Original Research Article

A Note on Sum Formulas of Generalized Pentanacci Sequence: Closed Forms of the Sum Formulas $\sum_{k=0}^{n} kx^k W_k$ and $\sum_{k=1}^{n} kx^k W_{-k}$

Abstract

In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} kx^k W_k$ and $\sum_{k=1}^{n} kx^k W_{-k}$ for generalized Pentanacci numbers are presented. As special cases, we give summation formulas of Pentanacci, Pentanacci-Lucas, and other fifth-order recurrence sequences.

Keywords: Pentanacci numbers; Pentanacci-Lucas numbers; sum formulas; summing formulas

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1 Introduction

The generalized Pentanacci sequence $\{W_n(W_0, W_1, W_2, W_3, W_4; r, s, t, u, v)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$\begin{array}{rcl} W_n & = & rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}, \\ W_0 & = & c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, W_4 = c_4, n \geq 5 \end{array} \tag{1.1}$$

where W_0, W_1, W_2, W_3, W_4 are arbitrary real or complex numbers and r, s, t, u, v are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{u}{v}W_{-n+1} - \frac{t}{v}W_{-n+2} - \frac{s}{v}W_{-n+3} - \frac{r}{v}W_{-n+4} + \frac{1}{v}W_{-n+5}$$

for $n = 1, 2, 3, ...$ when $v \neq 0$. Therefore, recurrence (1.1) holds for all integer n. Pentanacci sequence has been studied by many authors, see for example [(8)], [(9)], [(11)], [(26)].

Table 1: A few special case of generalized Pentanacci sequences.

No	Sequences (Numbers)	Notation	Ref
1	Generalized Pentanacci	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 1, 1, 1, 1, 1)\}$	[(26)]
2	Generalized Fifth order Pell	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 2, 1, 1, 1, 1)\}$	[(27)]
3	Generalized Fifth order Jacobsthal	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 1, 1, 1, 1, 2)\}$	[(28)]
4	Generalized 5-primes	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 2, 3, 5, 7, 11)\}$	[(29)]

For some specific values of W_0, W_1, W_2, W_3, W_4 and r, s, t, u, v it is worth presenting these special Pentanacci numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 2) are used for the special cases of r, s, t, u, v and initial values.

Table 2: A few members of generalized Pentanacci sequences.

Sequences (Numbers)	Notation	OEIS [(12)]	Ref
Pentanacci	$\{P_n\} = \{W_n(0, 1, 1, 2, 4; 1, 1, 1, 1, 1)\}$	A001591	[(26)]
Pentanacci-Lucas	$\{Q_n\} = \{W_n(5, 1, 3, 7, 15; 1, 1, 1, 1, 1)\}$	A074048	[(26)]
fifth order Pell	$\{P_n^{(5)}\} = \{W_n(0, 1, 2, 5, 13; 2, 1, 1, 1, 1)\}$	A141448	[(27)]
fifth order Pell-Lucas	$\{Q_n^{(5)}\} = \{W_n(5, 2, 6, 17, 46; 2, 1, 1, 1, 1)\}$		[(27)]
modified fifth-order Pell	$\{E_n^{(5)}\} = \{W_n(0, 1, 1, 3, 8; 2, 1, 1, 1, 1)\}$		[(27)]
fifth order Jacobsthal	$\{J_n^{(5)}\} = \{W_n(0, 1, 1, 1, 1; 1, 1, 1, 1, 2)\}$	A226310	[(28),(2)]
fifth order Jacobsthal-Lucas	$\{j_n^{(5)}\} = \{W_n(2, 1, 5, 10, 20; 1, 1, 1, 1, 2)\}$	A226311	[(28),(2)]
modified fifth order Jacobsthal	$\{K_n^{(5)}\} = \{W_n(3, 1, 3, 10, 20; 1, 1, 1, 1, 2)\}$		[(28)]
fifth-order Jacobsthal Perrin	$\{Q_n^{(5)}\} = \{W_n(3, 0, 2, 8, 16; 1, 1, 1, 1, 2)\}$		[(28)]
adjusted fifth-order Jacobsthal	$\{S_n^{(5)}\} = \{W_n(0, 1, 1, 2, 4; 1, 1, 1, 1, 2)\}$		[(28)]
modified fifth-order Jacobsthal-Lucas	$\{R_n^{(5)}\} = \{W_n(5, 1, 3, 7, 15; 1, 1, 1, 1, 2)\}$		[(28)]
5-primes	$\{G_n\} = \{W_n(0, 0, 0, 1, 2; 2, 3, 5, 7, 11)\}$		[(29)]
Lucas 5-primes	$\{H_n\} = \{W_n(5, 2, 10, 41, 150; 2, 3, 5, 7, 11)\}$		[(29)]
modified 5-primes	$\{E_n\} = \{W_n(0, 0, 0, 1, 1; 2, 3, 5, 7, 11)\}$		[(29)]

For easy writing, from now on, we drop the superscripts from the sequences, for example we write P_n for $P_n^{(5)}$.

We present some works on summing formulas of the numbers in the following Table 3.

Table 3: A few special study of sum formulas.

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[(1),(4),(32),(6),(7)]
Generalized Fibonacci	[(5),(13),(14),(15),(16),(17),(19)]
Generalized Tribonacci	[(3),(10),(18)]
Generalized Tetranacci	[(20),(25),(33)]
Generalized Pentanacci	[(21),(22)]
Generalized Hexanacci	[(23),(24)]

The following Theorem presents some linear summing formulas of generalized Pentanacci numbers with positive subscripts.

Theorem 1.1. *Let x be a real (or complex) number. For $n \geq 0$ we have the following formulas:*

(a) *If $rx + sx^2 + tx^3 + ux^4 + vx^5 - 1 \neq 0$ then*

$$\begin{aligned} \sum_{k=0}^{n} x^k W_k &= \frac{\Theta_1(x)}{rx + sx^2 + tx^3 + ux^4 + vx^5 - 1} \\ &= \frac{\Theta_1(x)}{\Theta(x)} \end{aligned}$$

where

$\Theta_1(x) = x^{n+4}W_{n+4} - (rx-1)x^{n+3}W_{n+3} - (sx^2 + rx - 1)x^{n+2}W_{n+2} - (sx^2 + tx^3 + rx -$

$1)x^{n+1}W_{n+1}+vx^{n+5}W_n-x^4W_4+x^3(rx-1)W_3+x^2(sx^2+rx-1)W_2+x(sx^2+tx^3+rx-1)W_1+(sx^2+tx^3+ux^4+rx-1)W_0.$

(b) *If* $r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1\neq 0$ *then*

$$\sum_{k=0}^{n}x^kW_{2k}$$
$$=\frac{\Theta_2(x)}{r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1}$$

where

$\Theta_2(x)=-(ux^2+sx-1)x^{n+1}W_{2n+2}+(t+rs+vx+rux)x^{n+2}W_{2n+1}+(u+t^2x-u^2x^2+v^2x^3+rt+2tvx^2+rvx-sux)x^{n+2}W_{2n}+(v+ru-svx+tux)x^{n+2}W_{2n-1}+v(r+vx^2+tx)x^{n+2}W_{2n-2}+x^2(ux^2+sx-1)W_4-x^3(t+rs+vx+rux)W_3+x(r^2x+ux^2-s^2x^2+2sx+rtx^2+rvx^3-sux^3-1)W_2-x^3(v+ru-svx+tux)W_1+(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+2sx+2rtx^2+rvx^3-2sux^3+tvx^4-1)W_0.$

(c) *If* $r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1=0$ *then*

$$\sum_{k=0}^{n}x^kW_{2k+1}$$
$$=\frac{\Theta_3(x)}{r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1}$$

where

$\Theta_3(x)=(r+vx^2+tx)x^{n+1}W_{2n+2}+(s-s^2x+t^2x^2-u^2x^3+v^2x^4+ux+rvx^2-2sux^2+2tvx^3+rtx)x^{n+1}W_{2n+1}+(t+vx-svx^2+rux-stx)x^{n+1}W_{2n}+(u-u^2x^2+v^2x^3+tvx^2+rvx-sux)x^{n+1}W_{2n-1}-v\left(ux^2+sx-1\right)x^{n+1}W_{2n-2}-x^2(r+vx^2+tx)W_4+x(r^2x+ux^2+sx+rtx^2+rvx^3-1)W_3-x^2(t+vx-svx^2+rux-stx)W_2+(r^2x+ux^2-s^2x^2+t^2x^3+2sx+2rtx^2+rvx^3-sux^3+tvx^4-1)W_1+vx^2(ux^2+sx-1)W_0$

Proof. It is given in Soykan [(30), Theorem 2.1]. □

The following Theorem presents some linear summing formulas of generalized Pentanacci numbers with negative subscripts.

Theorem 1.2. *For* $n\geq 1$ *we have the following formulas:If* $v+rx^4+sx^3+tx^2+ux-x^5\neq 0,$ *then*

$$\sum_{k=1}^{n}x^kW_{-k}=\frac{\Theta_4(x)}{v+rx^4+sx^3+tx^2+ux-x^5}.$$

$\Theta_4(x)=-x^{n+1}W_{4-n}+(r-x)x^{n+1}W_{-n+3}+(s+rx-x^2)x^{n+1}W_{-n+2}+(t+rx^2+sx-x^3)x^{n+1}W_{-n+1}+(u+rx^3+sx^2+tx-x^4)x^{n+1}W_{-n}+xW_4-x(r-x)W_3+x(-s-rx+x^2)W_2+x(-t-rx^2-sx+x^3)W_1+x(-u-rx^3-sx^2-tx+x^4)W_0.$

Proof. It is given in Soykan [(30), Theorem 4.1]. □

In this work, we investigate summation formulas of generalized Pentanacci numbers.

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2 Sum Formulas of Generalized Pentanacci Numbers with Positive Subscripts

The following Theorem presents some summing formulas of generalized Pentanacci numbers with positive subscripts.

