^{\eta}x^{\lambda}A_{2\lambda+1}).$$

In a similar way, using the following recurrence relation

$$A_{\eta}=rA_{\eta-1}+sA_{\eta-2}+tA_{\eta-3}+uA_{\eta-4}+vA_{\eta-5},$$

i.e.

$$rA_{\eta-1}=A_{\eta}-sA_{\eta-2}-tA_{\eta-3}-uA_{\eta-4}-vA_{\eta-5}.$$

We obtain the following obvious equalities;

$$\begin{aligned}
r\times1\times x^1A_2 &= 1\times x^1A_3-s\times1\times x^1A_1-t\times1\times x^1A_0\\
&\quad -u\times1\times x^1A_{-1}-v\times1\times x^1A_{-2},\\
r\times2\times x^2A_4 &= 2\times x^2A_5-s\times2\times x^2A_3-t\times2\times x^2A_2\\
&\quad -u\times2\times x^2A_1-v\times2\times x^2A_0,\\
&\ \ \vdots\\
r\times(\eta-1)\times x^{\eta-1}A_{2\eta-2} &= (\eta-1)\times x^{\eta-1}A_{2\eta-1}-s\times(\eta-1)\times x^{\eta-1}A_{2\eta-3}\\
&\quad -t\times(\eta-1)\times x^{\eta-1}A_{2\eta-4}-u\times(\eta-1)\times x^{\eta-1}A_{2\eta-5}\\
&\quad -v\times(\eta-1)\times x^{\eta-1}A_{2\eta-6},\\
r\times\eta\times x^{\eta}A_{2\eta} &= \eta\times x^{\eta}A_{2\eta+1}-s\times\eta\times x^{\eta}A_{2\eta-1}\\
&\quad -t\times\eta\times x^{\eta}A_{2\eta-2}-u\times\eta\times x^{\eta}A_{2\eta-3}-v\times\eta\times x^{\eta}A_{2\eta-4}.
\end{aligned}$$

By adding side by side the above equalities, we have that

$$r(-0\times x^0A_0+\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda})=(-0\times x^0A_1+\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda+1})$$
$$-s(-(\eta+1)x^{\eta+1}A_{2\eta+1}+\sum_{\lambda=0}^{\eta}(\lambda+1)x^{\lambda+1}A_{2\lambda+1})-t(-(\eta+1)x^{\eta+1}A_{2\eta}$$
$$+\sum_{\lambda=0}^{\eta}(\lambda+1)x^{\lambda+1}A_{2\lambda})-u(-(\eta+2)x^{\eta+2}A_{2\eta+1}-(\eta+1)x^{\eta+1}A_{2\eta-1}$$
$$+1\times x^1A_{-1}+\sum_{\lambda=0}^{\eta}(\lambda+2)x^{\lambda+2}A_{2\lambda+1})-v(-(\eta+2)x^{\eta+2}A_{2\eta}-(\eta+1)x^{\eta+1}A_{2\eta-2}$$
$$+1\times x^1A_{-2}+\sum_{\lambda=0}^{\eta}(\lambda+2)x^{\lambda+2}A_{2\lambda}).$$

Since

$$A_{-1} = -\frac{u}{v}A_0-\frac{t}{v}A_1-\frac{s}{v}A_2-\frac{r}{v}A_3+\frac{1}{v}A_4,$$
$$A_{-2} = -\frac{u}{v}(-\frac{u}{v}A_0-\frac{t}{v}A_1-\frac{s}{v}A_2-\frac{r}{v}A_3+\frac{1}{v}A_4)-\frac{t}{v}A_0-\frac{s}{v}A_1-\frac{r}{v}A_2+\frac{1}{v}A_3.$$

We have that

$$r(-0\times x^0A_0+\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda})=(-0\times x^0A_1+\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda+1}) \tag{2.2}$$
$$-s(-(\eta+1)x^{\eta+1}A_{2\eta+1}+x^1\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda+1}+x^1\sum_{\lambda=0}^{\eta}x^{\lambda}A_{2\lambda+1})-t(-(\eta+1)x^{\eta+1}A_{2\eta}$$
$$+x^1\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda}+x^1\sum_{\lambda=0}^{\eta}x^{\lambda}A_{2\lambda})-u(-(\eta+2)x^{\eta+2}A_{2\eta+1}-(\eta+1)x^{\eta+1}A_{2\eta-1}$$
$$+1\times x^1(-\frac{u}{v}A_0-\frac{t}{v}A_1-\frac{s}{v}A_2-\frac{r}{v}A_3+\frac{1}{v}A_4)+x^2\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda+1}+2x^2\sum_{\lambda=0}^{\eta}x^{\lambda}A_{2\lambda+1})$$
$$-v(-(\eta+2)x^{\eta+2}A_{2\eta}-(\eta+1)x^{\eta+1}A_{2\eta-2}+1\times x^1(-\frac{u}{v}(-\frac{u}{v}A_0-\frac{t}{v}A_1-\frac{s}{v}A_2-\frac{r}{v}A_3$$
$$+\frac{1}{v}A_4)-\frac{t}{v}A_0-\frac{s}{v}A_1-\frac{r}{v}A_2+\frac{1}{v}A_3)+x^2\sum_{\lambda=0}^{\eta}\lambda x^{\lambda}A_{2\lambda}+2x^2\sum_{\lambda=0}^{\eta}x^{\lambda}A_{2\lambda}).$$

Then, by using parts (b) and (c) of theorem 1.1, and solving the system of equations (2.1)-(2.2), the desired result follows.

□

3 Results for special z values in non-negative subscripts

In this section, we handle the closed-form solutions (identities) for sums $\sum_{\lambda=0}^{\eta}\lambda z^{\lambda}A_{\lambda}$, $\sum_{\lambda=0}^{\eta}\lambda z^{\lambda}A_{2\lambda}$ and $\sum_{\lambda=0}^{\eta}\lambda z^{\lambda}A_{2\lambda+1}$ for particular cases of $z=-1$ and $z=i$, specifically considering the sequence $\{A_{\eta}\}_{\eta\geq 0}$.

3.1 The case $z = 1$

The case $z = 1$ of theorem 2.1 is provided in Soykan [(31)].

3.2 The case $z = -1$

During this subsection, we overcome the closed-form solutions (identities) for sums $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} A_{\lambda}$, $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} A_{2\lambda}$ and $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} A_{2\lambda+1}$ specifically considering the sequence $\{A_{\eta}\}$.

Here, recall theorem 2.1 and set $r = s = t = u = v = 1$; this gives the result below.

Proposition 3.1. *For $\eta \geq 0$ and setting $r, s, t, u, v = 1$ in theorem 2.1 leads to the next outcomes.*

(a) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} A_{\lambda} = \frac{1}{4}((-1)^{\eta} (-(2\eta+5)A_{\eta+4} + (4\eta+8)A_{\eta+3} - (2\eta-1)A_{\eta+2} + (4\eta+2)A_{\eta+1} + (2\eta+7)A_{\eta}) + 5A_4 - 8A_3 - A_2 - 2A_1 - 7A_0)$.

(b) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} A_{2\lambda} = \frac{1}{4}((-1)^{\eta} ((2\eta-1)A_{2\eta+2} + 4A_{2\eta+1} - (2\eta-1)A_{2\eta} - (4\eta+6)A_{2\eta-1} - (2\eta+5) A_{2\eta-2}) + A_4 + 4A_3 - 5A_2 - 10A_1 - 7A_0)$.

