Original ResearchArticle

ANoteonSumFormulasofGeneralizedPentanacciSequence:ClosedFormsoftheSumFormulas $\sum_{k=0}^{n} kx^kW_k$ and $\sum_{k=1}^{n} kx^kW_{-k}$

Abstract

Inthispaper,closedformsofthesumformulas $\sum_{k=0}^{n} kx^kW_k$ and $\sum_{k=1}^{n} kx^kW_{-k}$ forgeneralized Pentanaccinumbersarepresented.Asspecialcases,wegivesummationformulasofPentanacci, Pentanacci-Lucas,andotherfifth-orderrecurrencesequences.

Keywords:Pentanaccinumbers;Pentanacci-Lucasnumbers;sumformulas;summingformulas

2010MathematicsSubjectClassification:11B37;11B39;11B8

1 Introduction

The generalized Pentanaccisequence $\{W_n(W_0, W_1, W_2, W_3, W_4; r, s, t, u, v)\}_{n\geq 0}$ (or $\{W_n\}_{n\geq 0}$)isdefinedasfollows:

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}+vW_{n-5},$$
$$W_0 = c_0, W_1=c_1, W_2=c_2, W_3=c_3, W_4=c_4, n\geq 5$$

where W_0,W_1,W_2,W_3,W_4 arearbitraryrealorcomplexnumbersand r,s,t,u,v arerealnumbers.

Thesequence $\{W_n\}_{n\geq 0}$ canbeextendedtonegativesubscriptsbydefining

$$W_{-n} = -\frac{u}{v}W_{-n+1} - \frac{t}{v}W_{-n+2} - \frac{s}{v}W_{-n+3} - \frac{r}{v}W_{-n+4} + \frac{1}{v}W$$

for $n=1,2,3,\ldots$ when v Therefore,recurrence(1.1)holdsforallinteger n.Pentanaccis edbymanyauthors,seeforexample[(8)],[(9)],[(11)],[(26)].

Table1:Afewspecial case ofgeneralizedPentanaccisequences.

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where,

No	Sequences(Numbers)	Notation	Ref
1	GeneralizedPentanacci	$\{V_n\}=\{W_n(W_0,W_1,W_2,W_3,W_4;1,1,1,1,1)\}$	[(26)]
2	GeneralizedFifthorderPell	$\{V_n\}=\{W_n(W_0,W_1,W_2,W_3,W_4;2,1,1,1,1)\}$	[(27)]
3	GeneralizedFifthorderJacobsthal	$\{V_n\}=\{W_n(W_0,W_1,W_2,W_3,W_4;1,1,1,1,2)\}$	[(28)]
4	Generalized5-primes	$\{V_n\}=\{W_n(W_0,W_1,W_2,W_3,W_4;2,3,5,7,11)\}$	[(29)]

ForsomespecificvaluesofW_0,W_1,W_2,W_3,W_4andr,s,t,u,vitisworthpresentingthesesequences ersinatableasaspecificname.Inliterature,forexample,thefollowingnamesand notations(seeTable2)areusedforthespecialcasesofr,s,t,u,vandinitialvalues.

Table2:AfewmembersofgeneralizedPentanaccisequences.

Sequences(Numbers)	Notation	OEIS	Ref
Pentanacci	$\{P_n\}=\{W_n(0,1,1,2,4;1,1,1,1,1)\}$		
Pentanacci-Lucas	$\{Q_n\}=\{W_n(5,1,3,7,15;1,1,1,1,1)\}$		
fifthorderPell	$\{P^{(5)}_n\}=\{W_n(0,1,2,5,13;2,1,1,1,1)\}$		
fifthorderPell-Lucas	$\{Q^{(5)}_n\}=\{W_n(5,2,6,17,46;2,1,1,1,1)\}$		[(27)]
modifiedfifth-orderPell	$\{E^{(5)}_n\}=\{W_n(0,1,1,3,8;2,1,1,1,1)\}$		[(27)]
fifthorderJacobsthal	$\{J^{(5)}_n\}=\{W_n(0,1,1,1,1;1,1,1,1,2)\}$	A226310	[(28),(2)]
fifthorderJacobsthal-Lucas	$\{j^{(5)}_n\}=\{W_n(2,1,5,10,20;1,1,1,1,2)\}$	A226311	[(28),(2)]
modifiedfifthorderJacobsthal	$\{K^{(5)}_n\}=\{W_n(3,1,3,10,20;1,1,1,1,2)\}$		[(28)]
fifth-orderJacobsthalPerrin	$\{Q^{(5)}_n\}=\{W_n(3,0,2,8,16;1,1,1,1,2)\}$		[(28)]
adjustedfifth-orderJacobsthal	$\{S^{(5)}_n\}=\{W_n(0,1,1,2,4;1,1,1,1,2)\}$		[(28)]
modifiedfifth-orderJacobsthal-Lucas	$\{R^{(5)}_n\}=\{W_n(5,1,3,7,15;1,1,1,1,2)\}$		[(28)]
5-primes	$\{G_n\}=\{W_n(0,0,0,1,2;2,3,5,7,11)\}$		[(29)]
Lucas5-primes	$\{H_n\}=\{W_n(5,2,10,41,150;2,3,5,7,11)\}$		[(29)]
modified 5-primes	$\{E_n\}=\{W_n(0,0,0,1,1;2,3,5,7,11)\}$		[(29)]

Foreasywriting, wedropthesuperscriptsfromthesequences, forexamplewewrite $P_n$for$P^{(5)}_n$.

Wepresentsome works onsummingformulasofthenumbersinthefollowingTable

Table3: Afewspecial study ofsumformulas.

Name of sequence	Paperswhichdealwithsummingformulas
PellandPell-Lucas	[(1),(4),(32),(6),(7)]
GeneralizedFibonacci	[(5),(13),(14),(15),(16),(17),(19)]
GeneralizedTribonacci	[(3),(10),(18)]
GeneralizedTetranacci	[(20),(25),(33)]
GeneralizedPentanacci	[(21),(22)]
GeneralizedHexanacci	[(23),(24)]

The following presents some linear summing formulas of generalized numberswithpositivesubscripts.

Theorem1.1.*Letxbeareal(orcomplex)number.Forn≥0wehave [illegible]efollowingformu*

(a)*If* [illegible] $\neq 0$ *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{\Theta_1(x)}{rx+sx^2+tx^3+ux^4+vx^5-1} = \frac{\Theta_1(x)}{\Theta(x)}$$

where

$$\Theta_1(x)=x^{n+4}W_{n+4}-(rx-1)x^{n+3}W_{n+3}-(sx^2+rx-1)x^{n+2}W_{n+2}-(sx^2+tx^3$$

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LENOVO 2024-09-26 17:32:49 theorem

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$1)x^{n+1}W_{n+1} + vx^{n+5}W_n - x^4W_4 + x^3(rx- 1)W_3 + x^2(sx^2 + rx- 1)W_2 + x(sx^2 + tx^3 + rx-1)W_1+(sx^2+tx^3+ux^4+rx-1)W_0.$

(b) *If* $r^2x +2ux^2-s^2x^2+t^2x^3-u^2x^4+ v^2x^5+ 2sx+ 2rtx^2+ 2rvx^3-2sux^3+ 2tvx^4-1=0$

$$\sum_{k=0}^{n} x^kW_{2k} = \frac{\Theta_2(x)}{r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx}$$

where

$\Theta_2(x)=-(ux^2+sx-1)x^{n+1}W_{2n+2}+(t+rs+vx+rux)x^{n+2}W_{2n+1}+(u+t$ $rt+2tvx^2+rvx-sux)x^{n+2}W_{2n}+(v+ru-svx+tux)x^{n+2}W_{2n-1}+v(r+vx$ $x^2(ux^2+sx-1)W_4-x^3(t+rs+vx+rux)W_3+x(r^2x+ux^2-s^2x^2+2sx+rt$ $W_2 - x^3(v + ru- svx+ tux)W_1 + (r^2x + 2ux^2- s^2x^2 + t^2x^3- u$ $rvx^3-2sux^3+tvx^4-1)W_0.$

(c) *If* $r^2x+ 2ux^2-s^2x^2+ t^2x^3-u^2x^4+ v^2x^5+ 2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1=0$

$$\sum_{k=0}^{n} x^kW_{2k+1} = \frac{\Theta_3(x)}{r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx}$$

where

$\Theta_3(x) = (r + vx^2 + tx)x^{n+1}W_{2n+2} + (s - s^2x + t^2x^2- u^2x^3 + v^2x^4$ $+2tvx^3+rtx)x^{n+1}W_{2n+1}+(t+vx-svx^2+rux-stx)x^{n+1}W_{2n}+(u-u^2x^2+v^2x^3$ $rvx-sux)x^{n+1}W_{2n-1}- v\ \ ux^2+sx-1x^{n+1}W_{2n-2}-x^2(r+vx^2+tx)W_4+$ $sx+rtx^2+rvx^3-1)W_3-x^2(t+vx-svx^2+rux-stx)W_2+(r^2x+ux^2-s^2x^2+t^2x$ $2sx+2rtx^2+rvx^3-sux^3+tvx^4-1)W_1+vx^2(ux^2+sx-1)W_0$

Proof. ItisgiveninSoykan [(30), Theorem 2.1].

ThefollowingT[illegible]esentssomelinearsummingformulasofgeneralizedPent ivesubscripts.

