Referee's Report for

A Note on Sum Formulas of Generalized Pentanacci Sequence: Closed Forms of the Sum Formulas $\sum_{k=0}^{n} kx^k W_k$ and $\sum_{k=1}^{n} kx^k W_{-k}$.

(... for the *Asian Research Journal of Mathematics*.)

This paper is not suitable for publication.

To explain what this paper is attempting to do, I'll need to go back to the Fibonacci numbers F_n, which have the simple recurrence

$$F_n = F_{n-1} + F_{n-2}$$

and initial values $F_0 = 0$ and $F_1 = 1$. Thanks to the magic of *generating functions*, we can write this power series

$$F_0 + F_1x + F_2x^2 + F_3x^3 + F_4x^4 + \cdots$$

as the much more compact expression

$$\frac{x}{1 - x - x^2}$$

with the understanding that we would expand the above expression as a Taylor series (well, a Maclaurin series to be precise) and we would discover that the coefficients of those terms would be the Fibonacci numbers. In other words,

$$\frac{x}{1 - x - x^2} = F_0 + F_1x + F_2x^2 + F_3x^3 + F_4x^4 + \cdots .$$

This is quite lovely, and quite well-understood. Generating functions have been around for almost 300 years, and there are many good books and videos and webpages about them.

If we wanted a finite series, say up to the F_9x^9 term (which is $34x^9$) we would simply take another series starting with $F_{10}x^{10} = 55x^{10}$. It is an easy problem in linear algebra to discover that the appropriate generating function will be

$$\frac{x^{10}(34x + 55)}{1 - x - x^2} = F_{10}x^{10} + F_{11}x^{11} + F_{12}x^{12} + F_{13}x^{13} + F_{14}x^{14} + \cdots$$

Now, if we subtract these two, we will get

$$\frac{x - x^{10}(34x + 55)}{1 - x - x^2} = F_0 + F_1x + F_2x^2 + F_3x^3 + F_4x^4 + \cdots + F_8x^8 + F_9x^9,$$

and so we have just discovered a closed-form identity for the sum

$$\sum_{k=0}^{9} x^k F_k.$$

What's more, if we take the derivative (with respect to x) of both sides of the identity

$$\sum_{k=0}^{9} x^k F_k = \frac{x - x^{10}(34x+55)}{1-x-x^2}$$

and then multiply both sides by x, we will then obtain a closed-form identity for the sum

$$\sum_{k=1}^{9} kx^k F_k.$$

None of this is new. And in the paper under review, the authors did *exactly* these steps but with the pentanacci numbers W_n, which satisfy

$$W_n = W_{n-1} + W_{n-2} + W_{n-3} + W_{n-4} + W_{n-5}.$$

This gives the authors some *very* complicated formulas for sums like

$$\sum_{k=1}^{9} kx^k W_k,$$

and when I say "complicated" I just mean "with lots of terms". And when I say "lots of terms", this is what I mean:

(c) *If* $r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1\neq 0$ *then*

$$\sum_{k=0}^{n} kx^k W_{2k+1} = \frac{\Omega_3}{(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)^2}$$

where

$$\Omega_3 = x^{n+1}(n(r+vx^2+tx)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)-3vx^2-t^3x^4-2v^3x^7-2tx-r-2rux^2+2stx^2+4svx^3+2uvx^4+rs^2x^2-r^2tx^2-2rt^2x^3+3ru^2x^4-2r^2vx^3-4rv^2x^5-s^2vx^4+2tu^2x^5-4t^2vx^5-5tv^2x^6+u^2vx^6+4rsux^3-6rtvx^4+2stux^4)W_{2n+2}+x^{n+1}(n(s-s^2x+t^2x^2-u^2x^3+v^2x^4+ux+rvx^2-2sux^2+2tvx^3+rtx)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+2s^2x-s-s^3x^2-3t^2x^2+4u^2x^3-2u^3x^5-5v^2x^4-r^2s^2x^2+2r^2t^2x^3-3r^2u^2x^4+4r^2v^2x^5-3rvx^2+6sux^2-8tvx^3+r^3tx^2+r^2ux^2+rt^3x^4+2st^2x^3-4s^2ux^3-5su^2x^4+2r^3vx^3+t^2ux^4+4sv^2x^5+2rv^3x^7+3uv^2x^6+6stvx^4+4tuvx^5+rs^2vx^4-2rtu^2x^5+6r^2tvx^4+4rt^2vx^5+5rtv^2x^6-ru^2vx^6-4r^2sux^3-2rstux^4-2rtx-2ux)W_{2n+1}+x^{n+1}(n(t+vx-svx^2+rux-stx)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+5svx^2-2t^3x^3-3v^3x^6-2vx-t-2tux^2-2rt^2x^2-s^2tx^2+r^3ux^2+r^2vx^2+st^3x^4+2ru^3x^5-2rv^2x^4-4s^2vx^3+3tu^2x^4+s^3vx^4-7t^2vx^4-8tv^2x^5+2sv^3x^7+2u^2vx^5-2rux+2stx+2rsux^2-4rtvx^3+4stux^3-r^2stx^2-2r^2svx^3-rt^2ux^4-2s^2tux^4-2stu^2x^5-2r^2uvx^4+4st^2vx^5-3ruv^2x^6+5stv^2x^6-su^2vx^6+2rsu^2x^4-4rtuvx^5)W_{2n}+x^{n+1}(n(u-u^2x^2+v^2x^3+tvx^2+rvx-sux)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+u^2x^2-u+u^3x^4-4v^2x^3-u^4x^6-v^4x^8-2r^2u^2x^3+r^2v^2x^4-s^2u^2x^4-2s^2v^2x^5-t^2v^2x^6+2u^2v^2x^7-3tvx^2-s^2ux^2+r^3vx^2-2t^2ux^3-2su^3x^5+6sv^2x^4-rv^3x^6-2tv^3x^7-2rvx+2sux+2rsvx^2-2rtux^2-4ruvx^3+4stvx^3-4tuvx^4-r^2sux^2-2rtu^2x^4+2r^2tvx^3+rt^2vx^4+st^2ux^4-s^2tvx^4+2ru^2vx^5+suv^2x^6+3tu^2vx^6+4rsuvx^4+4stuvx^5)W_{2n-1}-vx^{n+1}(n(ux^2+sx-1)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+1+s^2x^2+2t^2x^3-ux^2-u^2x^4+u^3x^6+4v^2x^5-2sx+2rtx^2+4rvx^3+6tvx^4+r^2sx^2+2r^2ux^3-st^2x^4+s^2ux^4+2su^2x^5-3sv^2x^6-2uv^2x^7-2rsvx^4+2rtux^4-4stvx^5-2tuvx^6)W_{2n-2}+x^2(2r-r^3x+4vx^2+v^3x^7+3tx-4stx^2-6svx^3-2tux^3-4uvx^4-2r^2tx^2-rt^2x^3+s^2tx^3-2ru^2x^4-r^2vx^3+rv^2x^5+2s^2vx^4-tu^2x^5+t^2vx^5+2tv^2x^6-2rsx-2rsux^3+2suvx^5)$$