(c) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} A_{2\lambda+1} = \frac{1}{4}((-1)^{\eta} ((2\eta+3)A_{2\eta+2} - (2\eta+7)A_{2\eta} - 6A_{2\eta-1} + (2\eta-1)A_{2\eta-2}) + 5A_4 - 4A_3 - 9A_2 - 6A_1 + A_0)$.

The next corollary follows if we replace A_{η} in the proposition 3.1 with P_{η}. We additionally apply the initial conditions where $P_0 = 0$, $P_1 = 1$, $P_2 = 1$, $P_3 = 2$ and $P_4 = 4$.

Corollary 3.1. *For $\eta \geq 0$, Pentanacci numbers get the followings:*

(a) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} P_{\lambda} = \frac{1}{4}((-1)^{\eta} (-(2\eta+5)P_{\eta+4} + (4\eta+8)P_{\eta+3} - (2\eta-1)P_{\eta+2} + (4\eta+2)P_{\eta+1} + (2\eta+7)P_{\eta}) + 1)$.

(b) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} P_{2\lambda} = \frac{1}{4}((-1)^{\eta} ((2\eta-1)P_{2\eta+2} + 4P_{2\eta+1} - (2\eta-1)P_{2\eta} - (4\eta+6)P_{2\eta-1} - (2\eta+5) P_{2\eta-2}) - 3)$.

(c) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} P_{2\lambda+1} = \frac{1}{4}((-1)^{\eta} ((2\eta+3)P_{2\eta+2} - (2\eta+7)P_{2\eta} - 6P_{2\eta-1} + (2\eta-1)P_{2\eta-2}) - 3)$.

The next corollary follows if we replace A_{η} in the proposition 3.1 with Q_{η}. We additionally apply the initial conditions where $Q_0 = 5$, $Q_1 = 1$, $Q_2 = 3$, $Q_3 = 7$ and $Q_4 = 15$.

Corollary 3.2. *For $\eta \geq 0$, Pentanacci-Lucas numbers satisfy the followings:*

(a) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} Q_{\lambda} = \frac{1}{4}((-1)^{\eta} (-(2\eta+5)Q_{\eta+4} + (4\eta+8)Q_{\eta+3} - (2\eta-1)Q_{\eta+2} + (4\eta+2)Q_{\eta+1} + (2\eta+7)Q_{\eta}) - 21)$.

(b) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} Q_{2\lambda} = \frac{1}{4}((-1)^{\eta} ((2\eta-1)Q_{2\eta+2} + 4Q_{2\eta+1} - (2\eta-1)Q_{2\eta} - (4\eta+6)Q_{2\eta-1} - (2\eta+5) Q_{2\eta-2}) - 17)$.

(c) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} Q_{2\lambda+1} = \frac{1}{4}((-1)^{\eta} ((2\eta+3)Q_{2\eta+2} - (2\eta+7)Q_{2\eta} - 6Q_{2\eta-1} + (2\eta-1)Q_{2\eta-2}) + 19)$.

Here, recall theorem 2.1 and set $r = 2$, $s = t = u = v = 1$; this gives the result below.

Proposition 3.2. *For $\eta \geq 0$ and setting $r = 2, s = t = u = v = 1$, in theorem 2.1 leads to the next outcomes.*

(a) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} A_{\lambda} = \frac{1}{9}((-1)^{\eta} (-(3\eta+8)A_{\eta+4} + (9\eta+21)A_{\eta+3} - (6\eta+4)A_{\eta+2} + (9\eta+6)A_{\eta+1} + (3\eta+11)A_{\eta}) + 8A_4 - 21A_3 + 4A_2 - 6A_1 - 11A_0)$.

(b) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} A_{2\lambda} = \frac{1}{25}((-1)^{\eta} ((5\eta-6)A_{2\eta+2} + 15A_{2\eta+1} - (5\eta-11)A_{2\eta} - (10\eta+13)A_{2\eta-2} - (15\eta+12)A_{2\eta-1}) - A_4 + 15A_3 - 19A_2 - 27A_1 - 23A_0)$.

(c) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} A_{2\lambda+1} = \frac{1}{25}((-1)^{\eta} ((10\eta+3)A_{2\eta+2} + 5A_{2\eta+1} - (10\eta+18)A_{2\eta} + (5\eta-6)A_{2\eta-2} - (5\eta+19)A_{2\eta-1}) + 13A_4 - 20A_3 - 28A_2 - 24A_1 - A_0)$.

The next corollary follows if we replace A_η in the proposition 3.2 with P_η. We additionally apply the initial conditions where $P_0 = 0$, $P_1 = 1$, $P_2 = 2$, $P_3 = 5$ and $P_4 = 13$.

Corollary 3.3. *For $\eta \geq 0$, the fifth-order Pell numbers satisfy the followings:*

(a) $\sum_{\lambda=0}^{\eta} \lambda(-1)^\lambda P_\lambda = \frac{1}{9}((-1)^\eta (-(3\eta+8)P_{\eta+4} + (9\eta+21)P_{\eta+3} - (6\eta+4)P_{\eta+2} + (9\eta+6)P_{\eta+1} + (3\eta+11)P_\eta) + 1)$.

(b) $\sum_{\lambda=0}^{\eta} \lambda(-1)^\lambda P_{2\lambda} = \frac{1}{25}((-1)^\eta ((5\eta-6)P_{2\eta+2} + 15P_{2\eta+1} - (5\eta-11)P_{2\eta} - (10\eta+13)P_{2\eta-2} - (15\eta+12)P_{2\eta-1}) - 3)$.

(c) $\sum_{\lambda=0}^{\eta} \lambda(-1)^\lambda P_{2\lambda+1} = \frac{1}{25}((-1)^\eta ((10\eta+3)P_{2\eta+2} + 5P_{2\eta+1} - (10\eta+18)P_{2\eta} + (5\eta-6)P_{2\eta-2} - (5\eta+19)P_{2\eta-1}) - 11)$.

The next corollary follows if we replace A_η in the proposition 3.2 with P_η. We additionally apply the initial conditions where $Q_0 = 5$, $Q_1 = 2$, $Q_2 = 6$, $Q_3 = 17$ and $Q_4 = 46$.

Corollary 3.4. *For $\eta \geq 0$, the fifth-order Pell-Lucas numbers have the followings:*

(a) $\sum_{\lambda=0}^{\eta} \lambda(-1)^\lambda Q_\lambda = \frac{1}{9}((-1)^\eta (-(3\eta+8)Q_{\eta+4} + (9\eta+21)Q_{\eta+3} - (6\eta+4)Q_{\eta+2} + (9\eta+6)Q_{\eta+1} + (3\eta+11)Q_\eta) - 32)$.

(b) $\sum_{\lambda=0}^{\eta} \lambda(-1)^\lambda Q_{2\lambda} = \frac{1}{25}((-1)^\eta ((5\eta-6)Q_{2\eta+2} + 15Q_{2\eta+1} - (5\eta-11)Q_{2\eta} - (10\eta+13)Q_{2\eta-2} - (15\eta+12)Q_{2\eta-1}) - 74)$.

(c) $\sum_{\lambda=0}^{\eta} \lambda(-1)^\lambda Q_{2\lambda+1} = \frac{1}{25}((-1)^\eta ((10\eta+3)Q_{2\eta+2} + 5Q_{2\eta+1} - (10\eta+18)Q_{2\eta} + (5\eta-6)Q_{2\eta-2} - (5\eta+19)Q_{2\eta-1}) + 37)$.