Theorem1.2. *Forn≥1w[illegible]havethefollowingformulas:If* $v+rx^4+sx^3+tx^2+ux-x^5=0$, *t*

$$\sum_{k=1}^{n} x^kW_{-k} = \frac{\Theta_4(x)}{v+rx^4+sx^3+tx^2+ux-x^5}.$$

[illegible]$(x)=-x^{n+1}W_{4-n}+(r-x)x^{n+1}W_{-n+3}+(s+rx-x^2)x^{n+1}W_{-n+2}+(t+rx^2+sx-$ $+ rx^3 + sx^2 + tx- x^4)x^{n+1}W_{-n} + xW_4 - x(r - x)W_3 + x(-s - rx+ x^2)$ $sx+x^3)W_1+x(-u-rx^3-sx^2-tx+x^4)W_0.$

Proof. ItisgiveninSoykan [(30), Theorem 4.1].

Inthiswork,weinvestigatesummationformulasofgeneralizedPentanaccinum

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where $\beta_4(x)$

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2 SumFormulasofGeneralizedPentanacciNumberswithPositiveSubscripts

Thefollowingpositheorem presentssomesummingformulasofgeneralizedPentana vesubscripts.

Theorem2.1. *Letxbeareal(orcomplex)number.Forn≥0wehave thefollowingformu*

(a) *If* $vx^5+ux^4+tx^3+sx^2+rx-1$ *then*

$$\sum_{k=0}^{n} kx^k W_k = \frac{\Omega_1}{(vx^5+ux^4+tx^3+sx^2+rx-1)^2}$$

where

$\Omega_1=x^{n+4}(n(sx^2+tx^3+ux^4+vx^5+rx-1)-4+2sx^2+tx^3-vx^5+3rx)W_{n+4}-x$ $+ux^4+vx^5+rx-1)+3-sx^2+ux^4+2vx^5+3r^2x^2-6rx+2rsx^3+rtx^4-rvx^6)W$ $rx-1)(sx^2+tx^3+ux^4+vx^5+rx-1)+2$ $2ux^4+3vx^5+2r^2x^2+2s^2x^4-4rx+4rsx^3-rux^5+stx^5-2rvx^6-svx^7)W_{n+2}$ $-x^{n+1}(n(sx^2+tx^3+rx-1)(sx^2+tx^3+ux^4+vx^5+rx-1)+1+2rs$ $+2stx^5-3rvx^6-sux^6-2svx^7-tvx^8-2sx^2-2tx^3+3ux^4+4vx^5+$ $2rx)W_{n+1}+$ $vx^{n+5}(n(sx^2+tx^3+ux^4+vx^5+rx-1)+3sx^2+2tx^3+ux^4+4rx-5)W_n-x^4(2s$ $4)W_4+x^3(-sx^2+ux^4+2vx^5+3r^2x^2-6rx+2rsx^3+rtx^4-rvx^6+3)W_3$ $+x^2(-4sx^2+tx^3+2ux^4+3vx^5+2r^2x^2+2s^2x^4-4rx+4rsx^3-rux^5+stx^5-2rvx^6-svx^7+2)W_2+$ $x(-2sx^2-2tx^3+3ux^4+4vx^5+r^2x^2+s^2x^4+t^2x^6-2rx+2rsx^3+2rtx^4-2rux^5$ $+2stx^5-3rvx^6-sux^6-2svx^7-tvx^8+1)W_1-vx^5(3sx^2+2tx^3+ux^4+4rx-5)W$

(b) *If* $r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1\neq 0$

$$\sum_{k=0}^{n} kx^k W_{2k} = \frac{\Omega_2}{(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2}$$

where

$\Omega_2=-x^{n+1}(n(ux^2+sx-1)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2$ $2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+1+s^2x^2+2t^2x^3-u^2x^4+u^3x^6+4v^2x^5-2sx$ $+2r^2ux^3-st^2x^4+s^2ux^4+2su^2x^5-3sv^2x^6-2uv^2x^7-2rsvx^4+2rtux^4-4st$ $r^2sx^2-ux^2)W_{2n+2}+x^{n+2}(n(t+rs+vx+rux)(r^2x+2ux^2-s^2x^2+$ $+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+2rs^2x-t^3x^3-2v^3x^6-2rs-3vx-2$ $+$ $2uvx^3+2ru^2x^3+2r^3ux^2+2r^2vx^2+ru^3x^5-s^2vx^3+2tu^2x^4-4t^2vx^4-5tv^2x$ $rst^2x^3+rs^2ux^3+2rsu^2x^4-2r^2svx^3+2r^2tux^3-3rsv^2x^5-2ruv^2x^6-3rux+2stx+4rsux^2-$ $4rstvx^4-2rtuvx^5)W_{2n+1}+x^{n+2}(n(u+t^2x-u^2x^2+v^2x^3+rt+2tvx^2+rvx-$ $sux)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)$ $+4u^2x^2-3t^2x-2u-2u^3x^4-5v^2x^3-2rt+2r^2t^2x^2-3r^2u^2x^3-s^2t^2x^3+4r^2v^2x^4+$ $2s^2u^2x^4-3s^2v^2x^5+r^3tx+r^2ux-8tvx^2+rt^3x^3+4st^2x^2-4s^2ux^2-6su^2x^3+2r^3vx^2+s^3ux^3+t^2ux^3+su^3x^5+8sv^2x^4+2rv^3x^6+3uv^2x^5+4rsvx^2+12stvx^3-2r^2sux^2-rs^2vx^3-2rtu^2x^4$ $+6r^2tvx^3+4rt^2vx^4+5rtv^2x^5-4s^2tvx^4-ru^2vx^5-2suv^2x^6+2rstx-2stuvx^5-3rvx+5sux+4tuvx^4)W_{2n}+x^{n+2}(n(v+ru-svx+tux)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+$ $2rvx^3-2sux^3+2tvx^4-1)+r^3ux-3v^3x^5-2ru-2v+r^2vx+2ru^3x^4-2rv^2x^3-4s^2vx^2+2tu^2x^3+s^3vx^3-$ $t^2vx^3+tu^3x^5-4tv^2x^4+2sv^3x^6+2u^2vx^4+4stux^2+2rsu^2x^3-2r^2svx^2+$ $2r^2tux^2+rt^2ux^3-s^2tux^3-2r^2uvx^3-3ruv^2x^5+2stv^2x^5-su^2vx^5-2t^2uvx^5-2tuv^2x^6-2rstvx^3-$

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$4rtuvx^4+5svx-3tux+2rsux)W_{2n-1}+vx^{n+2}(n(r+vx^2+tx)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+r^3x-2r-4vx^2-v^3x^7-3tx+4stx^2+6svx^3+2tux^3+4uvx^4+2r^2tx^2+rt^2x^3-s^2tx^3+2ru^2x^4+r^2vx^3-rv^2x^5-2s^2vx^4+tu^2x^5-t^2vx^5-2tv^2x^6+2rsx+2rsux^3-2suvx^5)W_{2n-2}+x^2(-r^2x-4ux^2+4s^2x^2-s^3x^3+t^2x^3+2u^2x^4+3v^2x^5-5sx+2rvx^3+6sux^3+4tvx^4+2r^2sx^2+3r^2ux^3-2s^2ux^4-su^2x^5+t^2ux^5-2sv^2x^6-uv^2x^7+2rstx^3+4rtux^4+2ruvx^5-2stvx^5+2)W_4+x^3(3t+v^3x^6+3rs+4vx-4rs^2x-2r^3sx-2r^2tx-6svx^2-2tux^2-4uvx^3+rs^3x^2-2rt^2x^2+s^2tx^2-4ru^2x^3-3r^3ux^2-3r^2vx^2-2rv^2x^4+2s^2vx^3-tu^2x^4+t^2vx^4+2tv^2x^5+4rux-4stx-8rsux^2-4rtvx^3+2suvx^4-2r^2stx^2+2rs^2ux^3+rsu^2x^4-4r^2tux^3-rt^2ux^4+2rsv^2x^5-2r^2uvx^4+ruv^2x^6+2rstvx^4)W_3+x(r^4x^2+2r^3tx^3+r^3vx^4-2r^2s^2x^3-r^2sux^4+4r^2sx^2+r^2t^2x^4+2r^2u^2x^5+2r^2ux^3-r^2v^2x^6-2r^2x-3rs^2tx^4-2rs^2vx^5-4rstux^5+4rstx^3-4rsuvx^6+2rsvx^4-rt^2vx^6+rtu^2x^6+4rtux^4-2rtv^2x^7-rtx^2+4ruvx^5-rv^3x^8+s^4x^4+2s^3ux^5-4s^3x^3+2s^2tvx^6+s^2u^2x^6-5s^2ux^4+2s^2v^2x^7+6s^2x^2-st^2ux^6-2st^2x^4-8stvx^5+suv^2x^8+4sux^3-6sv^2x^6-4sx+2t^2x^3-2tuvx^6+6tvx^4+u^3x^6-u^2x^4-2uv^2x^7-ux^2+4v^2x^5+1)W_2+x^3(3v+2v^3x^5+3ru-2r^3ux-2r^2vx-2uvx^2-2ru^2x^2-ru^3x^4+7s^2vx^2-4tu^2x^3-2s^3vx^3-t^3ux^4+2tv^2x^4-sv^3x^6-u^2vx^4-8svx+4tux-2rtvx^2-6stux^2+4suvx^3+rs^2ux^2+3r^2svx^2-5r^2tux^2-4rt^2ux^3+2rsv^2x^4+2s^2tux^3+2stu^2x^4+st^2vx^4+2ruv^2x^5-2s^2uvx^4+tuv^2x^6-4rsux+4rstvx^3)W_1+vx^3(3r-2r^3x+5vx^2-t^3x^4+4tx-2rux^2-6stx^2-8svx^3-4tux^3-6uvx^4+rs^2x^2-5r^2tx^2-4rt^2x^3+2s^2tx^3-ru^2x^4-4r^2vx^3-2rv^2x^5+3s^2vx^4-2t^2vx^5-tv^2x^6+u^2vx^6-4rsx-6rtvx^4+2stux^4+4suvx^5)W_0.$