Theorem 2.1. *Let x be a real (or complex) number. For $n \geq 0$ we have the following formulas:*

(a) *If $vx^5 + ux^4 + tx^3 + sx^2 + rx - 1 \neq 0$ then*

$$\sum_{k=0}^{n} kx^k W_k = \frac{\Omega_1}{(vx^5 + ux^4 + tx^3 + sx^2 + rx - 1)^2}$$

where

$\Omega_1 = x^{n+4}(n(sx^2+tx^3+ux^4+vx^5+rx-1)-4+2sx^2+tx^3-vx^5+3rx)W_{n+4}-x^{n+3}(n(rx-1)(sx^2+tx^3+ux^4+vx^5+rx-1)+3-sx^2+ux^4+2vx^5+3r^2x^2-6rx+2rsx^3+rtx^4-rvx^6)W_{n+3}-x^{n+2}(n(sx^2+rx-1)(sx^2+tx^3+ux^4+vx^5+rx-1)+2-4sx^2+tx^3+2ux^4+3vx^5+2r^2x^2+2s^2x^4-4rx+4rsx^3-rux^5+stx^5-2rvx^6-svx^7)W_{n+2}-x^{n+1}(n(sx^2+tx^3+rx-1)(sx^2+tx^3+ux^4+vx^5+rx-1)+1+2rsx^3+2rtx^4-2rux^5+2stx^5-3rvx^6-sux^6-2svx^7-tvx^8-2sx^2-2tx^3+3ux^4+4vx^5+r^2x^2+s^2x^4+t^2x^6-2rx)W_{n+1}+v x^{n+5}(n(sx^2+tx^3+ux^4+vx^5+rx-1)+3sx^2+2tx^3+ux^4+4rx-5)W_n-x^4(2sx^2+tx^3-vx^5+3rx-4)W_4+x^3(-sx^2+ux^4+2vx^5+3r^2x^2-6rx+2rsx^3+rtx^4-rvx^6+3)W_3+x^2(-4sx^2+tx^3+2ux^4+3vx^5+2r^2x^2+2s^2x^4-4rx+4rsx^3-rux^5+stx^5-2rvx^6-svx^7+2)W_2+x(-2sx^2-2tx^3+3ux^4+4vx^5+r^2x^2+s^2x^4+t^2x^6-2rx+2rsx^3+2rtx^4-2rux^5+2stx^5-3rvx^6-sux^6-2svx^7-tvx^8+1)W_1-vx^5(3sx^2+2tx^3+ux^4+4rx-5)W_0.$

(b) *If $r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1 \neq 0$ then*

$$\sum_{k=0}^{n} kx^k W_{2k}$$
$$= \frac{\Omega_2}{(r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1)^2}$$

where

$\Omega_2 = -x^{n+1}(n(ux^2+sx-1)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+1+s^2x^2+2t^2x^3-u^2x^4+u^3x^6+4v^2x^5-2sx+2rtx^2+4rvx^3+6tvx^4+2r^2ux^3-st^2x^4+s^2ux^4+2su^2x^5-3sv^2x^6-2uv^2x^7-2rsvx^4+2rtux^4-4stvx^5-2tuvx^6+r^2sx^2-ux^2)W_{2n+2}+x^{n+2}(n(t+rs+vx+rux)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+2rs^2x-t^3x^3-2v^3x^6-2rs-3vx-2t+r^3sx+r^2tx+4svx^2+2uvx^3+2ru^2x^3+2r^3ux^2+2r^2vx^2+ru^3x^5-s^2vx^3+2tu^2x^4-4t^2vx^4-5tv^2x^5+u^2vx^5+2stux^3-rst^2x^3+rs^2ux^3+2rsu^2x^4-2r^2svx^3+2r^2tux^3-3rsv^2x^5-2ruv^2x^6-3rux+2stx+4rsu x^2-4rstvx^4-2rtuvx^5)W_{2n+1}+x^{n+2}(n(u+t^2x-u^2x^2+v^2x^3+rt+2tvx^2+rvx-sux)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+4u^2x^2-3t^2x-2u-2u^3x^4-5v^2x^3-2rt+2r^2t^2x^2-3r^2u^2x^3-s^2t^2x^3+4r^2v^2x^4+2s^2u^2x^4-3s^2v^2x^5+r^3tx+r^2ux-8tvx^2+rt^3x^3+4st^2x^2-4s^2ux^2-6su^2x^3+2r^3vx^2+s^3ux^3+t^2ux^3+s u^3x^5+8sv^2x^4+2rv^3x^6+3uv^2x^5+4rsvx^2+12stvx^3-2r^2sux^2-rs^2vx^3-2rtu^2x^4+6r^2tvx^3+4rt^2vx^4+5rtv^2x^5-4s^2tvx^4-ru^2vx^5-2suv^2x^6+2rstx-2stuvx^5-3rvx+5sux+4tuvx^4)W_{2n}+x^{n+2}(n(v+ru-svx+tux)(r^2x+2ux^2-s^2x^2+t^2x^3-u^2x^4+v^2x^5+2sx+2rtx^2+2rvx^3-2sux^3+2tvx^4-1)+r^3ux-3v^3x^5-2ru-2v+r^2vx+2ru^3x^4-2rv^2x^3-4s^2vx^2+2tu^2x^3+s^3v x^3-t^2vx^3+tu^3x^5-4tv^2x^4+2sv^3x^6+2u^2vx^4+4stux^2+2rsu^2x^3-2r^2svx^2+2r^2tux^2+rt^2ux^3-s^2tux^3-2r^2uvx^3-3ruv^2x^5+2stv^2x^5-su^2vx^5-2t^2uvx^5-2tuv^2x^6-2rstvx^3-$

$$4rtuvx^4 + 5svx - 3tux + 2rsux)W_{2n-1} + vx^{n+2}(n(r + vx^2 + tx)(r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1) + r^3x - 2r - 4vx^2 - v^3x^7 - 3tx + 4stx^2 + 6svx^3 + 2tux^3 + 4uvx^4 + 2r^2tx^2 + rt^2x^3 - s^2tx^3 + 2ru^2x^4 + r^2vx^3 - rv^2x^5 - 2s^2vx^4 + tu^2x^5 - t^2v x^5 - 2tv^2x^6 + 2rsx + 2rsux^3 - 2suvx^5)W_{2n-2} + x^2(-r^2x - 4ux^2 + 4s^2x^2 - s^3x^3 + t^2x^3 + 2u^2x^4 + 3v^2x^5 - 5sx + 2rvx^3 + 6sux^3 + 4tvx^4 + 2r^2sx^2 + 3r^2ux^3 - 2s^2ux^4 - su^2x^5 + t^2ux^5 - 2sv^2x^6 - uv^2x^7 + 2rstx^3 + 4rtux^4 + 2ruvx^5 - 2stvx^5 + 2)W_4 + x^3(3t + v^3x^6 + 3rs + 4vx - 4rs^2x - 2r^3sx - 2r^2tx - 6svx^2 - 2tux^2 - 4uvx^3 + rs^3x^2 - 2rt^2x^2 + s^2tx^2 - 4ru^2x^3 - 3r^3ux^2 - 3r^2vx^2 - 2rv^2x^4 + 2s^2vx^3 - tu^2x^4 + t^2vx^4 + 2tv^2x^5 + 4rux - 4stx - 8rsux^2 - 4rtvx^3 + 2suvx^4 - 2r^2stx^2 + 2rs^2ux^3 + rsu^2x^4 - 4r^2tux^3 - rt^2ux^4 + 2rsv^2x^5 - 2r^2uvx^4 + ruv^2x^6 + 2rstvx^4) W_3 + x(r^4x^2 + 2r^3tx^3 + r^3vx^4 - 2r^2s^2x^3 - r^2sux^4 + 4r^2sx^2 + r^2t^2x^4 + 2r^2u^2x^5 + 2r^2ux^3 - r^2v^2x^6 - 2r^2x - 3rs^2tx^4 - 2rs^2vx^5 - 4rstux^5 + 4rstx^3 - 4rsuvx^6 + 2rsvx^4 - rt^2vx^6 + rtu^2x^6 + 4rtux^4 - 2rtv^2x^7 - rtx^2 + 4ruvx^5 - rv^3x^8 + s^4x^4 + 2s^3ux^5 - 4s^3x^3 + 2s^2tvx^6 + s^2u^2x^6 - 5s^2ux^4 + 2s^2v^2x^7 + 6s^2x^2 - st^2ux^6 - 2st^2x^4 - 8stvx^5 + suv^2x^8 + 4sux^3 - 6sv^2x^6 - 4sx + 2t^2x^3 - 2tuvx^6 + 6 tvx^4 + u^3x^6 - u^2x^4 - 2uv^2x^7 - ux^2 + 4v^2x^5 + 1)W_2 + x^3(3v + 2v^3x^5 + 3ru - 2r^3ux - 2r^2vx - 2uvx^2 - 2ru^2x^2 - ru^3x^4 + 7s^2vx^2 - 4tu^2x^3 - 2s^3vx^3 - t^3ux^4 + 2tv^2x^4 - sv^3x^6 - u^2vx^4 - 8svx + 4tux - 2rtvx^2 - 6stux^2 + 4suvx^3 + rs^2ux^2 + 3r^2svx^2 - 5r^2tux^2 - 4rt^2ux^3 + 2rsv^2x^4 + 2s^2tux^3 + 2stu^2x^4 + st^2vx^4 + 2ruv^2x^5 - 2s^2uvx^4 + tuv^2x^6 - 4rsux + 4rstvx^3)W_1 + vx^3(3r - 2r^3x + 5vx^2 - t^3x^4 + 4tx - 2rux^2 - 6stx^2 - 8svx^3 - 4tux^3 - 6uvx^4 + rs^2x^2 - 5r^2tx^2 - 4rt^2x^3 + 2s^2tx^3 - ru^2x^4 - 4r^2vx^3 - 2rv^2x^5 + 3s^2vx^4 - 2t^2vx^5 - tv^2x^6 + u^2vx^6 - 4rsx - 6rtvx^4 + 2stux^4 + 4suvx^5)W_0.$$