Here, recall theorem 2.1 and set $r = s = t = u = 1$, $v = 2$; this gives the result below.

Proposition 3.3. *For $\eta \geq 0$ and setting $r, s, t, u = 1, v = 2$, in theorem 2.1 leads to the next outcomes.*

(a) $\sum_{\lambda=0}^{\eta} \lambda(-1)^\lambda A_\lambda = \frac{1}{9}((-1)^\eta (-(3\eta+4)A_{\eta+4} + (6\eta+5)A_{\eta+3} - (3\eta-5)A_{\eta+2} + (6\eta-4)A_{\eta+1} + 2(3\eta+7)A_\eta) + 4A_4 - 5A_3 - 5A_2 + 4A_1 - 14A_0)$.

(b) $\sum_{\lambda=0}^{\eta} \lambda(-1)^\lambda A_{2\lambda} = \frac{1}{25}((-1)^\eta ((5\eta-14)A_{2\eta+2} + (5\eta+11)A_{2\eta+1} + (5\eta+36)A_{2\eta} - (20\eta-11)A_{2\eta-1} - 2(10\eta+7)A_{2\eta-2}) - 9A_4 + 16A_3 + 16A_2 - 9A_1 - 34A_0)$.

(c) $\sum_{\lambda=0}^{\eta} \lambda(-1)^\lambda A_{2\lambda+1} = \frac{1}{25}((-1)^\eta ((10\eta-3)A_{2\eta+2} + (10\eta+22)A_{2\eta+1} - (15\eta+3)A_{2\eta} + 2(5\eta-14)A_{2\eta-2} - (15\eta+28)A_{2\eta-1}) + 7A_4 + 7A_3 - 18A_2 - 43A_1 - 18A_0)$.

The next corollary follows if we replace A_η in the proposition 3.3 with J_η. We additionally apply the initial conditions where $J_0 = 0$, $J_1 = 1$, $J_2 = 1$, $J_3 = 1$ and $J_4 = 1$.

Corollary 3.5. *For $\eta \geq 0$, the fifth-order Jacobsthal numbers get the followings:*

(a) $\sum_{\lambda=0}^{\eta} \lambda(-1)^\lambda J_\lambda = \frac{1}{9}((-1)^\eta (-(3\eta+4)J_{\eta+4} + (6\eta+5)J_{\eta+3} - (3\eta-5)J_{\eta+2} + (6\eta-4)J_{\eta+1} + 2(3\eta+7)J_\eta) - 2)$.

(b) $\sum_{\lambda=0}^{\eta} \lambda(-1)^\lambda J_{2\lambda} = \frac{1}{25}((-1)^\eta ((5\eta-14)J_{2\eta+2} + (5\eta+11)J_{2\eta+1} + (5\eta+36)J_{2\eta} - (20\eta-11)J_{2\eta-1} - 2(10\eta+7)J_{2\eta-2}) + 14)$.

(c) $\sum_{\lambda=0}^{\eta} \lambda(-1)^\lambda J_{2\lambda+1} = \frac{1}{25}((-1)^\eta ((10\eta-3)J_{2\eta+2} + (10\eta+22)J_{2\eta+1} - (15\eta+3)J_{2\eta} + 2(5\eta-14)J_{2\eta-2} - (15\eta+28)J_{2\eta-1}) - 47)$.

The next corollary follows if we replace A_η in the proposition 3.3 with j_η. We additionally apply the initial conditions where $j_0 = 2$, $j_1 = 1$, $j_2 = 5$, $j_3 = 10$ and $j_4 = 20$.

Corollary 3.6. *For $\eta \geq 0$, the fifth-order Jacobsthal-Lucas numbers satisfy the followings:*

(a) $\sum_{\lambda=0}^{\eta} \lambda(-1)^\lambda j_\lambda = \frac{1}{9}((-1)^\eta (-(3\eta+4)j_{\eta+4} + (6\eta+5)j_{\eta+3} - (3\eta-5)j_{\eta+2} + (6\eta-4)j_{\eta+1} + 2(3\eta+7)j_\eta) - 19)$.

(b) $\sum_{\lambda=0}^{\eta}\lambda(-1)^{\lambda}j_{2\lambda}=\frac{1}{25}((-1)^{\eta}((5\eta-14)j_{2\eta+2}+(5\eta+11)j_{2\eta+1}+(5\eta+36)j_{2\eta}-(20\eta-11)j_{2\eta-1}-2(10\eta+7)j_{2\eta-2})-17)$.

(c) $\sum_{\lambda=0}^{\eta}\lambda(-1)^{\lambda}j_{2\lambda+1}=\frac{1}{25}((-1)^{\eta}((10\eta-3)j_{2\eta+2}+(10\eta+22)j_{2\eta+1}-(15\eta+3)j_{2\eta}+2(5\eta-14)j_{2\eta-2}-(15\eta+28)j_{2\eta-1})+41)$.

The next corollary follows if we replace A_η in the proposition 3.3 with K_η. We additionally apply the initial conditions where $K_0=3$, $K_1=1$, $K_2=3$, $K_3=10$ and $K_4=20$.

Corollary 3.7. *For $\eta\geq 0$, the modified fifth-order Jacobsthal numbers have the following property:*

(a) $\sum_{\lambda=0}^{\eta}\lambda(-1)^{\lambda}K_{\lambda}=\frac{1}{9}((-1)^{\eta}(-(3\eta+4)K_{\eta+4}+(6\eta+5)K_{\eta+3}-(3\eta-5)K_{\eta+2}+(6\eta-4)K_{\eta+1}+2(3\eta+7)K_{\eta})-23)$.

(b) $\sum_{\lambda=0}^{\eta}\lambda(-1)^{\lambda}K_{2\lambda}=\frac{1}{25}((-1)^{\eta}((5\eta-14)K_{2\eta+2}+(5\eta+11)K_{2\eta+1}+(5\eta+36)K_{2\eta}-(20\eta-11)K_{2\eta-1}-2(10\eta+7)K_{2\eta-2})-83)$.

(c) $\sum_{\lambda=0}^{\eta}\lambda(-1)^{\lambda}K_{2\lambda+1}=\frac{1}{25}((-1)^{\eta}((10\eta-3)K_{2\eta+2}+(10\eta+22)K_{2\eta+1}-(15\eta+3)K_{2\eta}+2(5\eta-14)K_{2\eta-2}-(15\eta+28)K_{2\eta-1})+59)$.

The next corollary follows if we replace A_η in the proposition 3.3 with Q_η. We additionally apply the initial conditions where $Q_0=3$, $Q_1=0$, $Q_2=2$, $Q_3=8$ and $Q_4=16$.

Corollary 3.8. *For $\eta\geq 0$, the fifth-order Jacobsthal Perrin numbers get the followings:*

(a) $\sum_{\lambda=0}^{\eta}\lambda(-1)^{\lambda}Q_{\lambda}=\frac{1}{9}((-1)^{\eta}(-(3\eta+4)Q_{\eta+4}+(6\eta+5)Q_{\eta+3}-(3\eta-5)Q_{\eta+2}+(6\eta-4)Q_{\eta+1}+2(3\eta+7)Q_{\eta})-28)$.