(c) If $r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1\neq$

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$$\sum_{k=0}^{n} kx^kW_{2k+1}=\frac{\Omega_3}{(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2\ldots}$$

where

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$\Omega_3=x^{n+1}(n(r+vx^2+tx)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+\ldots 2tvx^4-1)-3vx^2-t^3x^4-2v^3x^7-2tx-r-2rux^2+2stx^2\ldots rs^2x^2-r^2tx^2-2rt^2x^3+3ru^2x^4-2r^2vx^3-4rv^2x^5-s^2vx^4+2tu^2x^5-4t^2vx^5-\ldots x^3-6rtvx^4+2stux^4)W_{2n+2}+x^{n+1}(n(s-s^2x+t^2x^2-u^2x^3+v^2x^4+ux+rvx^2-2\ldots (r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)\ldots {}^2+4u^2x^3-2u^3x^5-5v^2x^4-r^2s^2x^2+2r^2t^2x^3-3r^2u^2x^4+4r^2v^2x^5-3rvx^2+6\ldots r^2ux^2+rt^3x^4+2st^2x^3-4s^2ux^3-5su^2x^4+2r^3vx^3+t^2ux^4+4sv^2x^5+2r\ldots +4tuvx^5+rs^2vx^4-2rtu^2x^5+6r^2tvx^4+4rt^2vx^5+5rtv^2x^6-ru^2vx^6-4r^2sux^3-2rstux^4-2rtx-2ux)W_{2n+1}+x^{n+1}(n(t+vx-svx^2+rux-stx)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+5svx^2-2t^3x^3-3v^3x^6-2vx-t-2tux^2-2rt^2x^2-s^2tx^2+r^3ux^2+r^2vx^2+st^3x^4+2ru^3x^5-2rv^2x^4-4s^2vx^3+3tu^2x^4+s^3vx^4-7t^2vx^4-8tv^2x^5+2sv^3x^7+2u^2vx^5-2rux+2stx+2rsux^2-4rtvx^3+4stux^3-r^2stx^2-2r^2svx^3-rt^2ux^4-2s^2tux^4-2stu^2x^5-2r^2uvx^4+4st^2vx^5-3ruv^2x^6+5stv^2x^6-su^2vx^6+2rsu^2x^4-4rtuvx^5)W_{2n}+x^{n+1}(n(u-u^2x^2+v^2x^3+tvx^2+rvx-sux)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+u^2x^2-u+u^3x^4-4v^2x^3-u^4x^6-v^4x^8-2r^2u^2x^3+r^2v^2x^4-s^2u^2x^4-2s^2v^2x^5-t^2v^2x^6+2u^2v^2x^7-3tvx^2-s^2ux^2+r^3vx^2-2t^2ux^3-2su^3x^5+6sv^2x^4-rv^3x^6-2tv^3x^7-2rvx+2sux+2rsvx^2-2rtux^2-4ruvx^3+4stvx^3-4tuvx^4-r^2sux^2-2rtu^2x^4+2r^2tvx^3+rt^2vx^4+st^2ux^4-s^2tvx^4+2ru^2vx^5+suv^2x^6+3tu^2vx^6+4rsuvx^4+4stuvx^5)W_{2n-1}-vx^{n+1}(n(ux^2+sx-1)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+1+s^2x^2+2t^2x^3-ux^2-u^2x^4+u^3x^6+4v^2x^5-2sx+2rtx^2+4rvx^3+6tvx^4+r^2sx^2+2r^2ux^3-st^2x^4+s^2ux^4+2su^2x^5-3sv^2x^6-2uv^2x^7-2rsvx^4+2rtux^4-4stvx^5-2tuvx^6)W_{2n-2}+x^2(2r-r^3x+4vx^2+v^3x^7+3tx-4stx^2-6svx^3-2tux^3-4uvx^4-2r^2tx^2-rt^2x^3+s^2tx^3-2ru^2x^4-r^2vx^3+rv^2x^5+2s^2vx^4-tu^2x^5+t^2vx^5+2tv^2x^6-2rsx-2rsux^3+2suvx^5)$

$W_4+x(r^4x^2+2r^3tx^3+r^3vx^4+2r^2sux^4+3r^2sx^2+r^2t^2x^4+2r^2u^2x^5+2r^2ux^3-r^2v^2x^6-2r^2x - rs^2tx^4-2rs^2vx^5+4rstx^3-2rsuvx^6+4rsvx^4-rt^2vx^6+rtu^2x^6+4rtux^4-2rtv^2x^7-rtx^2+4ruvx^5-rv^3x^8+s^2ux^4+s^2x^2-st^2x^4-4stvx^5+2su^2x^5-3sv^2x^6-2sx+2t^2x^3-2tuvx^6+6tvx^4+u^3x^6-u^2x^4-2uv^2x^7-ux^2+4v^2x^5+1)W_3+x^2(-2r^3ux^2+2r^2stx^2+3r^2svx^3-2r^2tux^3-r^2tx-2r^2vx^2+rs^2ux^3+2rst^2x^3+4rstvx^4-4rsux^2+2rsv^2x^5+2rtuvx^5-ru^3x^5-2ru^2x^3+2ruv^2x^6+3rux-s^3tx^3-2s^3vx^4+4s^2tx^2-2s^2uvx^5+7s^2vx^3-st^2vx^5+stu^2x^5-2stv^2x^6-5stx+4suvx^4-sv^3x^7-8svx^2+t^3x^3+4t^2vx^4-2tu^2x^4+5tv^2x^5+2t-u^2vx^5-2uvx^3+2v^3x^6+3vx)W_2+x^2(-2r^3vx^2+2r^2sux^2-5r^2tvx^3+3r^2u^2x^3-r^2ux-4r^2v^2x^4+rs^2vx^3+2rstux^3-4rsvx^2-4rt^2vx^4+4rtu^2x^4-6rtv^2x^5+ru^2vx^5-2rv^3x^6+3rvx-s^3ux^3+2s^2tvx^4-2s^2u^2x^4+4s^2ux^2+3s^2v^2x^5-6stvx^3-su^3x^5+6su^2x^3+2suv^2x^6-5sux-8sv^2x^4-t^3vx^5+t^2u^2x^5+t^2ux^3-2t^2v^2x^6-tv^3x^7+4tvx^2+2u^3x^4-4u^2x^2-3uv^2x^5+2u+5v^2x^3)W_1+vx^2(-r^2x-4ux^2+4s^2x^2-s^3x^3+t^2x^3+2u^2x^4+3v^2x^5-5sx+2rvx^3+6sux^3+4tvx^4+2r^2sx^2+3r^2ux^3-2s^2ux^4-su^2x^5+t^2ux^5-2sv^2x^6-uv^2x^7+2rstx^3+4rtux^4+2ruvx^5-2stvx^5+2)W_0.$

Proof. **(a)** Using therecurrence relation

$$W_n=rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}+$$

i.e.

$$vW_{n-5}=W_n-rW_{n-1}-sW_{n-2}-tW_{n-3}-uW_{n-4}$$

obtain

$$v\times 0\times x^0W_0 = 0\times x^0W_5-r\times 0\times x^0W_4-s\times 0\times x^0W_3-t$$
$$-u\times 0\times x^0$$
$$v\times 1\times x^1W_1 = 1\times x^1W_6-r\times 1\times x^1W_5-s\times 1\times x^1W_4-t$$
$$-u\times 1\times xW_2$$
$$v\times 2\times x^2W_2 = 2\times x^2W_7-r\times 2\times x^2W_6-s\times 2\times x^2W_5-t$$
$$-u\times 2\times xW_3$$

.

$$v(n-2)x^{n-2}W_{n-2} = (n-2)x^{n-2}W_{n+3}-r(n-2)x^{n-2}W_{n+2}-s($$
$$-t(n-2)x^{n-2}W_n-u(n-2)x^{n-2}W$$
$$v(n-1)x^{n-1}W_{n-1} = (n-1)x^{n-1}W_{n+4}-r(n-1)x^{n-1}W_{n+3}-s($$
$$-t(n-1)x^{n-1}W_{n+1}-u(n-1)x^{n-1}W$$
$$v\times n\times x^nW_n = n\times x^nW_{n+5}-r\times n\times x^nW_{n+4}-s\times n\times x^nW$$
$$-t\times n\times xW^n_{n+2}-u\times n\times x^n$$

Ifwe add theequations side byside (and usingTh 1(a)), weget(a)

(b)and(c) Usingthe recurrence relation

$$W_n=rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}+vW_n$$

i.e.

$$rW_{n-1}=W_n-sW_{n-2}-tW_{n-3}-uW_{n-4}-vW_n$$

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weobtain

$$r\times 1\times x^1 W_3 = 1\times x^1 W_4 - s\times 1\times x^1 W_2 - t\times 1\times x^1 W_1$$
$$- u\times 1\times x^1 W_0 - v\times 1\times x^1$$
$$r\times 2\times x^2 W_5 = 2\times x^2 W_6 - s\times 2\times x^2 W_4 - t\times 2\times x^2 W$$
$$- u\times 2\times x^2 W_2 - v\times 2\times x^2 W_1$$

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$$\vdots$$
$$r\times (n-1)\times x^{n-1} W_{2n-1} = (n-1)\times x^{n-1} W_{2n} - s\times (n-1)\times x^{n-1} W_2$$
$$- t\times (n-1)\times x^{n-1} W_{2n-3} - u\times (n-1)\times$$
$$- v\times (n-1)\times x^{n-1} W_{2n-}$$