(c) *If* $r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1 \neq 0$ *then*

$$\sum_{k=0}^{n} kx^k W_{2k+1}$$
$$= \frac{\Omega_3}{(r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1)^2}$$

where

$$\Omega_3 = x^{n+1}(n(r + vx^2 + tx)(r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1) - 3vx^2 - t^3x^4 - 2v^3x^7 - 2tx - r - 2rux^2 + 2stx^2 + 4svx^3 + 2uvx^4 + rs^2x^2 - r^2tx^2 - 2rt^2x^3 + 3ru^2x^4 - 2r^2vx^3 - 4rv^2x^5 - s^2vx^4 + 2tu^2x^5 - 4t^2vx^5 - 5tv^2x^6 + u^2vx^6 + 4rsux^3 - 6rtvx^4 + 2stux^4)W_{2n+2} + x^{n+1}(n(s - s^2x + t^2x^2 - u^2x^3 + v^2x^4 + ux + rvx^2 - 2sux^2 + 2tvx^3 + rtx) (r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1) + 2s^2x - s - s^3x^2 - 3t^2x^2 + 4u^2x^3 - 2u^3x^5 - 5v^2x^4 - r^2s^2x^2 + 2r^2t^2x^3 - 3r^2u^2x^4 + 4r^2v^2x^5 - 3rvx^2 + 6sux^2 - 8tvx^3 + r^3tx^2 + r^2ux^2 + rt^3x^4 + 2st^2x^3 - 4s^2ux^3 - 5su^2x^4 + 2r^3vx^3 + t^2ux^4 + 4sv^2x^5 + 2rv^3 x^7 + 3uv^2x^6 + 6stvx^4 + 4tuvx^5 + rs^2vx^4 - 2rtu^2x^5 + 6r^2tvx^4 + 4rt^2vx^5 + 5rtv^2x^6 - ru^2vx^6 - 4r^2 sux^3 - 2rstux^4 - 2rtx - 2ux)W_{2n+1} + x^{n+1}(n(t + vx - svx^2 + rux - stx)(r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1) + 5svx^2 - 2t^3x^3 - 3v^3x^6 - 2vx - t - 2tu x^2 - 2rt^2x^2 - s^2tx^2 + r^3ux^2 + r^2vx^2 + st^3x^4 + 2ru^3x^5 - 2rv^2x^4 - 4s^2vx^3 + 3tu^2x^4 + s^3vx^4 - 7t^2vx^4 - 8tv^2x^5 + 2sv^3x^7 + 2u^2vx^5 - 2rux + 2stx + 2rsux^2 - 4rtvx^3 + 4stux^3 - r^2stx^2 - 2r^2svx^3 - rt^2 ux^4 - 2s^2tux^4 - 2stu^2x^5 - 2r^2uvx^4 + 4st^2vx^5 - 3ruv^2x^6 + 5stv^2x^6 - su^2vx^6 + 2rsu^2x^4 - 4rtuvx^5)W_{2n} + x^{n+1}(n(u - u^2x^2 + v^2x^3 + tvx^2 + rvx - sux)(r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1) + u^2x^2 - u + u^3x^4 - 4v^2x^3 - u^4x^6 - v^4x^8 - 2r^2u^2x^3 + r^2v^2x^4 - s^2u^2x^4 - 2s^2v^2x^5 - t^2v^2x^6 + 2u^2v^2x^7 - 3tvx^2 - s^2ux^2 + r^3vx^2 - 2t^2ux^3 - 2su^3x^5 + 6sv^2x^4 - rv^3x^6 - 2tv^3x^7 - 2rvx + 2sux + 2rsvx^2 - 2rtux^2 - 4ruvx^3 + 4stvx^3 - 4tuvx^4 - r^2 sux^2 - 2rtu^2x^4 + 2r^2tvx^3 + rt^2vx^4 + st^2ux^4 - s^2tvx^4 + 2ru^2vx^5 + suv^2x^6 + 3tu^2vx^6 + 4rsuvx^4 + 4stuvx^5)W_{2n-1} - vx^{n+1}(n(ux^2 + sx - 1)(r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1) + 1 + s^2x^2 + 2t^2x^3 - ux^2 - u^2x^4 + u^3x^6 + 4v^2x^5 - 2sx + 2rtx^2 + 4rvx^3 + 6 tvx^4 + r^2sx^2 + 2r^2ux^3 - st^2x^4 + s^2ux^4 + 2su^2x^5 - 3sv^2x^6 - 2uv^2x^7 - 2rsvx^4 + 2rtux^4 - 4stvx^5 - 2tuvx^6)W_{2n-2} + x^2(2r - r^3x + 4vx^2 + v^3x^7 + 3tx - 4stx^2 - 6svx^3 - 2tux^3 - 4uvx^4 - 2r^2tx^2 - rt^2x^3 + s^2tx^3 - 2ru^2x^4 - r^2vx^3 + rv^2x^5 + 2s^2vx^4 - tu^2x^5 + t^2vx^5 + 2tv^2x^6 - 2rsx - 2rsux^3 + 2suvx^5)$$

$$W_4 + x(r^4x^2 + 2r^3tx^3 + r^3vx^4 + 2r^2sux^4 + 3r^2sx^2 + r^2t^2x^4 + 2r^2u^2x^5 + 2r^2ux^3 - r^2v^2x^6 - 2r^2x - rs^2tx^4 - 2rs^2vx^5 + 4rstx^3 - 2rsuvx^6 + 4rsvx^4 - rt^2vx^6 + rtu^2x^6 + 4rtux^4 - 2rtv^2x^7 - rtx^2 + 4ruvx^5 - rv^3x^8 + s^2ux^4 + s^2x^2 - st^2x^4 - 4stvx^5 + 2su^2x^5 - 3sv^2x^6 - 2sx + 2t^2x^3 - 2tuvx^6 + 6tvx^4 + u^3x^6 - u^2x^4 - 2uv^2x^7 - ux^2 + 4v^2x^5 + 1)W_3 + x^2(-2r^3ux^2 + 2r^2stx^2 + 3r^2svx^3 - 2r^2tux^3 - r^2tx - 2r^2vx^2 + rs^2ux^3 + 2rst^2x^3 + 4rstvx^4 - 4rsux^2 + 2rsv^2x^5 + 2rtuvx^5 - ru^3x^5 - 2ru^2x^3 + 2ruv^2x^6 + 3rux - s^3tx^3 - 2s^3vx^4 + 4s^2tx^2 - 2s^2uvx^5 + 7s^2vx^3 - st^2vx^5 + stu^2x^5 - 2stv^2x^6 - 5stx + 4suvx^4 - sv^3x^7 - 8svx^2 + t^3x^3 + 4t^2vx^4 - 2tu^2x^4 + 5tv^2x^5 + 2t - u^2vx^5 - 2uvx^3 + 2v^3x^6 + 3vx)W_2 + x^2(-2r^3vx^2 + 2r^2sux^2 - 5r^2tvx^3 + 3r^2u^2x^3 - r^2ux - 4r^2v^2x^4 + rs^2vx^3 + 2rstux^3 - 4rsvx^2 - 4rt^2vx^4 + 4rtu^2x^4 - 6rtv^2x^5 + ru^2vx^5 - 2rv^3x^6 + 3rvx - s^3ux^3 + 2s^2tvx^4 - 2s^2u^2x^4 + 4s^2ux^2 + 3s^2v^2x^5 - 6stvx^3 - su^3x^5 + 6su^2x^3 + 2suv^2x^6 - 5sux - 8sv^2x^4 - t^3vx^5 + t^2u^2x^5 + t^2ux^3 - 2t^2v^2x^6 - tv^3x^7 + 4tvx^2 + 2u^3x^4 - 4u^2x^2 - 3uv^2x^5 + 2u + 5v^2x^3)W_1 + vx^2(-r^2x - 4ux^2 + 4s^2x^2 - s^3x^3 + t^2x^3 + 2u^2x^4 + 3v^2x^5 - 5sx + 2rvx^3 + 6sux^3 + 4tvx^4 + 2r^2sx^2 + 3r^2ux^3 - 2s^2ux^4 - su^2x^5 + t^2ux^5 - 2sv^2x^6 - uv^2x^7 + 2rstx^3 + 4rtux^4 + 2ruvx^5 - 2stvx^5 + 2)W_0.$$

Proof. **(a)** Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}$$

i.e.

$$vW_{n-5} = W_n - rW_{n-1} - sW_{n-2} - tW_{n-3} - uW_{n-4}$$

we obtain

$$\begin{array}{rcl}
v \times 0 \times x^0 W_0 & = & 0 \times x^0 W_5 - r \times 0 \times x^0 W_4 - s \times 0 \times x^0 W_3 - t \times 0 \times x^0 W_2 \\
 & & -u \times 0 \times x^0 W_1 \\
v \times 1 \times x^1 W_1 & = & 1 \times x^1 W_6 - r \times 1 \times x^1 W_5 - s \times 1 \times x^1 W_4 - t \times 1 \times x^1 W_3 \\
 & & -u \times 1 \times x^1 W_2 \\
v \times 2 \times x^2 W_2 & = & 2 \times x^2 W_7 - r \times 2 \times x^2 W_6 - s \times 2 \times x^2 W_5 - t \times 2 \times x^2 W_4 \\
 & & -u \times 2 \times x^2 W_3 \\
 & & \vdots \\
v(n-2)x^{n-2}W_{n-2} & = & (n-2)x^{n-2}W_{n+3} - r(n-2)x^{n-2}W_{n+2} - s(n-2)x^{n-2}W_{n+1} \\
 & & -t(n-2)x^{n-2}W_n - u(n-2)x^{n-2}W_{n-1} \\
v(n-1)x^{n-1}W_{n-1} & = & (n-1)x^{n-1}W_{n+4} - r(n-1)x^{n-1}W_{n+3} - s(n-1)x^{n-1}W_{n+2} \\
 & & -t(n-1)x^{n-1}W_{n+1} - u(n-1)x^{n-1}W_n \\
v \times n \times x^n W_n & = & n \times x^n W_{n+5} - r \times n \times x^n W_{n+4} - s \times n \times x^n W_{n+3} \\
 & & -t \times n \times x^n W_{n+2} - u \times n \times x^n W_{n+1}
\end{array}$$

If we add the equations side by side (and using Theorem 1.1 (a)), we get (a).