(b) $\sum_{\lambda=0}^{\eta}\lambda(-1)^{\lambda}Q_{2\lambda}=\frac{1}{25}((-1)^{\eta}((5\eta-14)Q_{2\eta+2}+(5\eta+11)Q_{2\eta+1}+(5\eta+36)Q_{2\eta}-(20\eta-11)Q_{2\eta-1}-2(10\eta+7)Q_{2\eta-2})-86)$.

(c) $\sum_{\lambda=0}^{\eta}\lambda(-1)^{\lambda}Q_{2\lambda+1}=\frac{1}{25}((-1)^{\eta}((10\eta-3)Q_{2\eta+2}+(10\eta+22)Q_{2\eta+1}-(15\eta+3)Q_{2\eta}+2(5\eta-14)Q_{2\eta-2}-(15\eta+28)Q_{2\eta-1})+78)$.

The next corollary follows if we replace A_η in the proposition 3.3 with S_η. We additionally apply the initial conditions where $S_0=0$, $S_1=1$, $S_2=1$, $S_3=2$ and $S_4=4$.

Corollary 3.9. *For $\eta\geq 0$, the adjusted fifth-order Jacobsthal numbers satisfy the followings:*

(a) $\sum_{\lambda=0}^{\eta}\lambda(-1)^{\lambda}S_{\lambda}=\frac{1}{9}((-1)^{\eta}(-(3\eta+4)S_{\eta+4}+(6\eta+5)S_{\eta+3}-(3\eta-5)S_{\eta+2}+(6\eta-4)S_{\eta+1}+2(3\eta+7)S_{\eta})+5)$.

(b) $\sum_{\lambda=0}^{\eta}\lambda(-1)^{\lambda}S_{2\lambda}=\frac{1}{25}((-1)^{\eta}((5\eta-14)S_{2\eta+2}+(5\eta+11)S_{2\eta+1}+(5\eta+36)S_{2\eta}-(20\eta-11)S_{2\eta-1}-2(10\eta+7)S_{2\eta-2})+3)$.

(c) $\sum_{\lambda=0}^{\eta}\lambda(-1)^{\lambda}S_{2\lambda+1}=\frac{1}{25}((-1)^{\eta}((10\eta-3)S_{2\eta+2}+(10\eta+22)S_{2\eta+1}-(15\eta+3)S_{2\eta}+2(5\eta-14)S_{2\eta-2}-(15\eta+28)S_{2\eta-1})-19)$.

The next corollary follows if we replace A_η in the proposition 3.3 with R_η. We additionally apply the initial conditions where $R_0=5$, $R_1=1$, $R_2=3$, $R_3=7$ and $R_4=15$.

Corollary 3.10. *The following properties is satisfied by the modified fifth-order Jacobsthal-Lucas numbers, for $\eta\geq 0$:*

(a) $\sum_{\lambda=0}^{\eta}\lambda(-1)^{\lambda}R_{\lambda}=\frac{1}{9}((-1)^{\eta}(-(3\eta+4)R_{\eta+4}+(6\eta+5)R_{\eta+3}-(3\eta-5)R_{\eta+2}+(6\eta-4)R_{\eta+1}+2(3\eta+7)R_{\eta})-56)$.

(b) $\sum_{\lambda=0}^{\eta}\lambda(-1)^{\lambda}R_{2\lambda}=\frac{1}{25}((-1)^{\eta}((5\eta-14)R_{2\eta+2}+(5\eta+11)R_{2\eta+1}+(5\eta+36)R_{2\eta}-(20\eta-11)R_{2\eta-1}-2(10\eta+7)R_{2\eta-2})-154)$.

(c) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} R_{2\lambda+1} = \frac{1}{25}((-1)^{\eta}((10\eta-3)R_{2\eta+2} + (10\eta+22)R_{2\eta+1} - (15\eta+3)R_{2\eta} + 2(5\eta-14)R_{2\eta-2} - (15\eta+28)R_{2\eta-1}) - 33)$.

Here, recall theorem 2.1 and set $r = 2$, $s = 3$, $t = 5$, $u = 7$, $v = 11$; this gives the result below.

Proposition 3.4. *For $\eta \geq 0$ and setting $r = 2$, $s = 3$, $t = 5$, $u = 7$, $v = 11$, in theorem 2.1 leads to the next outcomes.*

(a) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} A_{\lambda} = \frac{1}{81}((-1)^{\eta}(-(9\eta-2)A_{\eta+4} + (27\eta-15)A_{\eta+3} + 36A_{\eta+2} + (45\eta-46)A_{\eta+1} + 11(9\eta+7)A_{\eta}) - 2A_4 + 15A_3 - 36A_2 + 46A_1 - 77A_0)$.

(b) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} A_{2\lambda} = \frac{1}{5329}((-1)^{\eta}(-(219\eta-184)A_{2\eta+2} + (1022\eta-1759)A_{2\eta+1} + (5037\eta+1827)A_{2\eta} - (1679\eta-6034)A_{2\eta-1} - 11(584\eta-807)A_{2\eta-2}) - 35A_4 - 737A_3 + 1535A_2 + 4355A_1 + 2453A_0)$.

(c) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} A_{2\lambda+1} = \frac{1}{5329}((-1)^{\eta}((584\eta-1391)A_{2\eta+2} + (4380\eta+2379)A_{2\eta+1} - (2774\eta-6954)A_{2\eta} - (7957\eta-10165)A_{2\eta-1} - 11(219\eta-184)A_{2\eta-2}) - 807A_4 + 1430A_3 + 4180A_2 + 2208A_1 - 385A_0)$.

The next corollary follows if we replace A_{η} in the proposition 3.4 with G_{η}. We additionally apply the initial conditions where $G_0 = 0$, $G_1 = 0$, $G_2 = 0$, $G_3 = 1$ and $G_4 = 2$.

Corollary 3.11. *For $\eta \geq 0$, the 5-primes numbers satisfy the followings:*

(a) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} G_{\lambda} = \frac{1}{81}((-1)^{\eta}(-(9\eta-2)G_{\eta+4} + (27\eta-15)G_{\eta+3} + 36G_{\eta+2} + (45\eta-46)G_{\eta+1} + 11(9\eta+7)G_{\eta}) + 11)$.

(b) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} G_{2\lambda} = \frac{1}{5329}((-1)^{\eta}(-(219\eta-184)G_{2\eta+2} + (1022\eta-1759)G_{2\eta+1} + (5037\eta+1827)G_{2\eta} - (1679\eta-6034)G_{2\eta-1} - 11(584\eta-807)G_{2\eta-2}) - 807)$.

(c) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} G_{2\lambda+1} = \frac{1}{5329}((-1)^{\eta}((584\eta-1391)G_{2\eta+2} + (4380\eta+2379)G_{2\eta+1} - (2774\eta-6954)G_{2\eta} - (7957\eta-10165)G_{2\eta-1} - 11(219\eta-184)G_{2\eta-2}) - 184)$.

The next corollary follows if we replace A_{η} in the proposition 3.4 with H_{η}. We additionally apply the initial conditions where $H_0 = 5$, $H_1 = 2$, $H_2 = 10$, $H_3 = 41$ and $H_4 = 150$.

Corollary 3.12. *The following properties is satisfied by the Lucas 5-primes numbers, for $\eta \geq 0$:*

(a) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} H_{\lambda} = \frac{1}{81}((-1)^{\eta}(-(9\eta-2)H_{\eta+4} + (27\eta-15)H_{\eta+3} + 36H_{\eta+2} + (45\eta-46)H_{\eta+1} + 11(9\eta+7)H_{\eta}) - 338)$.