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$$r\times n\times x^n W_{2n+1} = n\times x^n W_{2n+2} - s\times n\times x^n W_{2n} - t\times n\times x^n$$
$$- u\times n\times x^n W_{2n-2} - v\times n\times x^n$$

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Now,ifweaddtheaboveequationssidebyside,weget

$$r(-0\times x^0 W_1 + \sum_{k=0}^{n} kx^k W_{2k+1}) = (n\times x^n W_{2n+2} - 0\times x^0 W_2 - ($$
$$+ \sum_{k=0}^{n} (k-1)x^{k-1} W_{2k}) - s(-0\times x^0 W_0 + \sum_{k=0}^{n} kx^k W_{2k}) - t(-(n+1$$
$$+ \sum_{k=0}^{n} (k+1)x^{k+1} W_{2k+1}) - u(-(n+1)x^{n+1} W_{2n} + \sum_{k=0}^{n} (k+1)x^{k+1} W_{2k})$$
$$- v(-(n+2)x^{n+2} W_{2n+1} - (n+1)x^{n+1} W_{2n-1} + 1\times x^1 W_{-1}$$
$$+ \sum_{k=0}^{n} (k+2)x^{k+2} W_{2k+1}).$$

Since

$$W_{-1} = -\frac{u}{v} W_0 - \frac{t}{v} W_1 - \frac{s}{v} W_2 - \frac{r}{v} W_3 + \frac{1}{v} W$$

weobtain

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$$r(-0\times x^0 W_1 + \sum_{k=0}^{n} kx^k W_{2k+1}) = (n\times x^n W_{2n+2} - 0\times x^0 W_2 - (-1)\times x^{-1} W$$
$$+ x^{-1}\sum_{k=0}^{n} kx^k W_{2k} - x^{-1}\sum_{k=0}^{n} x^k W_{2k}) - s(-0\times x^0 W_0 + \sum_{k=0}^{n} kx^k W_{2k}) - t($$
$$+ x^1 \sum_{k=0}^{n} kx^k W_{2k+1} + x^1 \sum_{k=0}^{n} x^k W_{2k+1}) - u(-(n+1)x^{n+1} W_{2n} + x^1 \sum_{k=0}^{n} kx^k W_{2k}$$
$$+ x^1 \sum_{k=0}^{n} x^k W_{2k}) - v(-(n+2)x^{n+2} W_{2n+1} - (n+1)x^{n+1} W_{2n-1} + 1\times x^1(-\frac{u}{v} W_0 - \frac{t}{v} W_1$$
$$- \frac{s}{v} W_2 - \frac{r}{v} W_3 + \frac{1}{v} W_4) + x^2 \sum_{k=0}^{n} kx^k W_{2k+1} + 2x^2 \sum_{k=0}^{n} x^k W_{2k+1}).$$

Similarly,usingtherecurrencerelation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} +$$

i.e.

$$rW_{n-1}=W_n-sW_{n-2}-tW_{n-3}-uW_{n-4}-vW_n$$

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writethefollowingobviousequations;

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We write

$$\begin{aligned}
r\times 1\times x^1W_2 &= 1\times x^1W_3-s\times 1\times x^1W_1-t\times 1\times x^1W_0 \\
&\quad -u\times 1\times x^1W_{-1}-v\times 1\times x^1 \\
r\times 2\times x^2W_4 &= 2\times x^2W_5-s\times 2\times x^2W_3-t\times 2\times x^2W_2 \\
&\quad -u\times 2\times x^2W_1-v\times 2\times x^2 \\
&\;\;. \\
r\times(n-1)\times x^{n-1}W_{2n-2} &= (n-1)\times x^{n-1}W_{2n-1}-s\times(n-1)\times x^{n-1}W_{2n} \\
&\quad -t\times(n-1)\times x^{n-1}W_{2n-4}-u\times(n-1)\times x^{n} \\
&\quad -v\times(n-1)\times x^{n-1}W_2 \\
r\times n\times x^nW_{2n} &= n\times x^nW_{2n+1}-s\times n\times x^nW_{2n-1} \\
&\quad -t\times n\times x^nW_{2n-2}-u\times n\times x^nW_{2n-3}-v\times n
\end{aligned}$$

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Now,ifweaddtheaboveequationssidebyside,weobtain

$$\begin{aligned}
& r(-0\times x^0W_0+\sum_{k=0}^{n}kx^kW_{2k})=(-0\times x^0W_1+\sum_{k=0}^{n}kx^kW_{2k+1}) \\
& -s(-(n+1)x^{n+1}W_{2n+1}+\sum_{k=0}^{n}(k+1)x^{k+1}W_{2k+1})-t(-(n+1)x^{n+1}W_{2n} \\
& +\sum_{k=0}^{n}(k+1)x^{k+1}W_{2k})-u(-(n+2)x^{n+2}W_{2n+1}-(n+1)x^{n+1}W_{2n-1} \\
& +1\times x^1W_{-1}+\sum_{k=0}^{n}(k+2)x^{k+2}W_{2k+1})-v(-(n+2)x^{n+2}W_{2n}-(n+1)x^{n+1}W_{2n-2} \\
& +1\times x^1W_{-2}+\sum_{k=0}^{n}(k+2)x^{k+2}W
\end{aligned}$$

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Since

$$\begin{aligned}
W_{-1} &= -\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+ \\
W_{-2} &= -\frac{u}{v}(-\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+\frac{1}{v}W_4)-\frac{t}{v}W_0-\frac{s}{v}W_0-\frac{r}{v}W+
\end{aligned}$$

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have

$$r\left(-0\times x^0W_0+\sum_{k=0}^{n}kx^kW_{2k}\right)=\left(-0\times x^0W_1+\sum_{k=0}^{n}kx^kW_{2k+1}\right)$$

$$-s\left(-(n+1)x^{n+1}W_{2n+1}+x^1\sum_{k=0}^{n}kx^kW_{2k+1}+x^1\sum_{k=0}^{n}x^kW_{2k+1}\right)-t(-$$

$$+x^1\sum_{k=0}^{n}kx^kW_{2k}+x^1\sum_{k=0}^{n}x^kW_{2k})\ -u(-(n+2)x^{n+2}W_{2n+1}-(n+1)x\ \ W_{2n-1}$$

$$+1\times x^1\left(-\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3\right)+x\frac{1}{v}W_4\ \ 2\sum_{k=0}^{n}kx^kW_{2k+1}+2x\ 2\sum_{k=0}^{n}x^kW_{2k+1})$$

$$-v(-(n+2)x^{n+2}W_{2n}-(n+1)x^{n+1}W_{2n-2}+1\times x\ (-\ ^1(-\frac{u}{v}W_0-\frac{u}{v}W_1-\frac{t}{v}W_2-\frac{s}{v}W_3\ \frac{r}{}$$

$$+\frac{1}{v}W_4)-\frac{t}{v}W_0-\frac{s}{v}W_1-\frac{r}{v}W_2+\frac{1}{v}W_3)+x\frac{1}{}\ \ 2\sum_{k=0}^{n}kx^kW_{2k}+2x\ 2\sum_{k=0}^{n}x^kW_{2k}).$$

Then,usingTheorem1.1(b)and(c)andsolvingthesystem(2.1)-(2.2),thereq

(b) and(c)follow.□

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We have

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Then,byusingparts(b)and(c)oftheorem1.1 and solvingthesystem ofequations(2.1) - (2.2), the desired result follows.

3 SpecialCases

Inthissection,forthespecialcasesof$x$,werepresenttheclosedformsolutions(identitie $\sum_{k=0}^{n} kx^kW_k$, $\sum_{k=0}^{n} kx^kW_{2k}$and $\sum_{k=0}^{n} kx^kW$ orthespecific case ofseq

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cases

3.1 Thecase$x$=1

Inthis subsection weconsiderthespecialcase$x$=1.

Thecase$x$=1ofTheo givenin Soykan[(31)].

3.2 Thecase$x$=−1

Inthissubsectionweconsiderthespecialcase x= −1 andwepresentthecl (identities)ofthesums $\sum_{k=0}^{n} k(-1)^kW_k$, $\sum_{k=0}^{n} k(-1)^kW_{2k}$and $\sum_{k=0}^{n} k(-1)^k$ caseofthesequence$\{W_n\}$.

Taking$r=s=t=u=v=1$inTheorem2.1(a),(b)and(c),weobtainthefollowingPro

theorem

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Proposition3.1. *If$r=s=t=u=$ $v=1$ thenfor n havethefollowingform*

(a) $\sum_{k=0}^{n} k(-1)^kW_k=\frac{1}{}((-1)^n(-(2n+5)W_{n+4}+(4n+8)W_{n+3}-(2n-1)W_{n+2}+(4$ $(2n+7)W_n)+5W_4-8W_3-W_2-2W_1-7W_0)$.

(b) $\sum_{k=0}^{n} k(-1)^kW_{2k}=\frac{1}{4}((-1)^n((2n-1)W_{2n+3}+4W_{2n+1}-(2n-1)W_{2n}-(4n+6)W$ $W_{2n-2})+W_4+4W_3-5W_2-10W_1-7W_0)$.

(c) $\sum_{k=0}^{n} k(-1)^kW_{2k+1}=\frac{1}{4}((-1)^n((2n+3)W_{2n+2}-(2n+7)W_{2n}-6W_{2n-1}+(2n-1$ $5W_4-4W_3-9W_2-6W_1+W_0)$.

