# A Six-Point y-Function Hybrid Block Method for Direct Solution of Third Order Ordinary Differential Equations

## Abstract

In this article, a new continuous numerical method was developed for the numerical integration of general third-order initial value problems (IVPs) of ordinary differential equations (ODEs). The method was derived by adopting interpolation and collocation approach using a combination of power series and exponential functions as the basis function for the approximate solution to the ODEs. The approximate solution was interpolated at both nodal and off-nodal points while the differential systems from the approximate solution was collocated at all nodal points to obtain the set of methods with given step-numbers. This derived approach ensured that the hybrid points are obtained at the y-values of the methods. This class of hybrid linear multi-step method satisfied the conditions for usability and accuracy of any given method. These conditions were confirmed by testing the consistency, stability and convergence of the developed method. The method was applied to solve directly linear and non-linear third order ODEs without reduction to system of first-order ordinary differential equations. Results from the computation compared favorably with some existing numerical methods in literature with comparable properties to confirm the superiority of the newly developed method.

**Keywords:** linear multi-step method, hybrid points, nodal points, off nodal points, collocation, interpolation.

AMS Subject classification: 65L05, 65L06.

## 1 Introduction

This research provided an approximate solution to the general third order ordinary differential equations of the form:

$$y^{'''}(x) = f(x, y, y^{'}, y''), \;\; y(x) = c_0, \quad y^{'}(x) = c_1, \;\; y^{''}(x) = c_2. \tag{1}$$

where, f is a continuous real value function.

In many branches of engineering and the sciences, mathematical models frequently lead to the formulation of high-order linear and nonlinear initial value problems (IVPs). These high-order IVPs are particularly relevant in fields such as Biological Sciences and Control Theory, where their applications are of critical importance [1]. A common strategy employed in tackling these equations is the reduction method, which reduces the high-order IVP to a system of first-order ordinary differential equations (ODEs). This reduction approach has been widely utilized in the works of [2]-[5] and many others.

However, despite its effectiveness, the reduction method comes with several drawbacks. Notably, it leads to unnecessary computational overhead, wastage of computational resources, increased human effort, and high associated costs.

Recent developments suggest that direct methods, such as implicit linear multistep methods, offer significant improvements in terms of accuracy and computational efficiency when solving high-order IVPs. These direct methods circumvent the need for reducing the equations to first-order systems, thus addressing some of the inefficiencies inherent in the reduction strategy [6]-[10]. Among the direct methods, continuous collocation techniques have been the focus of extensive research. Several researchers have explored different approaches for solving higher-order ODEs. For instance, Kashkari and Alqarni [12] introduced an optimized two-step block method with three hybrid points, specifically designed to solve third-order IVPs with an order of accuracy of nine. Similarly, Kuboye et al. [14] proposed a single-step block integrator for the direct solution of third-order ODEs, while Obarhua [17] developed a three-step, four-point optimized hybrid block method tailored for the same class of problems.

The ongoing pursuit of higher accuracy and efficiency in numerical integration has motivated the development of the Six Points y-Function Hybrid Block Method. This method distinguishes itself from previous approaches by being problem-independent, allowing a high degree of flexibility in the choice of interpolation points relative to the order of the differential equation. Unlike traditional block methods that are problem-dependent, this new approach provides greater freedom in its application. The present study introduces an order-eight block integrator with six off-step points, designed to solve third-order ODEs. This method builds upon the successes of earlier works [10], [12], [17] by approximating the solution within the interval $[x_n, x_{n+2}]$, providing a significant advancement in the direct solution of high-order ODEs.

## 2 Mathematical Formulation

This work entirely focused on the numerical integration of third-order initial value problem of ordinary differential equation (1) which is approximated by a combination of power series polynomial and exponential functions of the type

$$p(x) = \sum_{j=0}^{n-1} a_j x^j + a_{c+i-1} \sum_{j=0}^{n} \frac{\alpha_j x^j}{j!} \tag{2}$$

The third derivatives of (2) is obtained as

$$p'''(x) = \sum_{j=3}^{n-4} j(j-1)(j-2)a_j x^{j-3} + a_{c+i-1} \sum_{j=3}^{n} \frac{\alpha_j x^{j-3}}{(j-3)!} \tag{3}$$

where $n = c + i$. Equations (1) and (3) yields a differential system:

$$f(x,y,y',y'') = \sum_{j=3}^{n-4} j(j-1)(j-2)a_j x^{j-1} + a_{c+i-1}\sum_{j=3}^{n}\frac{\alpha_j x^{j-3}}{(j-3)!} \tag{4}$$

where $x$ is continuous and differentiable, parameters $a_j$ 's in (2), and (4) are linear terms to be determined. To get the system of algebraic equations in equations (5) and (6), $x = x_{n+j}$, $j = 0, r_1, r_2, r_3, 1, s_1, s_2,$ and $s_3$ was applied to equation (2) and $x = x_{n+j}$, $j = 0\,(1)\,2$ applied to equation (4).