(b) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} H_{2\lambda} = \frac{1}{5329}((-1)^{\eta}(-(219\eta-184)H_{2\eta+2} + (1022\eta-1759)H_{2\eta+1} + (5037\eta+1827)H_{2\eta} - (1679\eta-6034)H_{2\eta-1} - 11(584\eta-807)H_{2\eta-2}) + 858)$.

(c) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} H_{2\lambda+1} = \frac{1}{5329}((-1)^{\eta}((584\eta-1391)H_{2\eta+2} + (4380\eta+2379)H_{2\eta+1} - (2774\eta-6954)H_{2\eta} - (7957\eta-10165)H_{2\eta-1} - 11(219\eta-184)H_{2\eta-2}) - 18129)$.

The next corollary follows if we replace A_{η} in the proposition 3.4 with E_{η}. We additionally apply the initial conditions where $E_0 = 0$, $E_1 = 0$, $E_2 = 0$, $E_3 = 1$ and $E_4 = 1$.

Corollary 3.13. *For $\eta \geq 0$, the modified 5-primes numbers get the followings:*

(a) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} E_{\lambda} = \frac{1}{81}((-1)^{\eta}(-(9\eta-2)E_{\eta+4} + (27\eta-15)E_{\eta+3} + 36E_{\eta+2} + (45\eta-46)E_{\eta+1} + 11(9\eta+7)E_{\eta}) + 13)$.

(b) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} E_{2\lambda} = \frac{1}{5329}((-1)^{\eta}(-(219\eta-184)E_{2\eta+2} + (1022\eta-1759)E_{2\eta+1} + (5037\eta+1827)E_{2\eta} - (1679\eta-6034)E_{2\eta-1} - 11(584\eta-807)E_{2\eta-2}) - 772)$.

(c) $\sum_{\lambda=0}^{\eta} \lambda(-1)^{\lambda} E_{2\lambda+1} = \frac{1}{5329}((-1)^{\eta}((584\eta-1391)E_{2\eta+2} + (4380\eta+2379)E_{2\eta+1} - (2774\eta-6954)E_{2\eta} - (7957\eta-10165)E_{2\eta-1} - 11(219\eta-184)E_{2\eta-2}) + 623)$.

3.3 The case $z=i$

During this subsection, we cope with the closed-form solutions (identities) for sums $\sum_{\lambda=0}^{\eta}\lambda i^{\lambda}A_{\lambda}$, $\sum_{\lambda=0}^{\eta}\lambda i^{\lambda}A_{2\lambda}$ and $\sum_{\lambda=0}^{\eta}\lambda i^{\lambda}A_{2\lambda+1}$ specifically considering the sequence $\{A_{\eta}\}$. Here, recall theorem 2.1 and set $z=i$, $r=s=t=u=v=1$; this gives the result below.

Proposition 3.5. *For $\eta\geq 0$ and setting $r,s,t,u,v=1$, in theorem 2.1 leads to the next outcomes.*

(a) $\sum_{\lambda=0}^{\eta}\lambda i^{\lambda}A_{\lambda}=\frac{1}{2i}(i^{\eta}(((1-i)\,\eta+6-i)A_{\eta+4}-2(\eta+(3+2i))A_{\eta+3}+(5i-10-(1-3i)\eta)A_{\eta+2}+((2+2i)\eta-4+10i)A_{\eta+1}+(((1+i)\eta+2+7i))A_{\eta})-(6-i)A_{4}+(6+4i)A_{3}+(10-5i)A_{2}+(4-10i)A_{1}-(2+7i)A_{0})$.

(b) $\sum_{\lambda=0}^{\eta}\lambda i^{\lambda}A_{2\lambda}=\frac{1}{18i}(i^{\eta}(((9+3i)\eta+6+7i)A_{2\eta+2}-(12\eta+16+12i)A_{2\eta+1}+(8+i-(3-3i)\eta)A_{2\eta}+(2i-8-(6-6i)\eta)A_{2\eta-1}+(2-13i-(3+3i)\eta)A_{2\eta-2})+(10-15i)A_{4}-(12-28i)A_{3}-(14+5i)A_{2}+(8+14i)A_{1}-(16-i)A_{0})$.

(c) $\sum_{\lambda=0}^{\eta}\lambda i^{\lambda}A_{2\lambda+1}=\frac{1}{18i}(i^{\eta}(((3i-3)\eta-10-5i)A_{2\eta+2}+((6+6i)\eta+14+8i)A_{2\eta+1}+((3+9i)\eta-2+9i)A_{2\eta}+(6\eta+8-6i)A_{2\eta-1}+((9+3i)\eta+6+7i)A_{2\eta-2})-(2-13i)A_{4}-(4+20i)A_{3}+(18-i)A_{2}-(6+14i)A_{1}+(10-15i)A_{0})$.

The next corollary follows if we replace A_{η} in the proposition 3.5 with P_{η}. We additionally apply the initial conditions where $P_0=0$, $P_1=1$, $P_2=1$, $P_3=2$ and $P_4=4$.

Corollary 3.14. *For $\eta\geq 0$, the Pentanacci numbers satisfy the followings.*

(a) $\sum_{\lambda=0}^{\eta}\lambda i^{\lambda}P_{\lambda}=\frac{1}{2i}(i^{\eta}(((1-i)\,\eta+6-i)P_{\eta+4}-2(\eta+(3+2i))P_{\eta+3}+(5i-10-(1-3i)\eta)P_{\eta+2}+((2+2i)\eta-4+10i)P_{\eta+1}+(((1+i)\eta+2+7i))P_{\eta})+(2-3i))$.

(b) $\sum_{\lambda=0}^{\eta}\lambda i^{\lambda}P_{2\lambda}=\frac{1}{18i}(i^{\eta}(((9+3i)\eta+6+7i)P_{2\eta+2}-(12\eta+16+12i)P_{2\eta+1}+(8+i-(3-3i)\eta)P_{2\eta}+(2i-8-(6-6i)\eta)P_{2\eta-1}+(2-13i-(3+3i)\eta)P_{2\eta-2})+(10+5i))$.

(c) $\sum_{\lambda=0}^{\eta}\lambda i^{\lambda}P_{2\lambda+1}=\frac{1}{18i}(i^{\eta}(((3i-3)\eta-10-5i)P_{2\eta+2}+((6+6i)\eta+14+8i)P_{2\eta+1}+((3+9i)\eta-2+9i)P_{2\eta}+(6\eta+8-6i)P_{2\eta-1}+((9+3i)\eta+6+7i)P_{2\eta-2})+(-4-3i))$.

The next corollary follows if we replace A_{η} in the proposition 3.5 with Q_{η}. We additionally apply the initial conditions where $Q_0=5$, $Q_1=1$, $Q_2=3$, $Q_3=7$ and $Q_4=15$.

Corollary 3.15. *For $\eta\geq 0$, the Pentanacci-Lucas numbers get the followings.*

(a) $\sum_{\lambda=0}^{\eta}\lambda i^{\lambda}Q_{\lambda}=\frac{1}{2i}(i^{\eta}(((1-i)\,\eta+6-i)Q_{\eta+4}-2(\eta+(3+2i))Q_{\eta+3}+(5i-10-(1-3i)\eta)Q_{\eta+2}+((2+2i)\eta-4+10i)Q_{\eta+1}+(((1+i)\eta+2+7i))Q_{\eta})+(-24-17i))$.