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proposition

Fromtheabove Proposition, wehavethefollowing Corollary whichgivesli Pentanaccinumbers(take$W_n=P_n$ with$P_0=0,P_1=1,P_2=1,P_3=2,P_4=4$).

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corollary

Corollary 3.1. *For $n \geq 0$, Pentanacci numbers have the following properties.*

(a) $\sum_{k=0}^{n} k(-1)^k P_k = \frac{1}{4}((-1)^n(-(2n+5)P_{n+4}+(4n+8)P_{n+3}-(2n-1)P_{n+2}+(4n+2)P_{n+1}+(2n+7)P_n)+1)$.

(b) $\sum_{k=0}^{n} k(-1)^k P_{2k} = \frac{1}{4}((-1)^n((2n-1)P_{2n+2}+4P_{2n+1}-(2n-1)P_{2n}-(4n+6)P_{2n-1}-(2n+5)P_{2n-2})-3)$.

(c) $\sum_{k=0}^{n} k(-1)^k P_{2k+1} = \frac{1}{4}((-1)^n((2n+3)P_{2n+2}-(2n+7)P_{2n}-6P_{2n-1}+(2n-1)P_{2n-2})-3)$.

Taking $W_n = Q_n$ with $Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 =$ in the ab[...] have the following Corollary which presents linear sum formulas of Pentanacci-Lu[...]

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corollary

Corollary 3.2. *For $n \geq 0$, Pentanacci-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n} k(-1)^k Q_k = \frac{1}{4}((-1)^n(-(2n+5)Q_{n+4}+(4n+8)Q_{n+3}-(2n-1)Q_{n+2}+(4n$ [...] $(2n+7)Q_n)-21)$.

(b) $\sum_{k=0}^{n} k(-1)^k Q_{2k} = \frac{1}{4}((-1)^n((2n-1)Q_{2n+2}+4Q_{2n+1}-(2n-1)Q_{2n}-(4n+6)Q_{2n-}$ [...] $Q_{2n-2})-17)$.

(c) $\sum_{k=0}^{n} k(-1)^k Q_{2k+1} = \frac{1}{4}((-1)^n((2n+3)Q_{2n+2}-(2n+7)Q_{2n}-6Q_{2n-1}+(2n-1)Q_{2n-2})+19)$.

Taking $r = 2, s = t = u = v = 1$ in Theorem 2 [...] ow[...] Proposition.

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Proposition 3.2. *If $r = 2, s = t = u = v = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{6}((-1)^n(-(3n+8)W_{n+4}+(9n+21)W_{n+3}-(6n+4)W_{n+2}+$ [...] $(3n+11)W_n)+8W_4-21W_3+4W_2-6W_1-11W_0)$.

(b) $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{25}((-1)^n((5n-6)W_{2n+2}+15W_{2n+1}-(5n-11)W_{2n}-(10n+13$ [...] $(15n+12)W_{2n-1})-W_4+15W_3-19W_2-27W_1-23W_0)$.

(c) $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{25}((-1)^n((10n+3)W_{2n+2}+5W_{2n+1}-(10n+18)W_{2n}+(5n-$ [...] $(5n+19)W_{2n-1})+13W_4-20W_3-28W_2-24W_1-W_0)$.

From the last Proposition, we have the following Corollary which gives lin[...] -order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13$).

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summation

Corollary 3.3. *For $n \geq 0$, fifth-[...]er Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k P_k = \frac{1}{6}((-1)^n(-(3n+8)P_{n+4}+(9n+21)P_{n+3}-(6n+4)P_{n+2}+(9n$ [...] $(3n+11)P_n)+1)$.

(b) $\sum_{k=0}^{n} k(-1)^k P_{2k} = \frac{1}{25}((-1)^n((5n-6)P_{2n+2}+15P_{2n+1}-(5n-11)P_{2n}-(10n+13)P$ [...] $(15n+12)P_{2n-1})-3)$.

(c) $\sum_{k=0}^{n} k(-1)^k P_{2k+1} = \frac{1}{25}((-1)^n((10n+3)P_{2n+2}+5P_{2n+1}-(10n+18)P_{2n}+(5n-6$ [...] $(5n+19)P_{2n-1})-11)$.

Taking $P_n = Q_n$ with $Q_0 = 5, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 =$ in the last [...] the following Corollary which presents linear sum formulas of fifth-order Pell-Lucas[...]

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corollary

Corollary 3.4. *For $n \geq 0$, fifth-[...]er Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k Q_k = \frac{1}{6}((-1)^n(-(3n+8)Q_{n+4}+(9n+21)Q_{n+3}-(6n+4)Q_{n+2}+(9n$ [...] $(3n+11)Q_n)-32)$.

(b) $\sum_{k=0}^{n} k(-1)^k Q_{2k} = \frac{1}{25}((-1)^n((5n-6)Q_{2n+2}+15Q_{2n+1}-(5n-11)Q_{2n}-(10n+13)$ [...] $(15n+12)Q_{2n-1})-74)$.

(c) $\sum_{k=0}^{n} k(-1)^k Q_{2k+1} = \frac{1}{25}((-1)^n((10n+3)Q_{2n+2}+5Q_{2n+1}-(10n+18)Q_{2n}+(5n-6)Q_{2n-2}-(5n+19)Q_{2n-1})+37).$

Taking $r=1, s=1, t=1, u=1, v=2$ in Theorem 2.1 Proposition.

Proposition 3.3. *If $r=1, s=1, t=1, u=1, v=2$ then for $n\geq 0$ we have the following formula*

(a) $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{6}((-1)^n(-(3n+4)W_{n+4}+(6n+5)W_{n+3}-(3n-5)W_{n+2}+($ … $(3n+7)W_n)+4W_4-5W_3-5W_2+4W_1-14W_0).$

(b) $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{25}((-1)^n((5n-14)W_{2n+2}+(5n+11)W_{2n+1}+(5n+36)W_{2n}-($ … $W_{2n-1}-2(10n+7)W_{2n-2})-9W_4+16W_3+16W_2-9W_1-34W_0).$

(c) $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{25}((-1)^n((10n-3)W_{2n+2}+(10n+22)W_{2n+1}-(15n+3)W_{2n}$ … $14)W_{2n-2}-(15n+28)W_{2n-1})+7W_4+7W_3-18W_2-43W_1-18W_0).$

Taking $W_n = J_n$ with $J_0=0, J_1=1, J_2=1, J_3=1, J_4=1$ in the las… Pro… following Corollary which presents linear sum formulas of fifth-order Jacobsthal num…

Corollary 3.5. *For $n\geq 0$, order Jacobsthal numbers have the following prope…*

(a) $\sum_{k=0}^{n} k(-1)^k J_k = \frac{1}{6}((-1)^n(-(3n+4)J_{n+4}+(6n+5)J_{n+3}-(3n-5)J_{n+2}+(6n-4)$ … $(3n+7)J_n)-2).$

(b) $\sum_{k=0}^{n} k(-1)^k J_{2k} = \frac{1}{25}((-1)^n((5n-14)J_{2n+2}+(5n+11)J_{2n+1}+(5n+36)J_{2n}-(20$ … $J_{2n-1}-2(10n+7)J_{2n-2})+14).$

(c) $\sum_{k=0}^{n} k(-1)^k J_{2k+1} = \frac{1}{25}((-1)^n((10n-3)J_{2n+2}+(10n+22)J_{2n+1}-(15n+3)J_{2n}+$ … $14)J_{2n-2}-(15n+28)J_{2n-1})-47).$

From the last Proposition, we have the following Corollary which gives linear … order Jacobsthal-Lucas numbers (take $W_n=j_n$ with $j_0=2, j_1=1, j_2=5, j_3=10, j_4=20$)…

Corollary 3.6. *For $n\geq 0$, order Jacobsthal-Lucas numbers have the followin…*

(a) $\sum_{k=0}^{n} k(-1)^k j_k = \frac{1}{9}((-1)^n(-(3n+4)j_{n+4}+(6n+5)j_{n+3}-(3n-5)j_{n+2}+(6n-4)j$ … $(3n+7)j_n)-19).$

(b) $\sum_{k=0}^{n} k(-1)^k j_{2k} = \frac{1}{25}((-1)^n((5n-14)j_{2n+2}+(5n+11)j_{2n+1}+(5n+36)j_{2n}-(20n$ … $j_{2n-1}-2(10n+7)j_{2n-2})-17).$

(c) $\sum_{k=0}^{n} k(-1)^k j_{2k+1} = \frac{1}{25}((-1)^n((10n-3)j_{2n+2}+(10n+22)j_{2n+1}-(15n+3)j_{2n}+2$ … $14)j_{2n-2}-(15n+28)j_{2n-1})+41).$

Taking $W_n=K_n$ with $K_0=3, K_1=1, K_2=3, K_3=10, K_4=$ 2… in the last… the following corollary which presents linear sum formula of modified fifth order…

Corollary 3.7. *For $n\geq 0$, mod… order Jacobsthal numbers have the following pr…*

(a) $\sum_{k=0}^{n} k(-1)^k K_k = \frac{1}{6}((-1)^n(-(3n+4)K_{n+4}+(6n+5)K_{n+3}-(3n-5)K_{n+2}+(6n$ … $(3n+7)K_n)-23).$

(b) $\sum_{k=0}^{n} k(-1)^k K_{2k} = \frac{1}{25}((-1)^n((5n-14)K_{2n+2}+(5n+11)K_{2n+1}+(5n+36)K_{2n}-($ … $K_{2n-1}-2(10n+7)K_{2n-2})-83).$

(c) $\sum_{k=0}^{n} k(-1)^k K_{2k+1} = \frac{1}{25}((-1)^n((10n-3)K_{2n+2}+(10n+22)K_{2n+1}-(15n+3)K_2$ … $14)K_{2n-2}-(15n+28)K_{2n-1})+59).$

From the last proposition, we have the following corollary which gives linear order Jacobsthal Perrin numbers (take $W_n=Q_n$ with $Q_0=3, Q_1=0, Q_2=2, Q_3=8, Q$…

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Corollary3.8. *Forn≥0,fifth-[illegible]erJacobsthalPerrinnumbershavethefollowingprop*

(a) $\sum_{k=0}^{n} k(-1)^k Q_k = \frac{1}{9}((-1)^n(-(3n+4)Q_{n+4}+(6n+5)Q_{n+3}-(3n-5)Q_{n+2}+(6n$ … $(3n+7)Q_n)-28)$.