(b) and (c) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5}$$

we obtain

$$\begin{aligned}
r\times 1\times x^{1}W_{3} &= 1\times x^{1}W_{4}-s\times 1\times x^{1}W_{2}-t\times 1\times x^{1}W_{1}\\
&\quad -u\times 1\times x^{1}W_{0}-v\times 1\times x^{1}W_{-1}\\
r\times 2\times x^{2}W_{5} &= 2\times x^{2}W_{6}-s\times 2\times x^{2}W_{4}-t\times 2\times x^{2}W_{3}\\
&\quad -u\times 2\times x^{2}W_{2}-v\times 2\times x^{2}W_{1}\\
&\ \ \vdots\\
r\times (n-1)\times x^{n-1}W_{2n-1} &= (n-1)\times x^{n-1}W_{2n}-s\times (n-1)\times x^{n-1}W_{2n-2}\\
&\quad -t\times (n-1)\times x^{n-1}W_{2n-3}-u\times (n-1)\times x^{n-1}W_{2n-4}\\
&\quad -v\times (n-1)\times x^{n-1}W_{2n-5}\\
r\times n\times x^{n}W_{2n+1} &= n\times x^{n}W_{2n+2}-s\times n\times x^{n}W_{2n}-t\times n\times x^{n}W_{2n-1}\\
&\quad -u\times n\times x^{n}W_{2n-2}-v\times n\times x^{n}W_{2n-3}
\end{aligned}$$

Now, if we add the above equations side by side, we get

$$\begin{aligned}
&r(-0\times x^{0}W_{1}+\sum_{k=0}^{n}kx^{k}W_{2k+1}) = (n\times x^{n}W_{2n+2}-0\times x^{0}W_{2}-(-1)\times x^{-1}W_{0}\\
&+\sum_{k=0}^{n}(k-1)x^{k-1}W_{2k})-s(-0\times x^{0}W_{0}+\sum_{k=0}^{n}kx^{k}W_{2k})-t(-(n+1)x^{n+1}W_{2n+1}\\
&+\sum_{k=0}^{n}(k+1)x^{k+1}W_{2k+1})-u(-(n+1)x^{n+1}W_{2n}+\sum_{k=0}^{n}(k+1)x^{k+1}W_{2k})\\
&-v(-(n+2)x^{n+2}W_{2n+1}-(n+1)x^{n+1}W_{2n-1}+1\times x^{1}W_{-1}\\
&+\sum_{k=0}^{n}(k+2)x^{k+2}W_{2k+1}).
\end{aligned}$$

Since

$$W_{-1}=-\frac{u}{v}W_{0}-\frac{t}{v}W_{1}-\frac{s}{v}W_{2}-\frac{r}{v}W_{3}+\frac{1}{v}W_{4}$$

we obtain

$$\begin{aligned}
&r(-0\times x^{0}W_{1}+\sum_{k=0}^{n}kx^{k}W_{2k+1}) = (n\times x^{n}W_{2n+2}-0\times x^{0}W_{2}-(-1)\times x^{-1}W_{0}\\
&+x^{-1}\sum_{k=0}^{n}kx^{k}W_{2k}-x^{-1}\sum_{k=0}^{n}x^{k}W_{2k})-s(-0\times x^{0}W_{0}+\sum_{k=0}^{n}kx^{k}W_{2k})-t(-(n+1)x^{n+1}W_{2n+1}\\
&+x^{1}\sum_{k=0}^{n}kx^{k}W_{2k+1}+x^{1}\sum_{k=0}^{n}x^{k}W_{2k+1})-u(-(n+1)x^{n+1}W_{2n}+x^{1}\sum_{k=0}^{n}kx^{k}W_{2k}\\
&+x^{1}\sum_{k=0}^{n}x^{k}W_{2k})-v(-(n+2)x^{n+2}W_{2n+1}-(n+1)x^{n+1}W_{2n-1}+1\times x^{1}(-\frac{u}{v}W_{0}-\frac{t}{v}W_{1}\\
&-\frac{s}{v}W_{2}-\frac{r}{v}W_{3}+\frac{1}{v}W_{4})+x^{2}\sum_{k=0}^{n}kx^{k}W_{2k+1}+2x^{2}\sum_{k=0}^{n}x^{k}W_{2k+1}).
\end{aligned}\tag{2.1}$$

Similarly, using the recurrence relation

$$W_{n}=rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}+vW_{n-5}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5}$$

we write the following obvious equations;

$$\begin{aligned}
r \times 1 \times x^1 W_2 &= 1 \times x^1 W_3 - s \times 1 \times x^1 W_1 - t \times 1 \times x^1 W_0 \\
&\quad -u \times 1 \times x^1 W_{-1} - v \times 1 \times x^1 W_{-2} \\
r \times 2 \times x^2 W_4 &= 2 \times x^2 W_5 - s \times 2 \times x^2 W_3 - t \times 2 \times x^2 W_2 \\
&\quad -u \times 2 \times x^2 W_1 - v \times 2 \times x^2 W_0 \\
&\quad \vdots \\
r \times (n-1) \times x^{n-1} W_{2n-2} &= (n-1) \times x^{n-1} W_{2n-1} - s \times (n-1) \times x^{n-1} W_{2n-3} \\
&\quad -t \times (n-1) \times x^{n-1} W_{2n-4} - u \times (n-1) \times x^{n-1} W_{2n-5} \\
&\quad -v \times (n-1) \times x^{n-1} W_{2n-6} \\
r \times n \times x^n W_{2n} &= n \times x^n W_{2n+1} - s \times n \times x^n W_{2n-1} \\
&\quad -t \times n \times x^n W_{2n-2} - u \times n \times x^n W_{2n-3} - v \times n \times x^n W_{2n-4}.
\end{aligned}$$

Now, if we add the above equations side by side, we obtain

$$\begin{aligned}
& r(-0 \times x^0 W_0 + \sum_{k=0}^{n} kx^k W_{2k}) = (-0 \times x^0 W_1 + \sum_{k=0}^{n} kx^k W_{2k+1}) \\
& -s(-(n+1)x^{n+1} W_{2n+1} + \sum_{k=0}^{n} (k+1)x^{k+1} W_{2k+1}) - t(-(n+1)x^{n+1} W_{2n} \\
& + \sum_{k=0}^{n} (k+1)x^{k+1} W_{2k}) - u(-(n+2)x^{n+2} W_{2n+1} - (n+1)x^{n+1} W_{2n-1} \\
& +1 \times x^1 W_{-1} + \sum_{k=0}^{n} (k+2)x^{k+2} W_{2k+1}) - v(-(n+2)x^{n+2} W_{2n} - (n+1)x^{n+1} W_{2n-2} \\
& +1 \times x^1 W_{-2} + \sum_{k=0}^{n} (k+2)x^{k+2} W_{2k})
\end{aligned}$$

Since

$$\begin{aligned}
W_{-1} &= -\frac{u}{v} W_0 - \frac{t}{v} W_1 - \frac{s}{v} W_2 - \frac{r}{v} W_3 + \frac{1}{v} W_4 \\
W_{-2} &= -\frac{u}{v}(-\frac{u}{v} W_0 - \frac{t}{v} W_1 - \frac{s}{v} W_2 - \frac{r}{v} W_3 + \frac{1}{v} W_4) - \frac{t}{v} W_0 - \frac{s}{v} W_1 - \frac{r}{v} W_2 + \frac{1}{v} W_3
\end{aligned}$$

---

we have

$$
\begin{aligned}
& r(-0\times x^{0}W_{0}+\sum_{k=0}^{n}kx^{k}W_{2k})=(-0\times x^{0}W_{1}+\sum_{k=0}^{n}kx^{k}W_{2k+1}) \qquad (2.2)\\
& -s(-(n+1)x^{n+1}W_{2n+1}+x^{1}\sum_{k=0}^{n}kx^{k}W_{2k+1}+x^{1}\sum_{k=0}^{n}x^{k}W_{2k+1})-t(-(n+1)x^{n+1}W_{2n}\\
& +x^{1}\sum_{k=0}^{n}kx^{k}W_{2k}+x^{1}\sum_{k=0}^{n}x^{k}W_{2k})-u(-(n+2)x^{n+2}W_{2n+1}-(n+1)x^{n+1}W_{2n-1}\\
& +1\times x^{1}(-\frac{u}{v}W_{0}-\frac{t}{v}W_{1}-\frac{s}{v}W_{2}-\frac{r}{v}W_{3}+\frac{1}{v}W_{4})+x^{2}\sum_{k=0}^{n}kx^{k}W_{2k+1}+2x^{2}\sum_{k=0}^{n}x^{k}W_{2k+1})\\
& -v(-(n+2)x^{n+2}W_{2n}-(n+1)x^{n+1}W_{2n-2}+1\times x^{1}(-\frac{u}{v}(-\frac{u}{v}W_{0}-\frac{t}{v}W_{1}-\frac{s}{v}W_{2}-\frac{r}{v}W_{3}\\
& +\frac{1}{v}W_{4})-\frac{t}{v}W_{0}-\frac{s}{v}W_{1}-\frac{r}{v}W_{2}+\frac{1}{v}W_{3})+x^{2}\sum_{k=0}^{n}kx^{k}W_{2k}+2x^{2}\sum_{k=0}^{n}x^{k}W_{2k}).
\end{aligned}
$$

Then, using Theorem 1.1 (b) and (c) and solving the system (2.1)-(2.2), the required result of (b) and (c) follow. □

□

3 Special Cases

In this section, for the special cases of x, we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n} kx^{k}W_{k}$, $\sum_{k=0}^{n} kx^{k}W_{2k}$ and $\sum_{k=0}^{n} kx^{k}W_{2k+1}$ for the specific case of sequence $\{W_{n}\}$.

3.1 The case $x = 1$

In this subsection we consider the special case $x = 1$.