(b) $\sum_{\lambda=0}^{\eta}\lambda i^{\lambda}Q_{2\lambda}=\frac{1}{18i}(i^{\eta}(((9+3i)\eta+6+7i)Q_{2\eta+2}-(12\eta+16+12i)Q_{2\eta+1}+(8+i-(3-3i)\eta)Q_{2\eta}+(2i-8-(6-6i)\eta)Q_{2\eta-1}+(2-13i-(3+3i)\eta)Q_{2\eta-2})+(-48-25i))$.

(c) $\sum_{\lambda=0}^{\eta}\lambda i^{\lambda}Q_{2\lambda+1}=\frac{1}{18i}(i^{\eta}(((3i-3)\eta-10-5i)Q_{2\eta+2}+((6+6i)\eta+14+8i)Q_{2\eta+1}+((3+9i)\eta-2+9i)Q_{2\eta}+(6\eta+8-6i)Q_{2\eta-1}+((9+3i)\eta+6+7i)Q_{2\eta-2})+(40-37i))$.

In a similar way, readers can calculate the corresponding sums of other generalized fifth-order Pentanacci numbers.

4 Sum $\sum_{\lambda=1}^{\eta}\lambda z^{\lambda}A_{-\lambda}$ with negative subscripts

In this section, we compute the following sum for the generalized Pentanacci sequences $\{A_{\eta}\}_{\eta\geq 0}$ with negative subscripts: $\sum_{\lambda=1}^{\eta}\lambda z^{\lambda}A_{-\lambda}$.

Theorem 4.1. *For $\eta \geq 1$ and let z be a real (or complex) number. Then, we get the following formula: If $v + rz^4 + sz^3 + tz^2 + uz - z^5 \neq 0$, then*

$$\sum_{\lambda=1}^{\eta} \lambda z^{\lambda} A_{-\lambda} = \frac{\Omega_4}{(v + rz^4 + sz^3 + tz^2 + uz - z^5)^2},$$

where,

$\Omega_4 = z^{\eta+1}(\eta(-v-rz^4-sz^3-tz^2-uz+z^5)-v+3rz^4+2sz^3+tz^2-4z^5)A_{-\eta+4}+z^{\eta+1}(\eta(r-z)(v+rz^4+sz^3+tz^2+uz-z^5)+6rz^5+sz^4-uz^2-3r^2z^4+rv-2vz-3z^6-2rsz^3-rtz^2)A_{-\eta+3}+z^{\eta+1}(\eta(s+rz-z^2)(v+rz^4+sz^3+tz^2+uz-z^5)+4rz^6+4sz^5-tz^4-2uz^3-3vz^2-2r^2z^5-2s^2z^3+sv-2z^7-4rsz^4+ruz^2-stz^2+2rvz)A_{-\eta+2}+z^{\eta+1}(\eta(t+rz^2+sz-z^3)(v+rz^4+sz^3+tz^2+uz-z^5)+2rz^7+2sz^6+2tz^5-3uz^4-4vz^3-r^2z^6-s^2z^4-t^2z^2+tv-z^8-2rsz^5-2rtz^4+2ruz^3-2stz^3+3rvz^2+suz^2+2svz)A_{-\eta+1}+z^{\eta+1}(\eta(u+rz^3+sz^2+tz-z^4)(v+rz^4+sz^3+tz^2+uz-z^5)-5vz^4+uv+4rvz^3+3svz^2+2tvz)A_{-\eta}+z(v-3rz^4-2sz^3-tz^2+4z^5)A_4+z(-6rz^5-sz^4+uz^2+3r^2z^4-rv+2vz+3z^6+2rsz^3+rtz^2)A_3+z(-4rz^6-4sz^5+tz^4+2uz^3+3vz^2+2r^2z^5+2s^2z^3-sv+2z^7+4rsz^4-ruz^2+stz^2-2rvz)A_2+z(-2rz^7-2sz^6-2tz^5+3uz^4+4vz^3+r^2z^6+s^2z^4+t^2z^2-tv+z^8+2rsz^5+2rtz^4-2ruz^3+2stz^3-3rvz^2-suz^2-2svz)A_1+vz(-u-4rz^3-3sz^2-2tz+5z^4)A_0.$

Proof. Using the following recurrence relation

$$\begin{aligned}
A_{\eta+5} &= rA_{\eta+4} + sA_{\eta+3} + tA_{\eta+2} + uA_{\eta+1} + vA_{\eta} \\
&\Rightarrow \ A_{-\eta+5} = rA_{-\eta+4} + sA_{-\eta+3} + tA_{-\eta+2} + uA_{-\eta+1} + vA_{-\eta} \\
&\Rightarrow \ A_{-\eta} = -\frac{u}{v}A_{-\eta+1} - \frac{t}{v}A_{-\eta+2} - \frac{s}{v}A_{-\eta+3} - \frac{r}{v}A_{-\eta+4} + \frac{1}{v}A_{-\eta+5} \\
&\Rightarrow \ A_{-\eta} = \frac{1}{v}A_{-\eta+5} - \frac{u}{v}A_{-\eta+1} - \frac{t}{v}A_{-\eta+2} - \frac{s}{v}A_{-\eta+3} - \frac{r}{v}A_{-\eta+4}
\end{aligned}$$

i.e.

$$vA_{-\eta} = A_{-\eta+5} - rA_{-\eta+4} - sA_{-\eta+3} - tA_{-\eta+2} - uA_{-\eta+1}.$$

We obtain

$$\begin{aligned}
v \times \eta \times z^{\eta} A_{-\eta} &= \eta \times z^{\eta} A_{-\eta+5} - r \times \eta \times z^{\eta} A_{-\eta+4} \\
&\quad -s \times \eta \times z^{\eta} A_{-\eta+3} - t \times \eta \times z^{\eta} A_{-\eta+2} - u \times \eta \times z^{\eta} A_{-\eta+1}, \\
v(\eta-1)z^{\eta-1}A_{-\eta+1} &= (\eta-1)z^{\eta-1}A_{-\eta+6} - r(\eta-1)z^{\eta-1}A_{-\eta+5} \\
&\quad -s(\eta-1)z^{\eta-1}A_{-\eta+4} - t(\eta-1)z^{\eta-1}A_{-\eta+3} - u(\eta-1)z^{\eta-1}A_{-\eta+2}, \\
v(\eta-2)z^{\eta-2}A_{-\eta+2} &= (\eta-2)z^{\eta-2}A_{-\eta+7} - r(\eta-2)z^{\eta-2}A_{-\eta+6} \\
&\quad -s(\eta-2)z^{\eta-2}A_{-\eta+5} - t(\eta-2)z^{\eta-2}A_{-\eta+4} - u(\eta-2)z^{\eta-2}A_{-\eta+3}, \\
&\ \ \vdots \\
v \times 3 \times z^3 A_{-3} &= 3 \times z^3 A_2 - r \times 3 \times z^3 A_1 \\
&\quad -s \times 3 \times z^3 A_0 - t \times 3 \times z^3 A_{-1} - u \times 3 \times z^3 A_{-2}, \\
v \times 2 \times z^2 A_{-2} &= 2 \times z^2 A_3 - r \times 2 \times z^2 A_2 \\
&\quad -s \times 2 \times z^2 A_1 - t \times 2 \times z^2 A_0 - u \times 2 \times z^2 A_{-1}, \\
v \times 1 \times z^1 A_{-1} &= 1 \times z^1 A_4 - r \times 1 \times z^1 A_3 \\
&\quad -s \times 1 \times z^1 A_2 - t \times 1 \times z^1 A_1 - u \times 1 \times z^1 A_0.
\end{aligned}$$

By adding the idendities side by side and applying theorem 1.2 (a), we arrive the result. □

5 Results for special z values in negative subscripts

This section provides the closed-form solutions (identities) for sums $\sum_{\lambda=1}^{\eta} \lambda z^{\lambda} A_{-\lambda}$, for particular cases of $z=-1$ and $z=i$, specifically considering the sequence $A_{\eta}\}_{\eta\geq 0}$.