(b) $\sum_{k=0}^{n} k(-1)^k Q_{2k} = \frac{1}{25}((-1)^n((5n-14)Q_{2n+2}+(5n+11)Q_{2n+1}+(5n+36)Q_{2n}-(20$ … $Q_{2n-1}-2(10n+7)Q_{2n-2})-86)$.

(c) $\sum_{k=0}^{n} k(-1)^k Q_{2k+1} = \frac{1}{25}((-1)^n((10n-3)Q_{2n+2}+(10n+22)Q_{2n+1}-(15n+3)Q_{2n}+$ … $14)Q_{2n-2}-(15n+28)Q_{2n-1})+78)$.

Taking $Q_n=W_n$ with $S_0=0, S_1=1, S_2=1, S_3=2, S_4=4$ in the propositi[illegible]n, we have … following corollary which presents linear sum formula of adjusted fifth-order Jaco…

Corollary 3.9. *For $n\geq 0$, adjusted fifth-order Jacobsthal numbers have the follow…*

(a) $\sum_{k=0}^{n} k(-1)^k S_k = \frac{1}{9}((-1)^n(-(3n+4)S_{n+4}+(6n+5)S_{n+3}-(3n-5)S_{n+2}+(6n-4)$ … $(3n+7)S_n)+5)$.

(b) $\sum_{k=0}^{n} k(-1)^k S_{2k} = \frac{1}{25}((-1)^n((5n-14)S_{2n+2}+(5n+11)S_{2n+1}+(5n+36)S_{2n}-(20n$ … $S_{2n-1}-2(10n+7)S_{2n-2})+3)$.

(c) $\sum_{k=0}^{n} k(-1)^k S_{2k+1} = \frac{1}{25}((-1)^n((10n-3)S_{2n+2}+(10n+22)S_{2n+1}-(15n+3)S_{2n}$ … $14)S_{2n-2}-(15n+28)S_{2n-1})-19)$.

From the last proposition, we have the following corollary which gives linear … modified fifth-order Jacobsthal-Lucas numbers (take $W_n=R_n$ with $R_0=5, R_1=1, R$…

Corollary3.10. *For $n\geq 0$, mod[illegible] order Jacobsthal-Lucas numbers have the foll…*

(a) $\sum_{k=0}^{n} k(-1)^k R_k = \frac{1}{9}((-1)^n(-(3n+4)R_{n+4}+(6n+5)R_{n+3}-(3n-5)R_{n+2}+(6n-4$ … $(3n+7)R_n)-56)$.

(b) $\sum_{k=0}^{n} k(-1)^k R_{2k} = \frac{1}{25}((-1)^n((5n-14)R_{2n+2}+(5n+11)R_{2n+1}+(5n+36)R_{2n}-(2$ … $R_{2n-1}-2(10n+7)R_{2n-2})-154)$.

(c) $\sum_{k=0}^{n} k(-1)^k R_{2k+1} = \frac{1}{25}((-1)^n((10n-3)R_{2n+2}+(10n+22)R_{2n+1}-(15n+3)R_{2n}$ … $14)R_{2n-2}-(15n+28)R_{2n-1})-33)$.

Taking $r=2, s=3, t=5, u=7, v=11$ in Theore[illegible]. [illegible] ow…

Proposition3.4. *If $r=2, s=3, t=5, u=7, v=11$ then for $n\geq 0$ w[illegible] have the foll…*

(a) $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{63}((-1)^n(-(9n-2)W_{n+4}+(27n-15)W_{n+3}+36W_{n+2}+(45n$ … $(9n+7)W_n)-2W_4+15W_3-36W_2+46W_1-77W_0)$.

(b) $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{5329}((-1)^n(-(219n-184)W_{2n+2}+(1022n-1759)W_{2n+1}+$ … $1827)W_{2n}-(1679n-6034)W_{2n-1}-11(584n-807)W_{2n-2})-35W_4-737W_3+$ … $4355W_1+2453W_0)$.

(c) $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{5329}((-1)^n((584n-1391)W_{2n+2}+(4380n+2379)W_{2n+1}-(2$ … $6954)W_{2n}-(7957n-10165)W_{2n-1}-11(219n-184)W_{2n-2})-807W_4+1430W$ … $2208W_1-385W_0)$.

From the last proposition, we have the following corollary which gives linear 5- sum formulas of primes numbers (take $W_n=G_n$ with $G_0=0, G_1=0, G_2=0, G_3=1, G_4=2$).

Corollary3.11. *For $n\geq 0$, 5-pri[illegible]es numbers have the following properties:*

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the fifth-order

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, in part (a), (b) and (c) of theorem (2.1),

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(a) $\sum_{k=0}^{n} k(-1)^k G_k = \frac{1}{81}((-1)^n(-(9n-2)G_{n+4}+(27n-15)G_{n+3}+36G_{n+2}+(45n-46)G_{n+1}+11(9n+7)G_n)+11).$

(b) $\sum_{k=0}^{n} k(-1)^k G_{2k} = \frac{1}{5329}((-1)^n(-(219n-184)G_{2n+2}+(1022n-1759)G_{2n+1}+(5037n+1827)G_{2n}-(1679n-6034)G_{2n-1}-11(584n-807)G_{2n-2})-807).$

(c) $\sum_{k=0}^{n} k(-1)^k G_{2k+1} = \frac{1}{5329}((-1)^n((584n-1391)G_{2n+2}+(4380n+2379)G_{2n+1}-(2774n-6954)G_{2n}-(7957n-10165)G_{2n-1}-11(219n-184)G_{2n-2})-184).$

Taking $W_n = H_n$ with $H_0 = 5, H_1 = 2, H_2 = 10, H_3 = 41, H_4 =$ 150 in th... have the following corollary which presents linear sum formulas of Lucas 5-prime...

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summation

Corollary 3.12. *For $n \geq 0$,* *Lucas 5-primes numbers have the following propertie...*

(a) $\sum_{k=0}^{n} k(-1)^k H_k = \frac{1}{81}((-1)^n(-(9n-2)H_{n+4}+(27n-15)H_{n+3}+36H_{n+2}+(45n-$ ... $(9n+7)H_n)-338).$

(b) $\sum_{k=0}^{n} k(-1)^k H_{2k} = \frac{1}{5329}((-1)^n(-(219n-184)H_{2n+2}+(1022n-1759)H_{2n+1}+(503$ ... $(1679n-6034)H_{2n-1}-11(584n-807)H_{2n-2})+858).$

(c) $\sum_{k=0}^{n} k(-1)^k H_{2k+1} = \frac{1}{5329}((-1)^n((584n-1391)H_{2n+2}+(4380n+2379)H_{2n+1}-(2$ ... $6954)H_{2n}-(7957n-10165)H_{2n-1}-11(219n-184)H_{2n-2})-18129).$

From the last proposition, we have the following corollary which gives linear ... modified 5-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 1, E_4 = 1$).

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summation

Corollary 3.13. *For $n \geq 0$,* *modified 5-primes numbers have the following proper...*

(a) $\sum_{k=0}^{n} k(-1)^k E_k = \frac{1}{81}((-1)^n(-(9n-2)E_{n+4}+(27n-15)E_{n+3}+36E_{n+2}+(45n-4$ ... $(9n+7)E_n)+13).$

(b) $\sum_{k=0}^{n} k(-1)^k E_{2k} = \frac{1}{5329}((-1)^n(-(219n-184)E_{2n+2}+(1022n-1759)E_{2n+1}+(50$ ... $(1679n-6034)E_{2n-1}-11(584n-807)E_{2n-2})-772).$

(c) $\sum_{k=0}^{n} k(-1)^k E_{2k+1} = \frac{1}{5329}((-1)^n((584n-1391)E_{2n+2}+(4380n+2379)E_{2n+1}-(2$ ... $6954)E_{2n}-(7957n-10165)E_{2n-1}-11(219n-184)E_{2n-2})+623).$

3.3 The case $x = i$

In this subsection we consider the special case $x = i$.

Taking $x = i, r = s = t = u = v = 1$ in Theorem 2.1 (a) ... g...

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Proposition 3.5. *If $r = s = t = u = v = 1$ then for* *we have the following formulas:*

(a) $\sum_{k=0}^{n} ki^k W_k = \frac{1}{8}(i^n(((1-i)n+6-i)W_{n+4}-2(n+(3+2i))W_{n+3}+(5i-10-(1-3i)$ ... $((2+2i)n-4+10i)W_{n+1}+(((1+i)n+2+7i))W_n)-(6-i)W_4+(6+4i)W_3+(10$ ... $(4-10i)W_1-(2+7i)W_0).$

(b) $\sum_{k=0}^{n} ki^k W_{2k} = \frac{1}{18}(i^n(((9+3i)n+6+7i)W_{2n+2}-(12n+16+12i)W_{2n+1}+(8+i-$ ... $2i-8-(6-6i)n)W_{2n-1}+(2-13i-(3+3i)n)W_{2n-2})+(10-15i)W_4-(12-28i)W$ ... $(14+5i)W_2+(8+14i)W_1-(16-i)W_0).$

(c) $\sum_{k=0}^{n} ki^k W_{2k+1} = \frac{1}{18}(i^n(((3i-3)n-10-5i)W_{2n+2}+((6+6i)n+14+8i)W_{2n+1}+($ ... $9i)n-2+9i)W_{2n}+(6n+8-6i)W_{2n-1}+((9+3i)n+6+7i)W_{2n-2})-(2-13i)W_4-(4+20i)W_3+(18-i)W_2-(6+14i)W_1+(10-15i)W_0).$

From the above Proposition, we have the following Corollary which gives linear sum formulas of Pentanacci numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4$).