The case $x = 1$ of Theorem 2.1 is given in Soykan [(31)].

3.2 The case $x = -1$

In this subsection we consider the special case $x = -1$ and we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n} k(-1)^{k}W_{k}$, $\sum_{k=0}^{n} k(-1)^{k}W_{2k}$ and $\sum_{k=0}^{n} k(-1)^{k}W_{2k+1}$ for the specific case of the sequence $\{W_{n}\}$.

Taking $r = s = t = u = v = 1$ in Theorem 2.1 (a), (b) and (c), we obtain the following Proposition.

Proposition 3.1. *If $r = s = t = u = v = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} k(-1)^{k}W_{k} = \frac{1}{4}((-1)^{n}\,(-(2n+5)W_{n+4}+(4n+8)W_{n+3}-(2n-1)W_{n+2}+(4n+2)W_{n+1}+(2n+7)W_{n})+5W_{4}-8W_{3}-W_{2}-2W_{1}-7W_{0})$.

(b) $\sum_{k=0}^{n} k(-1)^{k}W_{2k} = \frac{1}{4}((-1)^{n}\,((2n-1)W_{2n+2}+4W_{2n+1}-(2n-1)W_{2n}-(4n+6)W_{2n-1}-(2n+5)W_{2n-2})+W_{4}+4W_{3}-5W_{2}-10W_{1}-7W_{0})$.

(c) $\sum_{k=0}^{n} k(-1)^{k}W_{2k+1} = \frac{1}{4}((-1)^{n}\,((2n+3)W_{2n+2}-(2n+7)W_{2n}-6W_{2n-1}+(2n-1)W_{2n-2})+5W_{4}-4W_{3}-9W_{2}-6W_{1}+W_{0})$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Pentanacci numbers (take $W_{n} = P_{n}$ with $P_{0} = 0, P_{1} = 1, P_{2} = 1, P_{3} = 2, P_{4} = 4$).

Corollary 3.1. *For $n \geq 0,$ Pentanacci numbers have the following properties.*

(a) $\sum_{k=0}^{n} k(-1)^k P_k = \frac{1}{4}((-1)^n(-(2n+5)P_{n+4} + (4n+8)P_{n+3} - (2n-1)P_{n+2} + (4n+2)P_{n+1} + (2n+7)P_n) + 1).$

(b) $\sum_{k=0}^{n} k(-1)^k P_{2k} = \frac{1}{4}((-1)^n((2n-1)P_{2n+2} + 4P_{2n+1} - (2n-1)P_{2n} - (4n+6)P_{2n-1} - (2n+5) P_{2n-2}) - 3).$

(c) $\sum_{k=0}^{n} k(-1)^k P_{2k+1} = \frac{1}{4}((-1)^n((2n+3)P_{2n+2} - (2n+7)P_{2n} - 6P_{2n-1} + (2n-1)P_{2n-2}) - 3).$

Taking $W_n = Q_n$ with $Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 = 15$ in the above Proposition, we have the following Corollary which presents linear sum formulas of Pentanacci-Lucas numbers.

Corollary 3.2. *For $n \geq 0,$ Pentanacci-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n} k(-1)^k Q_k = \frac{1}{4}((-1)^n(-(2n+5)Q_{n+4} + (4n+8)Q_{n+3} - (2n-1)Q_{n+2} + (4n+2)Q_{n+1} + (2n+7)Q_n) - 21).$

(b) $\sum_{k=0}^{n} k(-1)^k Q_{2k} = \frac{1}{4}((-1)^n((2n-1)Q_{2n+2} + 4Q_{2n+1} - (2n-1)Q_{2n} - (4n+6)Q_{2n-1} - (2n+5) Q_{2n-2}) - 17).$

(c) $\sum_{k=0}^{n} k(-1)^k Q_{2k+1} = \frac{1}{4}((-1)^n((2n+3)Q_{2n+2} - (2n+7)Q_{2n} - 6Q_{2n-1} + (2n-1)Q_{2n-2}) + 19).$

Taking $r = 2, s = t = u = v = 1$ in Theorem 2.1 (a), (b) and (c), we obtain the following Proposition.

Proposition 3.2. *If $r = 2, s = t = u = v = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{9}((-1)^n(-(3n+8)W_{n+4} + (9n+21)W_{n+3} - (6n+4)W_{n+2} + (9n+6)W_{n+1} + (3n+11)W_n) + 8W_4 - 21W_3 + 4W_2 - 6W_1 - 11W_0).$

(b) $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{25}((-1)^n((5n-6)W_{2n+2} + 15W_{2n+1} - (5n-11)W_{2n} - (10n+13)W_{2n-2} - (15n+12)W_{2n-1}) - W_4 + 15W_3 - 19W_2 - 27W_1 - 23W_0).$

(c) $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{25}((-1)^n((10n+3)W_{2n+2} + 5W_{2n+1} - (10n+18)W_{2n} + (5n-6)W_{2n-2} - (5n+19)W_{2n-1}) + 13W_4 - 20W_3 - 28W_2 - 24W_1 - W_0).$

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13$).

Corollary 3.3. *For $n \geq 0,$ fifth-order Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k P_k = \frac{1}{9}((-1)^n(-(3n+8)P_{n+4} + (9n+21)P_{n+3} - (6n+4)P_{n+2} + (9n+6)P_{n+1} + (3n+11)P_n) + 1).$

(b) $\sum_{k=0}^{n} k(-1)^k P_{2k} = \frac{1}{25}((-1)^n((5n-6)P_{2n+2} + 15P_{2n+1} - (5n-11)P_{2n} - (10n+13)P_{2n-2} - (15n+12)P_{2n-1}) - 3).$

(c) $\sum_{k=0}^{n} k(-1)^k P_{2k+1} = \frac{1}{25}((-1)^n((10n+3)P_{2n+2} + 5P_{2n+1} - (10n+18)P_{2n} + (5n-6)P_{2n-2} - (5n+19)P_{2n-1}) - 11).$

Taking $P_n = Q_n$ with $Q_0 = 5, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 = 46$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Pell-Lucas numbers.

Corollary 3.4. *For $n \geq 0,$ fifth-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k Q_k = \frac{1}{9}((-1)^n(-(3n+8)Q_{n+4} + (9n+21)Q_{n+3} - (6n+4)Q_{n+2} + (9n+6)Q_{n+1} + (3n+11)Q_n) - 32).$

(b) $\sum_{k=0}^{n} k(-1)^k Q_{2k} = \frac{1}{25}((-1)^n((5n-6)Q_{2n+2} + 15Q_{2n+1} - (5n-11)Q_{2n} - (10n+13)Q_{2n-2} - (15n+12)Q_{2n-1}) - 74).$

(c) $\sum_{k=0}^{n} k(-1)^k Q_{2k+1} = \frac{1}{25}((-1)^n ((10n+3)Q_{2n+2} + 5Q_{2n+1} - (10n+18)Q_{2n} + (5n-6)Q_{2n-2} - (5n+19)Q_{2n-1}) + 37)$.

Taking $r = 1, s = 1, t = 1, u = 1, v = 2$ in Theorem 2.1 (a), (b) and (c), we obtain the following Proposition.

Proposition 3.3. *If* $r = 1, s = 1, t = 1, u = 1, v = 2$ *then for* $n \geq 0$ *we have the following formulas:*

(a) $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{9}((-1)^n (-(3n+4)W_{n+4} + (6n+5)W_{n+3} - (3n-5)W_{n+2} + (6n-4)W_{n+1} + 2(3n+7)W_n) + 4W_4 - 5W_3 - 5W_2 + 4W_1 - 14W_0)$.

(b) $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{25}((-1)^n ((5n-14)W_{2n+2} + (5n+11)W_{2n+1} + (5n+36)W_{2n} - (20n-11)W_{2n-1} - 2(10n+7)W_{2n-2}) - 9W_4 + 16W_3 + 16W_2 - 9W_1 - 34W_0)$.

(c) $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{25}((-1)^n ((10n-3)W_{2n+2} + (10n+22)W_{2n+1} - (15n+3)W_{2n} + 2(5n-14)W_{2n-2} - (15n+28)W_{2n-1}) + 7W_4 + 7W_3 - 18W_2 - 43W_1 - 18W_0)$.

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Jacobsthal numbers.

Corollary 3.5. *For* $n \geq 0$, *fifth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k J_k = \frac{1}{9}((-1)^n (-(3n+4)J_{n+4} + (6n+5)J_{n+3} - (3n-5)J_{n+2} + (6n-4)J_{n+1} + 2(3n+7)J_n) - 2)$.

(b) $\sum_{k=0}^{n} k(-1)^k J_{2k} = \frac{1}{25}((-1)^n ((5n-14)J_{2n+2} + (5n+11)J_{2n+1} + (5n+36)J_{2n} - (20n-11)J_{2n-1} - 2(10n+7)J_{2n-2}) + 14)$.

(c) $\sum_{k=0}^{n} k(-1)^k J_{2k+1} = \frac{1}{25}((-1)^n ((10n-3)J_{2n+2} + (10n+22)J_{2n+1} - (15n+3)J_{2n} + 2(5n-14)J_{2n-2} - (15n+28)J_{2n-1}) - 47)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20$).

Corollary 3.6. *For* $n \geq 0$, *fifth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k j_k = \frac{1}{9}((-1)^n (-(3n+4)j_{n+4} + (6n+5)j_{n+3} - (3n-5)j_{n+2} + (6n-4)j_{n+1} + 2(3n+7)j_n) - 19)$.

(b) $\sum_{k=0}^{n} k(-1)^k j_{2k} = \frac{1}{25}((-1)^n ((5n-14)j_{2n+2} + (5n+11)j_{2n+1} + (5n+36)j_{2n} - (20n-11)j_{2n-1} - 2(10n+7)j_{2n-2}) - 17)$.

(c) $\sum_{k=0}^{n} k(-1)^k j_{2k+1} = \frac{1}{25}((-1)^n ((10n-3)j_{2n+2} + (10n+22)j_{2n+1} - (15n+3)j_{2n} + 2(5n-14)j_{2n-2} - (15n+28)j_{2n-1}) + 41)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20$ in the last proposition, we have the following corollary which presents linear sum formula of modified fifth order Jacobsthal numbers.