5.1 The case $z=1$

See Soykan [(31)] for the case $z=1$ of theorem 4.1.

5.2 The case $z=-1$

This subsection considers the special case $z=-1$.

In this subsection, we give the closed-form solutions (identities) for sums $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} A_{-\lambda}$, specifically considering the sequence $A_{\eta}\}_{\eta\geq 0}$.

Now, recall theorem 4.1 and set $r=s=t=u=v=1$; this gives the result below.

Proposition 5.1. *For $\eta \geq 1$ and setting $r,s,t,u,v=1$, in theorem 4.1 leads to the next outcome.*
$\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} A_{-\lambda} = \frac{1}{4}((-1)^{\eta}((2\eta-5)A_{-\eta+4}-(4\eta-8)A_{-\eta+3}+(2\eta+1)A_{-\eta+2}-(4\eta-2)A_{-\eta+1}+(2\eta+7)A_{-\eta})+5A_4-8A_3-A_2-2A_1-7A_0)$.

The next corollary follows if we replace A_{η} in the proposition 5.1 with P_{η} and Q_{η} respectively. We additionally apply the initial conditions where $P_0=0$, $P_1=1$, $P_2=1$, $P_3=2$, $P_4=4$, $Q_0=5$, $Q_1=1$, $Q_2=3$, $Q_3=7$ and $Q_4=15$.

Corollary 5.1. *For $\eta \geq 1$, we get the followings.*

(a) $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} P_{-\lambda} = \frac{1}{4}((-1)^{\eta}((2\eta-5)P_{-\eta+4}-(4\eta-8)P_{-\eta+3}+(2\eta+1)P_{-\eta+2}-(4\eta-2)P_{-\eta+1}+(2\eta+7)P_{-\eta})+1)$.

(b) $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} Q_{-\lambda} = \frac{1}{4}((-1)^{\eta}((2\eta-5)Q_{-\eta+4}-(4\eta-8)Q_{-\eta+3}+(2\eta+1)Q_{-\eta+2}-(4\eta-2)Q_{-\eta+1}+(2\eta+7)Q_{-\eta})-21)$.

Here, recall theorem 4.1 and set $r=2$, $s=t=u=v=1$; this gives the result below.

Proposition 5.2. *For $\eta \geq 1$ and setting $r=2$ and $s,t,u,v=1$, in theorem 4.1 leads to the next outcome.*
$\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} A_{-\lambda} = \frac{1}{9}((-1)^{\eta}((3\eta-8)A_{-\eta+4}-(9\eta-21)A_{-\eta+3}+(6\eta-4)A_{-\eta+2}-(9\eta-6)A_{-\eta+1}+(6\eta+11)A_{-\eta})+8A_4-21A_3+4A_2-6A_1-11A_0)$.

The next corollary follows if we replace A_{η} in the proposition 5.2 with P_{η} and Q_{η} respectively. We additionally apply the initial conditions where $P_0=0$, $P_1=1$, $P_2=2$, $P_3=5$, $P_4=13$, $Q_0=5$, $Q_1=2$, $Q_2=6$, $Q_3=17$ and $Q_4=46$.

Corollary 5.2. *For $\eta \geq 1$, we have the followings:*

(a) $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} P_{-\lambda} = \frac{1}{9}((-1)^{\eta}((3\eta-8)P_{-\eta+4}-(9\eta-21)P_{-\eta+3}+(6\eta-4)P_{-\eta+2}-(9\eta-6)P_{-\eta+1}+(6\eta+11)P_{-\eta})+1)$.

(b) $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} Q_{-\lambda} = \frac{1}{9}((-1)^{\eta}((3\eta-8)Q_{-\eta+4}-(9\eta-21)Q_{-\eta+3}+(6\eta-4)Q_{-\eta+2}-(9\eta-6)Q_{-\eta+1}+(6\eta+11)Q_{-\eta})-32)$.

At this point, recall theorem 4.1 and set $r=s=t=u=1$ and $v=2$; this gives the result below.

Proposition 5.3. *For $\eta \geq 1$ and setting $r, s, t, u = 1$ and $v = 1$, in theorem 4.1 leads to the next outcomes.*

$\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} A_{-\lambda} = \frac{1}{9}((-1)^{\eta}((3\eta-4)A_{-\eta+4} - (6\eta-5)A_{-\eta+3} + (3\eta+5)A_{-\eta+2} - (6\eta+4) A_{-\eta+1} + (3\eta+14)A_{-\eta}) + 4A_4 - 5A_3 - 5A_2 + 4A_1 - 14A_0)$.

The next corollary follows if we replace A_{η} in the proposition 5.3 with followings respectively:
$A_{\eta} = J_{\eta}$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1,$
$A_{\eta} = j_{\eta}$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20,$
$A_{\eta} = K_{\eta}$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20,$
$A_{\eta} = Q_{\eta}$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16,$
$A_{\eta} = S_{\eta}$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4,$
$A_{\eta} = R_{\eta}$ with $R_0 = 5, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15.$

Corollary 5.3. *For $\eta \geq 1$, we get the followings:*

(a) $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} J_{-\lambda} = \frac{1}{9}((-1)^{\eta}((3\eta-4)J_{-\eta+4} - (6\eta-5)J_{-\eta+3} + (3\eta+5)J_{-\eta+2} - (6\eta+4)J_{-\eta+1} + (3\eta+14)J_{-\eta}) - 2)$.

(b) $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} j_{-\lambda} = \frac{1}{9}((-1)^{\eta}((3\eta-4)j_{-\eta+4} - (6\eta-5)j_{-\eta+3} + (3\eta+5)j_{-\eta+2} - (6\eta+4)j_{-\eta+1} + (3\eta+14)j_{-\eta}) - 19)$.

(c) $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} K_{-\lambda} = \frac{1}{9}((-1)^{\eta}((3\eta-4)K_{-\eta+4} - (6\eta-5)K_{-\eta+3} + (3\eta+5)K_{-\eta+2} - (6\eta+4) K_{-\eta+1} + (3\eta+14)K_{-\eta}) - 23)$.

(d) $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} Q_{-\lambda} = \frac{1}{9}((-1)^{\eta}((3\eta-4)Q_{-\eta+4} - (6\eta-5)Q_{-\eta+3} + (3\eta+5)Q_{-\eta+2} - (6\eta+4) Q_{-\eta+1} + (3\eta+14)Q_{-\eta}) - 28)$.

(e) $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} S_{-\lambda} = \frac{1}{9}((-1)^{\eta}((3\eta-4)S_{-\eta+4} - (6\eta-5)S_{-\eta+3} + (3\eta+5)S_{-\eta+2} - (6\eta+4) S_{-\eta+1} + (3\eta+14)S_{-\eta}) + 5)$.