$$
\begin{aligned}
&W_4 + x(r^4x^2 + 2r^3tx^3 + r^3vx^4 + 2r^2sux^4 + 3r^2sx^2 + r^2t^2x^4 + 2r^2u^2x^5 + 2r^2ux^3 - r^2v^2x^6 - \\
&2r^2x - rs^2tx^4 - 2rs^2vx^5 + 4rstx^3 - 2rsuvx^6 + 4rsvx^4 - rt^2vx^6 + rtu^2x^6 + 4rtux^4 - 2rtv^2x^7 - \\
&rtx^2 + 4ruvx^5 - rv^3x^8 + s^2ux^4 + s^2x^2 - st^2x^4 - 4stvx^5 + 2su^2x^5 - 3sv^2x^6 - 2sx + 2t^2x^3 - 2tuvx^6 + \\
&6tvx^4 + u^3x^6 - u^2x^4 - 2uv^2x^7 - ux^2 + 4v^2x^5 + 1)W_3 + x^2(-2r^3ux^2 + 2r^2stx^2 + 3r^2svx^3 - 2r^2tux^3 - \\
&r^2tx - 2r^2vx^2 + rs^2ux^3 + 2rst^2x^3 + 4rstvx^4 - 4rsux^2 + 2rsv^2x^5 + 2rtuvx^5 - ru^3x^5 - 2ru^2x^3 + \\
&2ruv^2x^6 + 3rux - s^3tx^3 - 2s^3vx^4 + 4s^2tx^2 - 2s^2uvx^5 + 7s^2vx^3 - st^2vx^5 + stu^2x^5 - 2stv^2x^6 - \\
&5stx + 4suvx^4 - sv^3x^7 - 8svx^2 + t^3x^3 + 4t^2vx^4 - 2tu^2x^4 + 5tv^2x^5 + 2t - u^2vx^5 - 2uvx^3 + 2v^3x^6 + \\
&3vx)W_2 + x^2(-2r^3vx^2 + 2r^2sux^2 - 5r^2tvx^3 + 3r^2u^2x^3 - r^2ux - 4r^2v^2x^4 + rs^2vx^3 + 2rstux^3 - \\
&4rsvx^2 - 4rt^2vx^4 + 4rtu^2x^4 - 6rtv^2x^5 + ru^2vx^5 - 2rv^3x^6 + 3rvx - s^3ux^3 + 2s^2tvx^4 - 2 \\
&s^2u^2x^4 + 4s^2ux^2 + 3s^2v^2x^5 - 6stvx^3 - su^3x^5 + 6su^2x^3 + 2suv^2x^6 - 5sux - 8sv^2x^4 - t^3 \\
&vx^5 + t^2u^2x^5 + t^2ux^3 - 2t^2v^2x^6 - tv^3x^7 + 4tvx^2 + 2u^3x^4 - 4u^2x^2 - 3uv^2x^5 + 2u + 5v^2x^3)W_1 + \\
&vx^2(-r^2x - 4ux^2 + 4s^2x^2 - s^3x^3 + t^2x^3 + 2u^2x^4 + 3v^2x^5 - 5sx + 2rvx^3 + 6sux^3 + 4tvx^4 + 2r^2sx^2 + \\
&3r^2ux^3 - 2s^2ux^4 - su^2x^5 + t^2ux^5 - 2sv^2x^6 - uv^2x^7 + 2rstx^3 + 4rtux^4 + 2ruvx^5 - 2stvx^5 + 2)W_0.
\end{aligned}
$$

I can say with 100% confidence that *nobody will ever read or use this formula.* I'm certainly not going to read it, nor am I going to verify whether or not it's true.

Let me emphasize this point again: *nobody will ever read this formula.* It has thirty-six lines of text in the definition of Ω_3, and each line has around 10 to 12 terms, so that's close to 400 terms. Nobody will ever read or type in or use an equation with 400 terms.

Furthermore, even if it (this formula) was true, it would not be interesting because first of all it's just way too long and second of all it's just a result of some easy applications of linear algebra as seen above.

There are some interesting formulas that come from the pentanacci numbers. Unfortunately, they are not here in this paper.