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summation

Corollary 3.14. *For $n \geq 0$, Pentanacci numbers have the following properties.*

(a) $\sum_{k=0}^{n} k i^k P_k = \frac{1}{2}(i^n(((1-i)n+6-i)P_{n+4}-2(n+(3+2i))P_{n+3}+(5i-10-(1-3i)n$ ... $((2+2i)n-4+10i)P_{n+1}+(((1+i)n+2+7i))P_n)+(2-3i))$.

(b) $\sum_{k=0}^{n} k i^k P_{2k} = \frac{1}{18}(i^n(((9+3i)n+6+7i)P_{2n+2}-(12n+16+12i)P_{2n+1}+(8+i$ ... $(2i-8-(6-6i)n)P_{2n-1}+(2-13i-(3+3i)n)P_{2n-2})+(10+5i))$.

(c) $\sum_{k=0}^{n} k i^k P_{2k+1} = \frac{1}{18}(i^n(((3i-3)n-10-5i)P_{2n+2}+((6+6i)n+14+8i)P_{2n+1}+((3+$ ... $2+9i)P_{2n}+(6n+8-6i)P_{2n-1}+((9+3i)n+6+7i)P_{2n-2})+(-4-3i))$.

Taking $P_n = Q_n$ with $Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 =$ in the ab... have the following Corollary which presents linear sum formulas of Pentanacci-L...

Corollary 3.15. *For $n \geq 0$, Pentanacci-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n} k i^k Q_k = \frac{1}{2}(i^n(((1-i)n+6-i)Q_{n+4}-2(n+(3+2i))Q_{n+3}+(5i-10-(1-3i)$ ... $((2+2i)n-4+10i)Q_{n+1}+(((1+i)n+2+7i))Q_n)+(-24-17i))$.

(b) $\sum_{k=0}^{n} k i^k Q_{2k} = \frac{1}{18}(i^n(((9+3i)n+6+7i)Q_{2n+2}-(12n+16+12i)Q_{2n+1}+(8+i$ ... $2i-8-(6-6i)n)Q_{2n-1}+(2-13i-(3+3i)n)Q_{2n-2})+(-48-25i))$.

(c) $\sum_{k=0}^{n} k i^k Q_{2k+1} = \frac{1}{18}(i^n(((3i-3)n-10-5i)Q_{2n+2}+((6+6i)n+14+8i)Q_{2n+1}+((3+9i)n-$ $2+9i)Q_{2n}+(6n+8-6i)Q_{2n-1}+((9+3i)n+6+7i)Q_{2n-2})+(40-37i))$.

Corresponding sums of the other fifth order generalized Pentanacci numbers sim... rly.

4 Sum Formulas of Generalized Pentanacci Num... ative Subscripts

The following Theorem presents some linear summing formulas generalize... with negative subscripts.

Theorem 4.1. *Let x be a real (or complex) number. For $n \geq 1$ we have the following fo... $v+rx^4+sx^3+tx^2+ux-x^5 = 0$, then*

$$\sum_{k=1}^{n} k x^k W_{-k} = \frac{\Omega_4}{(v+rx^4+sx^3+tx^2+ux-x^5)^2}$$

where

$\Omega_4 = x^{n+1}(n(-v - rx^4 - sx^3 - tx^2 - ux + x^5) - v + 3$ ... $4x^5)W_{-n+4} + x^{n+1}(n(r-x)(v+rx^4+sx^3+tx^2+ux-x^5)+6rx^5+sx^4-ux^2-3r^2x^4+rv-2$ ... $)W_{-n+3} + x^{n+1}(n(s+rx-x^2)(v+rx^4+sx^3+tx^2+ux-x^5)+4rx^6+4sx^5-tx^4-2ux$ ... $+sv-2x^7-4rsx^4+rux^2-stx^2+2rvx)W_{-n+2} + x^{n+1}(n(t+rx^2+sx-x^3)(v+rx^4+$ ... $tx^2+ux-x^5)+2rx^7+2sx^6+2tx^5-3ux^4-4vx^3-r^2x^6-s^2x^4-t^2x^2+tv-x^8-2rsx^5-2rt$ ... $3rvx^2+sux^2+2svx)W_{-n+1} + x^{n+1}(n(u+rx^3+sx^2+tx-x^4)(v+rx^4+sx^3+tx$ ... $uv+4rvx^3+3svx^2+2tvx)W_{-n} + x(v-3rx^4-2sx^3-tx^2+4x^5)W_4 + x(-6rx^5$ ... $rv+2vx+3x^6+2rsx^3+rtx^2)W_3 + x(-4rx^6-4sx^5+tx^4+2ux^3+3vx^2+2r^2x^{..}+2s^{..}x^{..}-sv+2x^{..}+4rsx^4-rux^2+stx^2-2rvx)W_2 + x(-2rx^7-2sx^6-2tx^5+3ux^4+4vx^3+r^2x^6+s^2x^4+t^2x^2-tv+x^8+2rsx^5+2rtx^4-2rux^3+2stx^3-3rvx^2-sux^2-2svx)W_1 + vx(-u-4rx^3-3sx^2-2tx+5x^4)W_0$.

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Similarly, can be calculated.

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Proof. Using the recurrence relation

$$W_{n+5} = rW_{n+4}+sW_{n+3}+tW_{n+2}+uW_{n+1}+vW_n$$
$$\Rightarrow W_{-n+5}=rW_{-n+4}+sW_{-n+3}+tW_{-n+2}+uW_{-n+1}+vW_{-n}$$
$$\Rightarrow W_{-n}=-\frac{u}{v}W_{-n+1}-\frac{t}{v}W_{-n+2}-\frac{s}{v}W_{-n+3}-\frac{r}{v}W_{-n+4}+\frac{1}{v}W_{-n+5}$$
$$\Rightarrow W_{-n}=\frac{1}{v}W_{-n+5}-\frac{u}{v}W_{-n+1}-\frac{t}{v}W_{-n+2}-\frac{s}{v}W_{-n+3}-\frac{r}{v}W_{-n+4}$$

i.e.

$$vW_{-n}=W_{-n+5}-rW_{-n+4}-sW_{-n+3}-tW_{-n+2}-uW_{-n+1}$$

obtain

$$v\times n\times x^nW_{-n} = n\times x^nW_{-n+5}-r\times n\times x^nW_{-n+4} -s\times n\times x^nW_{-n+3}-t\times n\times x^nW_{-n+2}-u\times n\times x$$
$$v(n-1)x^{n-1}W_{-n+1} = (n-1)x^{n-1}W_{-n+6}-r(n-1)x^{n-1}W_{-n+5} -s(n-1)x^{n-1}W_{-n+4}-t(n-1)x^{n-1}W_{-n+3}-u(n-1)$$
$$v(n-2)x^{n-2}W_{-n+2} = (n-2)x^{n-2}W_{-n+7}-r(n-2)x^{n-2}W_{-n+6} -s(n-2)x^{n-2}W_{-n+5}-t(n-2)x^{n-2}W_{-n+4}-u(n-2)$$
$$\vdots$$
$$v\times 3\times x^3W_{-3} = 3\times x^3W_2-r\times 3\times x^3W_1 -s\times 3\times x^3W_0-t\times 3\times x^3W_{-1}-u\times 3\times$$
$$v\times 2\times x^2W_{-2} = 2\times x^2W_3-r\times 2\times x^2W_2 -s\times 2\times x^2W_1-t\times 2\times x^2W_0-u\times 2\times$$
$$v\times 1\times x^1W_{-1} = 1\times x^1W_4-r\times 1\times x^1W_3 -s\times 1\times x^1W_2-t\times 1\times x^1W_1-u\times 1\times x^1W_0.$$

If we add the equations side by side (and using Th 2(a)), we get (a).

5 Specific Cases

In this section, for the specific cases of x, we present the closed form solutions (identitie $\sum_{k=1}^{n} kx^kW_{-k}$, $\sum_{k=1}^{n} kx^kW_{-2k}$ and $\sum_{k=1}^{n} kx^kW_{-2k+1}$ for the specific case of sed

5.1 The case $x=1$

In this subsection, we consider the special case $x=1$.

The case $x=1$ of The given in Soykan [(31)].

5.2 The case $x=-1$

In this subsection we consider the special case $x=-1$.

Taking $r=s=t=u=v=1$ in Theorem 4.1, we obtain the following Propositio

Proposition 5.1. *If $r=s=t=u=v=1$ then for $n\geq$ we have the following formulas:*

$\sum_{k=1}^{n} k(-1)^kW_{-k}=\frac{1}{4}((-1)^n((2n-5)W_{-n+4}-(4n-8)W_{-n+3}+(2n+1)W_{-n+2}-$ $2)W_{-n+1}+(2n+7)W_{-n})+5W_4-8W_3-W_2-2W_1-7W_0)$.

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Special

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theorem

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formulas.