(f) $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} R_{-\lambda} = \frac{1}{9}((-1)^{\eta}((3\eta-4)R_{-\eta+4} - (6\eta-5)R_{-\eta+3} + (3\eta+5)R_{-\eta+2} - (6\eta+4) R_{-\eta+1} + (3\eta+14)R_{-\eta}) - 56)$.

Proposition 5.4. *For $\eta \geq 1$ and setting $r = 2$, $s = 3$, $t = 5$, $u = 7$, $v = 11$, in theorem 4.1 leads to the next outcomes.*

$\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} A_{-\lambda} = \frac{1}{81}((-1)^{\eta}((9\eta+2)A_{-\eta+4} - (27\eta+15)A_{-\eta+3} + 36A_{-\eta+2} - (45\eta+46)A_{-\eta+1} - (18\eta-77)A_{-\eta}) - 2A_4 + 15A_3 - 36A_2 + 46A_1 - 77A_0)$.

The next corollary follows if we replace A_{η} in the proposition 5.4 with G_{η}, H_{η} and E_{η} respectively. We additionally apply the initial conditions where $G_0 = 0$, $G_1 = 0$, $G_2 = 0$, $G_3 = 1$, $G_4 = 2$, $H_0 = 5$, $H_1 = 2$, $H_2 = 10$, $H_3 = 41$, $H_4 = 150$, $E_0 = 0$, $E_1 = 0$, $E_2 = 0$, $E_3 = 1$ and $E_4 = 1$.

Corollary 5.4. *For $\eta \geq 1$, we get the followings:*

(a) $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} G_{-\lambda} = \frac{1}{81}((-1)^{\eta}((9\eta+2)G_{-\eta+4} - (27\eta+15)G_{-\eta+3} + 36G_{-\eta+2} - (45\eta+46)G_{-\eta+1} - (18\eta-77)G_{-\eta}) + 11)$.

(b) $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} H_{-\lambda} = \frac{1}{81}((-1)^{\eta}((9\eta+2)H_{-\eta+4} - (27\eta+15)H_{-\eta+3} + 36H_{-\eta+2} - (45\eta+46)H_{-\eta+1} - (18\eta-77)H_{-\eta}) - 338)$.

(c) $\sum_{\lambda=1}^{\eta} \lambda(-1)^{\lambda} E_{-\lambda} = \frac{1}{81}((-1)^{\eta}((9\eta+2)E_{-\eta+4} - (27\eta+15)E_{-\eta+3} + 36E_{-\eta+2} - (45\eta+46)E_{-\eta+1} - (18\eta-77)E_{-\eta}) + 13)$.

5.3 The case $z = i$

In this subsection, we present the closed-form solutions (identities) for sums $\sum_{\lambda=1}^{\eta} \lambda i^{\lambda} A_{-\lambda}$, specifically considering the sequence $A_{\eta}\}_{\eta \geq 0}$. At this point, recall theorem 4.1 and set $r = s = t = u = v = 1$; this gives the result below.

Proposition 5.5. *For $\eta \geq 1$ and setting $r, s, t, u, v = 1$, in theorem 4.1 leads to the next outcomes.*
$\sum_{\lambda=1}^{\eta} \lambda i^{\lambda} A_{-\lambda} = \frac{1}{2i}(i^{\eta}(((1+i)\eta - 6 - i)A_{-\eta+4} + (6 - 2\eta - 4i)A_{-\eta+3} + (10 + 5i - (1+3i)\eta)A_{-\eta+2} + ((2-2i)\eta + 4 + 10i)A_{-\eta+1} + ((1+i)\eta - 2 + 7i)A_{-\eta}) + (6+i)A_4 - (6-4i)A_3 - (10+5i)A_2 - (4+10i)A_1 + (2-7i)A_0)$.

The next corollary follows if we replace A_{η} in the proposition 5.5 with P_{η} and Q_{η} respectively. We additionally apply the initial conditions where $P_0 = 0$, $P_1 = 1$, $P_2 = 2$, $P_3 = 5$, $P_4 = 4$, $Q_0 = 5$, $Q_1 = 1$, $Q_2 = 3$, $Q_3 = 7$ and $Q_4 = 15$.

Corollary 5.5. *For $\eta \geq 1$, we obtain the followings.*

(a) $\sum_{\lambda=1}^{\eta} \lambda i^{\lambda} P_{-\lambda} = \frac{1}{2i}(i^{\eta}(((1+i)\eta - 6 - i)P_{-\eta+4} + (6 - 2\eta - 4i)P_{-\eta+3} + (10 + 5i - (1+3i)\eta)P_{-\eta+2} + ((2-2i)\eta + 4 + 10i)P_{-\eta+1} + ((1+i)\eta - 2 + 7i)P_{-\eta}) + (-2-3i))$.

(b) $\sum_{\lambda=1}^{\eta} \lambda i^{\lambda} Q_{-\lambda} = \frac{1}{2i}(i^{\eta}(((1+i)\eta - 6 - i)Q_{-\eta+4} + (6 - 2\eta - 4i)Q_{-\eta+3} + (10 + 5i - (1+3i)\eta)Q_{-\eta+2} + ((2-2i)\eta + 4 + 10i)Q_{-\eta+1} + ((1+i)\eta - 2 + 7i)Q_{-\eta}) + (24 - 17i))$.

Corresponding summations of the other fifth-order generalized Pentanacci numbers can be calculated in a similar way.

6 Conclusion

In this study, we initially revisited the definition of generalized Pentanacci numbers, laying out the formal expressions and recurrence relations. We also drew from several important publications in the literature to build a foundation for our paper. To enhance comprehension, we present three detailed tables. Table 1 showcases the notations of some special cases of the generalized Pentanacci sequences with the related references. Table 2 gives the notations of some members of generalized Pentanacci sequences with the OEIS numbers of some of them and related papers. Finally, Table 3 presents the some special studies of sum formulas related to sequences and related studies. Moreover, we highlight two crucial theorems conducted by Soykan in [(30)], which have proven invaluable in this research. These theorems, referenced frequently throughout this paper, provided key theorems that significantly contributed to the development of the results presented here.

In Section 2, we compute the following sums for the generalized Pentanacci sequences $\{A_{\eta}\}_{\eta \geq 0}$ with positive subscripts: $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{\lambda}$, $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{2\lambda}$ and $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{2\lambda}$.

In section 3, we provide the closed-form solutions (identities) for sums $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{\lambda}$, $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{2\lambda}$ and $\sum_{\lambda=0}^{\eta} \lambda z^{\lambda} A_{2\lambda+1}$ for particular cases of $z = -1$ and $z = i$, specifically considering the sequence $\{A_{\eta}\}_{\eta \geq 0}$. The above sums have been calculated for the sequences listed in Table 2 when $z = -1$ and $z = i$.

In section 4, we compute the following sum for the generalized Pentanacci sequences $\{A_{\eta}\}_{\eta \geq 0}$ with negative subscripts: $\sum_{\lambda=1}^{\eta} \lambda z^{\lambda} A_{-\lambda}$.

In section 5, we provide the closed-form solutions (identities) for sums $\sum_{\lambda=1}^{\eta} \lambda z^{\lambda} A_{-\lambda}$ for particular cases of $z = -1$ and $z = i$, specifically considering the sequence $\{A_{\eta}\}_{\eta \geq 0}$. The above sums have been computed for the sequences listed in Table 2 when $z = -1$ and $z = i$.

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