$$y_{n+j} = \sum_{j=0}^{9} a_j x^{j-1} + \alpha_{c+i-1}\sum_{j=0}^{10}\frac{x^{j-3}}{(j-3)!}, j = 0, r_1, r_2, r_3, 1, s_1, s_2, s_3, \tag{5}$$

$$f_{n+j} = \sum_{j=3}^{9} j(j-1)(j-2)a_j x^{j-3} + a_{c+i-1}\sum_{j=3}^{10}\frac{\alpha_j x^{j-3}}{(j-3)!}\; j = 0\,(1)\,2 \tag{6}$$

Using the relation $x_{n+\frac{j}{4}} = x_n + \frac{jh}{4}$, (5) and (6) were written as matrix form and solved using CAS in Wolfram Mathematica to obtain the parameters $a_j$'s for j = 0,1,2,$\cdots$, 10 which were then substituted back into (2) to yields the following continuous scheme after some simplifications:

$$y(t) = \sum_{j=0}^{8}(\alpha_j y_{n+\dfrac{j}{4}} + h^3\beta_j f_{n+\dfrac{j}{4}}) \tag{7}$$

where $x = x_{n+t} = x_n + th$, the coefficients $\alpha_j' s$ and $\beta_j' s$ define the continuous scheme. Evaluating (7) at $t = 2$ yields the main linear multistep formula of the proposed Six Points y-Function Hybrid Block method. This gives

$$\begin{aligned} &y_{n+2} - \frac{130208}{16873}y_{n+\frac{7}{4}} + \frac{500782}{16873}y_{n+\frac{3}{2}} - \frac{678432}{16873}y_{n+\frac{5}{4}} + \frac{678432}{16873}y_{n+\frac{3}{4}} - \frac{500782}{16873}y_{n+\frac{1}{2}} \\ &+ \frac{130208}{16873}y_{n+\frac{1}{4}} - y_n = \frac{h^3}{1079872}(1025f_n + 295310f_{n+1} + 1025f_{n+2}) \end{aligned} \tag{8}$$

## 2.1 Block formulation of the derived formula

In keeping with [10], the normalized form of the general block method is given by

$$AY_i = Ey_n + h^{\mu-\rho}df(y_n) + h^{\mu-\rho}BF(y_n) \tag{9}$$

To derive the block formula described in (9), we combine the formulas in (39) with the first, and second derivative formulas obtained from (7), and write them in block form using the definition of the implicit block method in (9)

$$h^a\sum_{j=0}^{q}\varphi_{m,j}y^{\phi}_{n+j} = h^{\phi}\sum_{r=0}^{q}\nabla_{m,j}y^{\phi}_n + h^{p-\phi}\left(\sum_{j=0}^{q}\Delta_{m,j}f_n + \sum_{r=0}^{q}\eta_{m,j}f_{n+j}\right) \tag{10}$$

where $\rho$ represent the power of the derivative of the continuous method and $p$ represent the order of the problem to be solved. Equation (10) was solved for $j = 0\left(\dfrac{1}{4}\right)2$ in order to obtain the following block formulas that constitute the derived six points y-function hybrid block Method.

$$y_{n+\frac{1}{4}} = y_n + \frac{1}{4}hy'_n + \frac{1}{32}h^2y''_n + \frac{h^3}{928972800}\Big(2218464f_{n+\frac{1}{4}} - 2539000f_{n+\frac{1}{2}} + 2552864f_{n+\frac{3}{4}} - 1746630f_{n+1} + 733568f_{n+\frac{5}{4}} - 151264f_{n+\frac{3}{2}} + 1347197f_n + 4001f_{n+2}\Big) \quad (11)$$