Corollary 3.7. *For* $n \geq 0$,*modified fifth order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n} k(-1)^k K_k = \frac{1}{9}((-1)^n (-(3n+4)K_{n+4} + (6n+5)K_{n+3} - (3n-5)K_{n+2} + (6n-4)K_{n+1} + 2(3n+7)K_n) - 23)$.

(b) $\sum_{k=0}^{n} k(-1)^k K_{2k} = \frac{1}{25}((-1)^n ((5n-14)K_{2n+2} + (5n+11)K_{2n+1} + (5n+36)K_{2n} - (20n-11)K_{2n-1} - 2(10n+7)K_{2n-2}) - 83)$.

(c) $\sum_{k=0}^{n} k(-1)^k K_{2k+1} = \frac{1}{25}((-1)^n ((10n-3)K_{2n+2} + (10n+22)K_{2n+1} - (15n+3)K_{2n} + 2(5n-14)K_{2n-2} - (15n+28)K_{2n-1}) + 59)$.

From the last proposition, we have the following corollary which gives linear sum formula of fifth-order Jacobsthal Perrin numbers (take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16$).

Corollary 3.8. *For $n \geq 0$, fifth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum_{k=0}^{n} k(-1)^k Q_k = \frac{1}{9}((-1)^n(-(3n+4)Q_{n+4} + (6n+5)Q_{n+3} - (3n-5)Q_{n+2} + (6n-4)Q_{n+1} + 2(3n+7)Q_n) - 28)$.

(b) $\sum_{k=0}^{n} k(-1)^k Q_{2k} = \frac{1}{25}((-1)^n((5n-14)Q_{2n+2} + (5n+11)Q_{2n+1} + (5n+36)Q_{2n} - (20n-11)Q_{2n-1} - 2(10n+7)Q_{2n-2}) - 86)$.

(c) $\sum_{k=0}^{n} k(-1)^k Q_{2k+1} = \frac{1}{25}((-1)^n((10n-3)Q_{2n+2} + (10n+22)Q_{2n+1} - (15n+3)Q_{2n} + 2(5n-14)Q_{2n-2} - (15n+28)Q_{2n-1}) + 78)$.

Taking $Q_n = W_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4$ in the proposition, we have the following corollary which presents linear sum formula of adjusted fifth-order Jacobsthal numbers.

Corollary 3.9. *For $n \geq 0$, adjusted fifth-order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n} k(-1)^k S_k = \frac{1}{9}((-1)^n(-(3n+4)S_{n+4} + (6n+5)S_{n+3} - (3n-5)S_{n+2} + (6n-4)S_{n+1} + 2(3n+7)S_n) + 5)$.

(b) $\sum_{k=0}^{n} k(-1)^k S_{2k} = \frac{1}{25}((-1)^n((5n-14)S_{2n+2} + (5n+11)S_{2n+1} + (5n+36)S_{2n} - (20n-11)S_{2n-1} - 2(10n+7)S_{2n-2}) + 3)$.

(c) $\sum_{k=0}^{n} k(-1)^k S_{2k+1} = \frac{1}{25}((-1)^n((10n-3)S_{2n+2} + (10n+22)S_{2n+1} - (15n+3)S_{2n} + 2(5n-14)S_{2n-2} - (15n+28)S_{2n-1}) - 19)$.

From the last proposition, we have the following corollary which gives linear sum formula of modified fifth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 5, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15$).

Corollary 3.10. *For $n \geq 0$, modified fifth-order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=0}^{n} k(-1)^k R_k = \frac{1}{9}((-1)^n(-(3n+4)R_{n+4} + (6n+5)R_{n+3} - (3n-5)R_{n+2} + (6n-4)R_{n+1} + 2(3n+7)R_n) - 56)$.

(b) $\sum_{k=0}^{n} k(-1)^k R_{2k} = \frac{1}{25}((-1)^n((5n-14)R_{2n+2} + (5n+11)R_{2n+1} + (5n+36)R_{2n} - (20n-11)R_{2n-1} - 2(10n+7)R_{2n-2}) - 154)$.

(c) $\sum_{k=0}^{n} k(-1)^k R_{2k+1} = \frac{1}{25}((-1)^n((10n-3)R_{2n+2} + (10n+22)R_{2n+1} - (15n+3)R_{2n} + 2(5n-14)R_{2n-2} - (15n+28)R_{2n-1}) - 33)$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.4. *If $r = 2, s = 3, t = 5, u = 7, v = 11$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{81}((-1)^n(-(9n-2)W_{n+4} + (27n-15)W_{n+3} + 36W_{n+2} + (45n-46)W_{n+1} + 11(9n+7)W_n) - 2W_4 + 15W_3 - 36W_2 + 46W_1 - 77W_0)$.

(b) $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{5329}((-1)^n(-(219n-184)W_{2n+2} + (1022n-1759)W_{2n+1} + (5037n+1827)W_{2n} - (1679n-6034)W_{2n-1} - 11(584n-807)W_{2n-2}) - 35W_4 - 737W_3 + 1535W_2 + 4355W_1 + 2453W_0)$.

(c) $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{5329}((-1)^n((584n-1391)W_{2n+2} + (4380n+2379)W_{2n+1} - (2774n-6954)W_{2n} - (7957n-10165)W_{2n-1} - 11(219n-184)W_{2n-2}) - 807W_4 + 1430W_3 + 4180W_2 + 2208W_1 - 385W_0)$.

From the last proposition, we have the following corollary which gives linear sum formulas of 5-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 1, G_4 = 2$).

Corollary 3.11. *For $n \geq 0$, 5-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k G_k = \frac{1}{81}((-1)^n(-(9n-2)G_{n+4}+(27n-15)G_{n+3}+36G_{n+2}+(45n-46)G_{n+1}+11(9n+7)G_n)+11)$.

(b) $\sum_{k=0}^{n} k(-1)^k G_{2k} = \frac{1}{5329}((-1)^n(-(219n-184)G_{2n+2}+(1022n-1759)G_{2n+1}+(5037n+1827)G_{2n}-(1679n-6034)G_{2n-1}-11(584n-807)G_{2n-2})-807)$.

(c) $\sum_{k=0}^{n} k(-1)^k G_{2k+1} = \frac{1}{5329}((-1)^n((584n-1391)G_{2n+2}+(4380n+2379)G_{2n+1}-(2774n-6954)G_{2n}-(7957n-10165)G_{2n-1}-11(219n-184)G_{2n-2})-184)$.

Taking $W_n = H_n$ with $H_0 = 5, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150$ in the last proposition, we have the following corollary which presents linear sum formulas of Lucas 5-primes numbers.

Corollary 3.12. *For $n \geq 0,$ Lucas 5-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k H_k = \frac{1}{81}((-1)^n(-(9n-2)H_{n+4}+(27n-15)H_{n+3}+36H_{n+2}+(45n-46)H_{n+1}+11(9n+7)H_n)-338)$.

(b) $\sum_{k=0}^{n} k(-1)^k H_{2k} = \frac{1}{5329}((-1)^n(-(219n-184)H_{2n+2}+(1022n-1759)H_{2n+1}+(5037n+1827)H_{2n}-(1679n-6034)H_{2n-1}-11(584n-807)H_{2n-2})+858)$.

(c) $\sum_{k=0}^{n} k(-1)^k H_{2k+1} = \frac{1}{5329}((-1)^n((584n-1391)H_{2n+2}+(4380n+2379)H_{2n+1}-(2774n-6954)H_{2n}-(7957n-10165)H_{2n-1}-11(219n-184)H_{2n-2})-18129)$.

From the last proposition, we have the following corollary which gives linear sum formulas of modified 5-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 1, E_4 = 1$).

Corollary 3.13. *For $n \geq 0,$ modified 5-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k E_k = \frac{1}{81}((-1)^n(-(9n-2)E_{n+4}+(27n-15)E_{n+3}+36E_{n+2}+(45n-46)E_{n+1}+11(9n+7)E_n)+13)$.

(b) $\sum_{k=0}^{n} k(-1)^k E_{2k} = \frac{1}{5329}((-1)^n(-(219n-184)E_{2n+2}+(1022n-1759)E_{2n+1}+(5037n+1827)E_{2n}-(1679n-6034)E_{2n-1}-11(584n-807)E_{2n-2})-772)$.

(c) $\sum_{k=0}^{n} k(-1)^k E_{2k+1} = \frac{1}{5329}((-1)^n((584n-1391)E_{2n+2}+(4380n+2379)E_{2n+1}-(2774n-6954)E_{2n}-(7957n-10165)E_{2n-1}-11(219n-184)E_{2n-2})+623)$.

3.3 The case $x = i$

In this subsection we consider the special case $x = i.$

Taking $x = i, r = s = t = u = v = 1$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.5. *If $r = s = t = u = v = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} ki^k W_k = \frac{1}{2i}(i^n(((1-i)n+6-i)W_{n+4}-2(n+(3+2i))W_{n+3}+(5i-10-(1-3i)n)W_{n+2}+((2+2i)n-4+10i)W_{n+1}+(((1+i)n+2+7i))W_n)-(6-i)W_4+(6+4i)W_3+(10-5i)W_2+(4-10i)W_1-(2+7i)W_0)$.

(b) $\sum_{k=0}^{n} ki^k W_{2k} = \frac{1}{18i}(i^n(((9+3i)n+6+7i)W_{2n+2}-(12n+16+12i)W_{2n+1}+(8+i-(3-3i)n)W_{2n}+(2i-8-(6-6i)n)W_{2n-1}+(2-13i-(3+3i)n)W_{2n-2})+(10-15i)W_4-(12-28i)W_3-(14+5i)W_2+(8+14i)W_1-(16-i)W_0)$.