From the above P[...] have the following Coroll[...] gives sum form[...] Pentanacci-Lucas numbers (take $W_n = P_n$ with $P_0 = 0$, $P_1 = 1$, $P_2 = 1$, $P_3 = 2$, [...] take $W_n = Q_n$ with $Q_0 = 5$, $Q_1 = 1$, $Q_2 = 3$, $Q_3 = 7$, $Q_4 = 15$, respectively).

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of the Pentanacci

Corollary 5.1. *For $n \geq 1$, we have the following properties.*

(a) $\sum_{k=1}^{n} k(-1)^k P_{-k} = \frac{1}{4}((-1)^n((2n-5)P_{-n+4} - (4n-8)P_{-n+3} + (2n+1)P_{-n+2} - ($ [...] $(2n+7)P_{-n}) + 1).$

(b) $\sum_{k=1}^{n} k(-1)^k Q_{-k} = \frac{1}{4}((-1)^n((2n-5)Q_{-n+4} - (4n-8)Q_{-n+3} + (2n+1)Q_{-n+2} - (4n-2)Q_{-n+1} + (2n+7)Q_{-n}) - 21).$

Taking $r=2, s=t=u=v=1$ in Theore[...]e obtain the following Proposi[...]

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Proposition 5.2. *If $r=2, s=t=u=v=1$ then for $n \geq 1$ we have the following formu*[...]

$\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{6}((-1)^n((3n-8)W_{-n+4} - (9n-21)W_{-n+3} + (6n-4)W_{-n+2}$ [...] $W_{-n+1} + (6n+11)W_{-n}) + 8W_4 - 21W_3 + 4W_2 - 6W_1 - 11W_0).$

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From the last Proposition, we have the following Corollary which gives sum[...] Pell and fifth-order Pell-Lucas numbers (take $W_n = P_n$ with $P_0 = 0$, $P_1 = 1$, $P_2 = 2$, P_3 [...] and take $W_n = Q_n$ with $Q_0 = 5$, $Q_1 = 2$, $Q_2 = 6$, $Q_3 = 17$, $Q_4 = 46$, respectively).

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Corollary 5.2. *For $n \geq 1$, we have the following properties:*

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(a) $\sum_{k=1}^{n} k(-1)^k P_{-k} = \frac{1}{6}((-1)^n((3n-8)P_{-n+4} - (9n-21)P_{-n+3} + (6n-4)P_{-n+2} - ($ [...] $P_{-n+1} + (6n+11)P_{-n}) + 1).$

(b) $\sum_{k=1}^{n} k(-1)^k Q_{-k} = \frac{1}{6}((-1)^n((3n-8)Q_{-n+4} - (9n-21)Q_{-n+3} + (6n-4)Q_{-n+2} -$ [...] $Q_{-n+1} + (6n+11)Q_{-n}) - 32).$

Taking $r=s=t=1, u=1, v=2$ in Theorem 4.1[...]in the following Proposition.

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Proposition 5.3. *If $r=s=t=1, u=1, v=2$ then for $n \geq 1$ we have the following formulas:*

$\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{9}((-1)^n((3n-4)W_{-n+4} - (6n-5)W_{-n+3} + (3n+5)W_{-n+2} -$ [...] $W_{-n+1} + (3n+14)W_{-n}) + 4W_4 - 5W_3 - 5W_2 + 4W_1 - 14W_0).$

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Taking, respectively,

$W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1$ (fifth-order Jacobsthal numbers),

$W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20$ (fifth order Jacobsthal-Lucas numbe[...]

$W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20$ (modified fifth order Jacobstha[...]

$W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16$ (fifth-order Jacobsthal Perrin[...]

$W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4$ (adjusted fifth-order Jacobsthal numbers),

$W_n = R_n$ with $R_0 = 5, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15$ (modified fifth-order Jacobsthal-Lucas numbers),

[...] the last Proposition, we have the following Corollary.

Corollary 5.3. *For $n \geq 1$, we have the following properties:*

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By using the last proposition, we have the following corollary.

(a) $\sum_{k=1}^{n} k(-1)^k J_{-k} = \frac{1}{9}((-1)^n((3n-4)J_{-n+4} - (6n-5)J_{-n+3} + (3n+5)J_{-n+2} - (6n$ [...] $J_{-n+1} + (3n+14)J_{-n}) - 2).$

(b) $\sum_{k=1}^{n} k(-1)^k j_{-k} = \frac{1}{9}((-1)^n((3n-4)j_{-n+4} - (6n-5)j_{-n+3} + (3n+5)j_{-n+2} - (6n$ [...] $(3n+14)j_{-n}) - 19).$

(c) $\sum_{k=1}^{n} k(-1)^k K_{-k} = \frac{1}{9}((-1)^n((3n-4)K_{-n+4} - (6n-5)K_{-n+3} + (3n+5)K_{-n+2} -$ [...] $K_{-n+1} + (3n+14)K_{-n}) - 23).$

(d) $\sum_{k=1}^{n} k(-1)^k Q_{-k} = \frac{1}{9}((-1)^n((3n-4)Q_{-n+4}-(6n-5)Q_{-n+3}+(3n+5)Q_{-n+2}-(6n+4)Q_{-n+1}+(3n+14)Q_{-n})-28)$.

(e) $\sum_{k=1}^{n} k(-1)^k S_{-k} = \frac{1}{9}((-1)^n((3n-4)S_{-n+4}-(6n-5)S_{-n+3}+(3n+5)S_{-n+2}-(6n+4)S_{-n+1}+(3n+14)S_{-n})+5)$.

(f) $\sum_{k=1}^{n} k(-1)^k R_{-k} = \frac{1}{9}((-1)^n((3n-4)R_{-n+4}-(6n-5)R_{-n+3}+(3n+5)R_{-n+2}-(6n+4)R_{-n+1}+(3n+14)R_{-n})-56)$.

Taking $r=2, s=3, t=5, u=7, v=11$ in Theorem 4.1 nthefollowing Propositi

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Proposition 5.4. *If $r=2, s=3, t=5, u=7, v=$ then for $n\geq 1$ we have the follo*

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$\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{81}((-1)^n((9n+2)W_{-n+4}-(27n+15)W_{-n+3}+36W_{-n+2}-($ $46)W_{-n+1}-(18n-77)W_{-n})-2W_4+15W_3-36W_2+46W_1-77W_0)$.

From the last Proposition, we have the following Corollary which gives sum

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summation formulas of the 5-prime

Lucas 5-primes and modified 5-primes numbers (take $W_n = G_n$ with $G_0 = 0,$ $G_4 = 2$, take $W_n = H_n$ with $H_0 = 5, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 1$ with $E_0=0, E_1=0, E_2=0, E_3=1, E_4=1$, respectively).

Corollary 5.4. *For $n\geq 1$, we have the following properties:*

(a) $\sum_{k=1}^{n} k(-1)^k G_{-k} = \frac{1}{81}((-1)^n((9n+2)G_{-n+4}-(27n+15)G_{-n+3}+36G_{-n+2}-(4$ $(18n-77)G_{-n})+11)$.

(b) $\sum_{k=1}^{n} k(-1)^k H_{-k} = \frac{1}{81}((-1)^n((9n+2)H_{-n+4}-(27n+15)H_{-n+3}+36H_{-n+2}-(45n+46)H_{-n+1}-(18n-77)H_{-n})-338)$.

(c) $\sum_{k=1}^{n} k(-1)^k E_{-k} = \frac{1}{81}((-1)^n((9n+2)E_{-n+4}-(27n+15)E_{-n+3}+36E_{-n+2}-(45n+46)E_{-n+1}-(18n-77)E_{-n})+13)$.

5.3 The case $x=i$

In this subsection, we consider the special case $x=i$.

Taking $r=s=t=u=v=1$ in Theorem 4 btain the following proposition.

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theorem 4.1,

Proposition 5.5. *If $r=s=t=u=v=1$ then for $n\geq 1$ we have the following formula:*

$\sum_{k=1}^{n} k i^k W_{-k} = \frac{1}{2}(i^n(((1+i)n-6-i)W_{-n+4}+(6-2n-4i)W_{-n+3}+(10+5i-(1+$ $((2-2i)n+4+10i)W_{-n+1}+((1+i)n-2+7i)W_{-n})+(6+i)W_4-(6-4i)W_3-(10+5i)W$ $(4+10i)W_1+(2-7i)W_0)$.

From the above P havethefollowing Coroll givessumform

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summation

Pentanacci-Lucas numbers (take $W_n = P_n$ with $P_0=0, P_1=1, P_2=1, P_3=2,$ take $W_n=Q_n$ with $Q_0=5, Q_1=1, Q_2=3, Q_3=7, Q_4=15$, respectively).

Corollary 5.5. *For $n\geq 1$, we have the following properties.*

(a) $\sum_{k=1}^{n} k i^k P_{-k} = \frac{1}{2}(i^n(((1+i)n-6-i)P_{-n+4}+(6-2n-4i)P_{-n+3}+(10+5i-(1+$ $(2-2i)n+4+10i)P_{-n+1}+((1+i)n-2+7i)P_{-n})+(-2-3i))$.

(b) $\sum_{k=1}^{n} k i^k Q_{-k} = \frac{1}{2}(i^n(((1+i)n-6-i)Q_{-n+4}+(6-2n-4i)Q_{-n+3}+(10+5i-(1+3i)n)Q_{-n+2}+((2-2i)n+4+10i)Q_{-n+1}+((1+i)n-2+7i)Q_{-n})+(24-17i))$.

Correspondingsums oftheotherfifthordergeneralizedPentanaccinumbersc imilarly.

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summations

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