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hy'_n + \frac{1}{8}h^2y''_n + \frac{h^3}{7257600}\Big(53703f_n + 153456f_{n+\frac{1}{4}} - 125720f_{n+\frac{1}{2}} + 127776f_{n+\frac{3}{4}} - 87270f_{n+1} + 36592f_{n+\frac{5}{4}} - 7536f_{n+\frac{3}{2}} + 199f_{n+2}\Big) \quad (12)$$

$$y_{n+\frac{3}{4}} = y_n + \frac{3}{4}hy'_n + \frac{9}{32}h^2y''_n + \frac{9h^3}{11468800}\Big(22899f_n + 80928f_{n+\frac{1}{4}} - 44400f_{n+\frac{1}{2}} + 55328f_{n+\frac{3}{4}} - 37890f_{n+1} + 15936f_{n+\frac{5}{4}} - 3288f_{n+\frac{3}{2}} + 87f_{n+2}\Big) \quad (13)$$

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{h^3}{226800}\Big(7523f_n + 29376f_{n+\frac{1}{4}} - 10840f_{n+\frac{1}{2}} + 20096f_{n+\frac{3}{4}} - 12600f_{n+1} + 5312f_{n+\frac{5}{4}} - 1096f_{n+\frac{3}{2}} + 29f_{n+2}\Big) \quad (14)$$

$$y_{n+\frac{5}{4}} = y_n + \frac{5}{4}hy'_n + \frac{25}{32}h^2y''_n + \frac{125h^3}{37158912}\Big(15753f_n + 65184f_{n+\frac{1}{4}} - 16760f_{n+\frac{1}{2}} + 47904f_{n+\frac{3}{4}} - 24270f_{n+1} + 11200f_{n+\frac{5}{4}} - 2304f_{n+\frac{3}{2}} + 61f_{n+2}\Big) \quad (15)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{3}{2}hy'_n + \frac{9}{8}h^2y''_n + \frac{9h^3}{89600}\Big(771f_n + 3312f_{n+\frac{1}{4}} - 600f_{n+\frac{1}{2}} + 2592f_{n+\frac{3}{4}} - 990f_{n+1} + 624f_{n+\frac{5}{4}} - 112f_{n+\frac{3}{2}} + 3f_{n+2}\Big) \quad (16)$$

$$y_{n+\frac{7}{4}} = y_n + \frac{7}{4}hy'_n + \frac{49}{32}h^2y''_n + \frac{343h^3}{132710400}\Big(41219f_n + 181728f_{n+\frac{1}{4}} - 22960f_{n+\frac{1}{2}} + 149408f_{n+\frac{3}{4}} - 42210f_{n+1} + 41216f_{n+\frac{5}{4}} - 2968f_{n+\frac{3}{2}} + 167f_{n+2}\Big) \quad (17)$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2y''_n + \frac{h^3}{14175}\Big(2001f_n + 8832f_{n+\frac{1}{4}} - 400f_{n+\frac{1}{2}} + 6912f_{n+\frac{3}{4}} - 660f_{n+1} + 1664f_{n+\frac{5}{4}} + 528f_{n+\frac{3}{2}} + 23f_{n+2}\Big) \quad (18)$$

$$y'_{n+\frac{1}{4}} = y'_n + \frac{1}{4}hy''_n + \frac{h^2}{116121600}\Big(1670723f_n + 3754488f_{n+\frac{1}{4}} - 3931520f_{n+\frac{1}{2}} + 3899984f_{n+\frac{3}{4}} - 2653050f_{n+1} + 1110712f_{n+\frac{5}{4}} - 228568f_{n+\frac{3}{2}} + 6031f_{n+2}\Big) \quad (19)$$

$$y'_{n+\frac{1}{2}} = y'_n + \frac{1}{2}hy''_n + \frac{h^2}{1814400}\Big(59989f_n + 220704f_{n+\frac{1}{4}} - 133420f_{n+\frac{1}{2}} + 146752f_{n+\frac{3}{4}} - 101310f_{n+1} + 42656f_{n+\frac{5}{4}} - 8804f_{n+\frac{3}{2}} + 233f_{n+2}\Big) \quad (20)$$

$$y'_{n+\frac{3}{4}} = y'_n + \frac{3}{4}hy''_n + \frac{3h^2}{358400}\Big(6157f_n + 25902f_{n+\frac{1}{4}} - 7365f_{n+\frac{1}{2}} + 15876f_{n+\frac{3}{4}} - 10485f_{n+1} + 4398f_{n+\frac{5}{4}} - 907f_{n+\frac{3}{2}} + 24f_{n+2}\Big) \quad (21)$$