(c) $\sum_{k=0}^{n} ki^k W_{2k+1} = \frac{1}{18i}(i^n(((3i-3)n-10-5i)W_{2n+2}+((6+6i)n+14+8i)W_{2n+1}+((3+9i)n-2+9i)W_{2n}+(6n+8-6i)W_{2n-1}+((9+3i)n+6+7i)W_{2n-2})-(2-13i)W_4-(4+20i)W_3+(18-i)W_2-(6+14i)W_1+(10-15i)W_0)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Pentanacci numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4$).

Corollary 3.14. *For $n \geq 0$, Pentanacci numbers have the following properties.*

(a) $\sum_{k=0}^{n} ki^k P_k = \frac{1}{2i}(i^n(((1-i)\,n+6-i)P_{n+4} - 2(n+(3+2i))P_{n+3} + (5i-10-(1-3i)n)P_{n+2} + ((2+2i)n-4+10i)P_{n+1} + (((1+i)n+2+7i))P_n) + (2-3i))$.

(b) $\sum_{k=0}^{n} ki^k P_{2k} = \frac{1}{18i}(i^n(((9+3i)n+6+7i)P_{2n+2} - (12n+16+12i)P_{2n+1} + (8+i-(3-3i)n)P_{2n} + (2i-8-(6-6i)n)P_{2n-1} + (2-13i-(3+3i)n)P_{2n-2}) + (10+5i))$.

(c) $\sum_{k=0}^{n} ki^k P_{2k+1} = \frac{1}{18i}(i^n(((3i-3)n-10-5i)P_{2n+2} + ((6+6i)n+14+8i)P_{2n+1} + ((3+9i)n-2+9i)P_{2n} + (6n+8-6i)P_{2n-1} + ((9+3i)n+6+7i)P_{2n-2}) + (-4-3i))$.

Taking $P_n = Q_n$ with $Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 = 15$ in the above Proposition, we have the following Corollary which presents linear sum formulas of Pentanacci-Lucas numbers.

Corollary 3.15. *For $n \geq 0$, Pentanacci-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n} ki^k Q_k = \frac{1}{2i}(i^n(((1-i)\,n+6-i)Q_{n+4} - 2(n+(3+2i))Q_{n+3} + (5i-10-(1-3i)n)Q_{n+2} + ((2+2i)n-4+10i)Q_{n+1} + (((1+i)n+2+7i))Q_n) + (-24-17i))$.

(b) $\sum_{k=0}^{n} ki^k Q_{2k} = \frac{1}{18i}(i^n(((9+3i)n+6+7i)Q_{2n+2} - (12n+16+12i)Q_{2n+1} + (8+i-(3-3i)n)Q_{2n} + (2i-8-(6-6i)n)Q_{2n-1} + (2-13i-(3+3i)n)Q_{2n-2}) + (-48-25i))$.

(c) $\sum_{k=0}^{n} ki^k Q_{2k+1} = \frac{1}{18i}(i^n(((3i-3)n-10-5i)Q_{2n+2} + ((6+6i)n+14+8i)Q_{2n+1} + ((3+9i)n-2+9i)Q_{2n} + (6n+8-6i)Q_{2n-1} + ((9+3i)n+6+7i)Q_{2n-2}) + (40-37i))$.

Corresponding sums of the other fifth order generalized Pentanacci numbers can be calculated similarly.

4 Sum Formulas of Generalized Pentanacci Numbers with Negative Subscripts

The following Theorem presents some linear summing formulas of generalized Pentanacci numbers with negative subscripts.

Theorem 4.1. *Let x be a real (or complex) number. For $n \geq 1$ we have the following formulas: If $v + rx^4 + sx^3 + tx^2 + ux - x^5 \neq 0$, then*

$$\sum_{k=1}^{n} kx^k W_{-k} = \frac{\Omega_4}{(v + rx^4 + sx^3 + tx^2 + ux - x^5)^2}$$

where

$\Omega_4 = x^{n+1}(n(-v - rx^4 - sx^3 - tx^2 - ux + x^5) - v + 3rx^4 + 2sx^3 + tx^2 - 4x^5)W_{-n+4} + x^{n+1}(n(r-x)(v + rx^4 + sx^3 + tx^2 + ux - x^5) + 6rx^5 + sx^4 - ux^2 - 3r^2x^4 + rv - 2vx - 3x^6 - 2rsx^3 - rtx^2)W_{-n+3} + x^{n+1}(n(s+rx-x^2)(v+rx^4+sx^3+tx^2+ux-x^5) + 4rx^6 + 4sx^5 - tx^4 - 2ux^3 - 3vx^2 - 2r^2x^5 - 2s^2x^3 + sv - 2x^7 - 4rsx^4 + rux^2 - stx^2 + 2rvx)W_{-n+2} + x^{n+1}(n(t+rx^2+sx-x^3)(v+rx^4+sx^3+tx^2+ux-x^5) + 2rx^7 + 2sx^6 + 2tx^5 - 3ux^4 - 4vx^3 - r^2x^6 - s^2x^4 - t^2x^2 + tv - x^8 - 2rsx^5 - 2rtx^4 + 2rux^3 - 2stx^3 + 3rvx^2 + sux^2 + 2svx)W_{-n+1} + x^{n+1}(n(u + rx^3 + sx^2 + tx - x^4)(v + rx^4 + sx^3 + tx^2 + ux - x^5) - 5vx^4 + uv + 4rvx^3 + 3svx^2 + 2tvx)W_{-n} + x(v - 3rx^4 - 2sx^3 - tx^2 + 4x^5)W_4 + x(-6rx^5 - sx^4 + ux^2 + 3r^2x^4 - rv + 2vx + 3x^6 + 2rsx^3 + rtx^2)W_3 + x(-4rx^6 - 4sx^5 + tx^4 + 2ux^3 + 3vx^2 + 2r^2x^5 + 2s^2x^3 - sv + 2x^7 + 4rsx^4 - rux^2 + stx^2 - 2rvx)W_2 + x(-2rx^7 - 2sx^6 - 2tx^5 + 3ux^4 + 4vx^3 + r^2x^6 + s^2x^4 + t^2x^2 - tv + x^8 + 2rsx^5 + 2rtx^4 - 2rux^3 + 2stx^3 - 3rvx^2 - sux^2 - 2svx)W_1 + vx(-u - 4rx^3 - 3sx^2 - 2tx + 5x^4)W_0.$

Proof. Using the recurrence relation

$$\begin{aligned}
W_{n+5} &= rW_{n+4}+sW_{n+3}+tW_{n+2}+uW_{n+1}+vW_n \\
&\Rightarrow W_{-n+5}=rW_{-n+4}+sW_{-n+3}+tW_{-n+2}+uW_{-n+1}+vW_{-n} \\
&\Rightarrow W_{-n}=-\frac{u}{v}W_{-n+1}-\frac{t}{v}W_{-n+2}-\frac{s}{v}W_{-n+3}-\frac{r}{v}W_{-n+4}+\frac{1}{v}W_{-n+5} \\
&\Rightarrow W_{-n}=\frac{1}{v}W_{-n+5}-\frac{u}{v}W_{-n+1}-\frac{t}{v}W_{-n+2}-\frac{s}{v}W_{-n+3}-\frac{r}{v}W_{-n+4}
\end{aligned}$$

i.e.

$$vW_{-n}=W_{-n+5}-rW_{-n+4}-sW_{-n+3}-tW_{-n+2}-uW_{-n+1}$$

we obtain

$$\begin{aligned}
v\times n\times x^{n}W_{-n} &= n\times x^{n}W_{-n+5}-r\times n\times x^{n}W_{-n+4} \\
&\quad -s\times n\times x^{n}W_{-n+3}-t\times n\times x^{n}W_{-n+2}-u\times n\times x^{n}W_{-n+1} \\
v(n-1)x^{n-1}W_{-n+1} &= (n-1)x^{n-1}W_{-n+6}-r(n-1)x^{n-1}W_{-n+5} \\
&\quad -s(n-1)x^{n-1}W_{-n+4}-t(n-1)x^{n-1}W_{-n+3}-u(n-1)x^{n-1}W_{-n+2} \\
v(n-2)x^{n-2}W_{-n+2} &= (n-2)x^{n-2}W_{-n+7}-r(n-2)x^{n-2}W_{-n+6} \\
&\quad -s(n-2)x^{n-2}W_{-n+5}-t(n-2)x^{n-2}W_{-n+4}-u(n-2)x^{n-2}W_{-n+3} \\
&\ \ \vdots \\
v\times 3\times x^{3}W_{-3} &= 3\times x^{3}W_{2}-r\times 3\times x^{3}W_{1} \\
&\quad -s\times 3\times x^{3}W_{0}-t\times 3\times x^{3}W_{-1}-u\times 3\times x^{3}W_{-2} \\
v\times 2\times x^{2}W_{-2} &= 2\times x^{2}W_{3}-r\times 2\times x^{2}W_{2} \\
&\quad -s\times 2\times x^{2}W_{1}-t\times 2\times x^{2}W_{0}-u\times 2\times x^{2}W_{-1} \\
v\times 1\times x^{1}W_{-1} &= 1\times x^{1}W_{4}-r\times 1\times x^{1}W_{3} \\
&\quad -s\times 1\times x^{1}W_{2}-t\times 1\times x^{1}W_{1}-u\times 1\times x^{1}W_{0}.
\end{aligned}$$

If we add the equations side by side (and using Theorem 1.2 (a)), we get (a). □

5 Specific Cases

In this section, for the specific cases of x, we present the closed form solutions (identities) of the sums $\sum_{k=1}^{n}kx^{k}W_{-k}$, $\sum_{k=1}^{n}kx^{k}W_{-2k}$ and $\sum_{k=1}^{n}kx^{k}W_{-2k+1}$ for the specific case of sequence $\{W_n\}$.

5.1 The case $x=1$

In this subsection, we consider the special case $x=1$.

The case $x=1$ of Theorem 4.1 is given in Soykan [(31)].

5.2 The case $x=-1$

In this subsection we consider the special case $x=-1$.

Taking $r=s=t=u=v=1$ in Theorem 4.1, we obtain the following Proposition.

Proposition 5.1. *If $r=s=t=u=v=1$ then for $n\geq 1$ we have the following formulas:*

$\sum_{k=1}^{n}k(-1)^{k}W_{-k}=\frac{1}{4}((-1)^{n}((2n-5)W_{-n+4}-(4n-8)W_{-n+3}+(2n+1)W_{-n+2}-(4n-2)W_{-n+1}+(2n+7)W_{-n})+5W_{4}-8W_{3}-W_{2}-2W_{1}-7W_{0})$.

From the above Proposition, we have the following Corollary which gives sum formulas of Pentanacci and Pentanacci-Lucas numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4$ and take $W_n = Q_n$ with $Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 = 15$, respectively).

Corollary 5.1. *For $n \geq 1,$ we have the following properties.*

- **(a)** $\sum_{k=1}^{n} k(-1)^k P_{-k} = \frac{1}{4}((-1)^n\left((2n-5)P_{-n+4} - (4n-8)P_{-n+3} + (2n+1)P_{-n+2} - (4n-2)P_{-n+1} + (2n+7)P_{-n}\right) + 1).$
- **(b)** $\sum_{k=1}^{n} k(-1)^k Q_{-k} = \frac{1}{4}((-1)^n\left((2n-5)Q_{-n+4} - (4n-8)Q_{-n+3} + (2n+1)Q_{-n+2} - (4n-2)Q_{-n+1} + (2n+7)Q_{-n}\right) - 21).$

Taking $r = 2, s = t = u = v = 1$ in Theorem 4.1, we obtain the following Proposition.

Proposition 5.2. *If $r = 2, s = t = u = v = 1$ then for $n \geq 1$ we have the following formulas:*
$\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{9}((-1)^n\left((3n-8)W_{-n+4} - (9n-21)W_{-n+3} + (6n-4)W_{-n+2} - (9n-6)W_{-n+1} + (6n+11)W_{-n}\right) + 8W_4 - 21W_3 + 4W_2 - 6W_1 - 11W_0).$

From the last Proposition, we have the following Corollary which gives sum formulas of fifth-order Pell and fifth-order Pell-Lucas numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13$ and take $W_n = Q_n$ with $Q_0 = 5, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 = 46$, respectively).

Corollary 5.2. *For $n \geq 1,$ we have the following properties:*

- **(a)** $\sum_{k=1}^{n} k(-1)^k P_{-k} = \frac{1}{9}((-1)^n\left((3n-8)P_{-n+4} - (9n-21)P_{-n+3} + (6n-4)P_{-n+2} - (9n-6)P_{-n+1} + (6n+11)P_{-n}\right) + 1).$
- **(b)** $\sum_{k=1}^{n} k(-1)^k Q_{-k} = \frac{1}{9}((-1)^n\left((3n-8)Q_{-n+4} - (9n-21)Q_{-n+3} + (6n-4)Q_{-n+2} - (9n-6)Q_{-n+1} + (6n+11)Q_{-n}\right) - 32).$

Taking $r = s = t = 1, u = 1, v = 2$ in Theorem 4.1, we obtain the following Proposition.

Proposition 5.3. *If $r = s = t = 1, u = 1, v = 2$ then for $n \geq 1$ we have the following formulas:*
$\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{9}((-1)^n\left((3n-4)W_{-n+4} - (6n-5)W_{-n+3} + (3n+5)W_{-n+2} - (6n+4)W_{-n+1} + (3n+14)W_{-n}\right) + 4W_4 - 5W_3 - 5W_2 + 4W_1 - 14W_0).$

Taking, respectively,
$W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1$ (fifth-order Jacobsthal numbers),
$W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20$ (fifth order Jacobsthal-Lucas numbers),
$W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20$ (modified fifth order Jacobsthal numbers),
$W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16$ (fifth-order Jacobsthal Perrin numbers),
$W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4$ (adjusted fifth-order Jacobsthal numbers),
$W_n = R_n$ with $R_0 = 5, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15$ (modified fifth-order Jacobsthal-Lucas numbers),
in the last Proposition, we have the following Corollary.

Corollary 5.3. *For $n \geq 1,$ we have the following properties:*

- **(a)** $\sum_{k=1}^{n} k(-1)^k J_{-k} = \frac{1}{9}((-1)^n\left((3n-4)J_{-n+4} - (6n-5)J_{-n+3} + (3n+5)J_{-n+2} - (6n+4)J_{-n+1} + (3n+14)J_{-n}\right) - 2).$
- **(b)** $\sum_{k=1}^{n} k(-1)^k j_{-k} = \frac{1}{9}((-1)^n\left((3n-4)j_{-n+4} - (6n-5)j_{-n+3} + (3n+5)j_{-n+2} - (6n+4)j_{-n+1} + (3n+14)j_{-n}\right) - 19).$
- **(c)** $\sum_{k=1}^{n} k(-1)^k K_{-k} = \frac{1}{9}((-1)^n\left((3n-4)K_{-n+4} - (6n-5)K_{-n+3} + (3n+5)K_{-n+2} - (6n+4)K_{-n+1} + (3n+14)K_{-n}\right) - 23).$

(d) $\sum_{k=1}^{n} k(-1)^{k}Q_{-k} = \frac{1}{9}((-1)^{n}((3n-4)Q_{-n+4}-(6n-5)Q_{-n+3}+(3n+5)Q_{-n+2}-(6n+4)Q_{-n+1}+(3n+14)Q_{-n})-28)$.

(e) $\sum_{k=1}^{n} k(-1)^{k}S_{-k} = \frac{1}{9}((-1)^{n}((3n-4)S_{-n+4}-(6n-5)S_{-n+3}+(3n+5)S_{-n+2}-(6n+4)S_{-n+1}+(3n+14)S_{-n})+5)$.

(f) $\sum_{k=1}^{n} k(-1)^{k}R_{-k} = \frac{1}{9}((-1)^{n}((3n-4)R_{-n+4}-(6n-5)R_{-n+3}+(3n+5)R_{-n+2}-(6n+4)R_{-n+1}+(3n+14)R_{-n})-56)$.

Taking $r=2, s=3, t=5, u=7, v=11$ in Theorem 4.1, we obtain the following Proposition.

Proposition 5.4. *If $r=2, s=3, t=5, u=7, v=11$ then for $n \geq 1$ we have the following formulas:*

$\sum_{k=1}^{n} k(-1)^{k}W_{-k} = \frac{1}{81}((-1)^{n}((9n+2)W_{-n+4}-(27n+15)W_{-n+3}+36W_{-n+2}-(45n+46)W_{-n+1}-(18n-77)W_{-n})-2W_{4}+15W_{3}-36W_{2}+46W_{1}-77W_{0})$.

From the last Proposition, we have the following Corollary which gives sum formulas of 5-primes, Lucas 5-primes and modified 5-primes numbers (take $W_n = G_n$ with $G_0=0, G_1=0, G_2=0, G_3=1, G_4=2$, take $W_n = H_n$ with $H_0=5, H_1=2, H_2=10, H_3=41, H_4=150$, take $W_n=E_n$ with $E_0=0, E_1=0, E_2=0, E_3=1, E_4=1$, respectively).

Corollary 5.4. *For $n \geq 1$, we have the following properties:*

(a) $\sum_{k=1}^{n} k(-1)^{k}G_{-k} = \frac{1}{81}((-1)^{n}((9n+2)G_{-n+4}-(27n+15)G_{-n+3}+36G_{-n+2}-(45n+46)G_{-n+1}-(18n-77)G_{-n})+11)$.

(b) $\sum_{k=1}^{n} k(-1)^{k}H_{-k} = \frac{1}{81}((-1)^{n}((9n+2)H_{-n+4}-(27n+15)H_{-n+3}+36H_{-n+2}-(45n+46)H_{-n+1}-(18n-77)H_{-n})-338)$.

(c) $\sum_{k=1}^{n} k(-1)^{k}E_{-k} = \frac{1}{81}((-1)^{n}((9n+2)E_{-n+4}-(27n+15)E_{-n+3}+36E_{-n+2}-(45n+46)E_{-n+1}-(18n-77)E_{-n})+13)$.

5.3 The case $x = i$

In this subsection, we consider the special case $x = i$.

Taking $r=s=t=u=v=1$ in Theorem 4.1, we obtain the following proposition.

Proposition 5.5. *If $r=s=t=u=v=1$ then for $n \geq 1$ we have the following formula:*

$\sum_{k=1}^{n} ki^{k}W_{-k} = \frac{1}{2i}(i^{n}(((1+i)n-6-i)W_{-n+4}+(6-2n-4i)W_{-n+3}+(10+5i-(1+3i)n)W_{-n+2}+((2-2i)n+4+10i)W_{-n+1}+((1+i)n-2+7i)W_{-n})+(6+i)W_{4}-(6-4i)W_{3}-(10+5i)W_{2}-(4+10i)W_{1}+(2-7i)W_{0})$.

From the above Proposition, we have the following Corollary which gives sum formulas of Pentanacci and Pentanacci-Lucas numbers (take $W_n = P_n$ with $P_0=0, P_1=1, P_2=1, P_3=2, P_4=4$ and take $W_n=Q_n$ with $Q_0=5, Q_1=1, Q_2=3, Q_3=7, Q_4=15$, respectively).

Corollary 5.5. *For $n \geq 1$, we have the following properties.*

(a) $\sum_{k=1}^{n} ki^{k}P_{-k} = \frac{1}{2i}(i^{n}(((1+i)n-6-i)P_{-n+4}+(6-2n-4i)P_{-n+3}+(10+5i-(1+3i)n)P_{-n+2}+((2-2i)n+4+10i)P_{-n+1}+((1+i)n-2+7i)P_{-n})+(-2-3i))$.

(b) $\sum_{k=1}^{n} ki^{k}Q_{-k} = \frac{1}{2i}(i^{n}(((1+i)n-6-i)Q_{-n+4}+(6-2n-4i)Q_{-n+3}+(10+5i-(1+3i)n)Q_{-n+2}+((2-2i)n+4+10i)Q_{-n+1}+((1+i)n-2+7i)Q_{-n})+(24-17i))$.

Corresponding sums of the other fifth order generalized Pentanacci numbers can be calculated similarly.

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