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{113400}\Big(7943f_n + 35328f_{n+\frac{1}{4}} - 4880f_{n+\frac{1}{2}} + 26624f_{n+\frac{3}{4}} - 12810f_{n+1} + 5632f_{n+\frac{5}{4}} - 1168f_{n+\frac{3}{2}} + 31f_{n+2}\Big) \quad (22)$$

$$y'_{n+\frac{5}{4}} = y'_n + \frac{5}{4}hy''_n + \frac{25h^2}{4644864}\Big(16453f_n + 75480f_{n+\frac{1}{4}} - 4600f_{n+\frac{1}{2}} + 63760f_{n+\frac{3}{4}} - 16350f_{n+1} + 12824f_{n+\frac{5}{4}} - 2480f_{n+\frac{3}{2}} + 65f_{n+2}\Big) \quad (23)$$

$$y'_{n+\frac{3}{2}} = y'_n + \frac{3}{2}hy''_n + \frac{3h^2}{22400}\Big(799f_n + 3744f_{n+\frac{1}{4}} - 60f_{n+\frac{1}{2}} + 3392f_{n+\frac{3}{4}} - 450f_{n+1} + 1056f_{n+\frac{5}{4}} - 84f_{n+\frac{3}{2}} + 3f_{n+2}\Big) \quad (24)$$

$$y'_{n+\frac{7}{4}} = y'_n + \frac{7}{4}hy''_n + \frac{49h^2}{8294400}\Big(21361f_n + 99996f_{n+\frac{1}{4}} + 4970f_{n+\frac{1}{2}} + 88648f_{n+\frac{3}{4}} + 3360f_{n+1} + 30044f_{n+\frac{5}{4}} + 10654f_{n+\frac{3}{2}} + 167f_{n+2}\Big) \quad (25)$$

$$y'_{n+2} = y'_n + 2hy''_n + \frac{2h^2}{14175}\Big(1081f_n + 4416f_{n+\frac{1}{4}} + 1880f_{n+\frac{1}{2}} + 1408f_{n+\frac{3}{4}} + 4170f_{n+1} - 1216f_{n+\frac{5}{4}} + 2344f_{n+\frac{3}{2}} + 92f_{n+2}\Big) \quad (26)$$

$$y''_{n+\frac{1}{4}} = y''_n + \frac{h}{3870720}\Big(295767f_n + 1107792f_{n+\frac{1}{4}} - 935876f_{n+\frac{1}{2}} + 908064f_{n+\frac{3}{4}} - 612126f_{n+1} + 254992f_{n+\frac{5}{4}} - 52308f_{n+\frac{3}{2}} + 1375f_{n+2}\Big) \quad (27)$$

$$y''_{n+\frac{1}{2}} = y''_n + \frac{h}{15120}\Big(1111f_n + 5832f_{n+\frac{1}{4}} - 527f_{n+\frac{1}{2}} + 2224f_{n+\frac{3}{4}} - 1647f_{n+1} + 712f_{n+\frac{5}{4}} - 149f_{n+\frac{3}{2}} + 4f_{n+2}\Big) \quad (28)$$

$$y''_{n+\frac{3}{4}} = y''_n + \frac{h}{143360}\Big(10645f_n + 54000f_{n+\frac{1}{4}} + 12276f_{n+\frac{1}{2}} + 44896f_{n+\frac{3}{4}} \\ -21114f_{n+1} + 8496f_{n+\frac{5}{4}} - 1724f_{n+\frac{3}{2}} + 45f_{n+2}\Big) \tag{29}$$

$$y''_{n+1} = y''_n + \frac{h}{3780}\Big(279f_n + 1440f_{n+\frac{1}{4}} + 244f_{n+\frac{1}{2}} + 1728f_{n+\frac{3}{4}} \\ -36f_{n+1} + 160f_{n+\frac{5}{4}} - 36f_{n+\frac{3}{2}} + nf_{n+2}\Big) \tag{30}$$

$$y''_{n+\frac{5}{4}} = y''_n + \frac{5h}{774144}\Big(11503f_n + 58320f_{n+\frac{1}{4}} + 12700f_{n+\frac{1}{2}} + 63520f_{n+\frac{3}{4}} \\ +25650f_{n+1} + 24208f_{n+\frac{5}{4}} - 2420f_{n+\frac{3}{2}} + 55f_{n+2}\Big) \tag{31}$$

$$y''_{n+\frac{3}{2}} = y''_n + \frac{1}{560}h\Big(41f_n + 216f_{n+\frac{1}{4}} + 27f_{n+\frac{1}{2}} + 272f_{n+\frac{3}{4}} \\ +27f_{n+1} + 216f_{n+\frac{5}{4}} + 41f_{n+\frac{3}{2}}\Big) \tag{32}$$

$$y''_{n+\frac{7}{4}} = y''_n + \frac{7h}{552960}\Big(6759f_n + 22608f_{n+\frac{1}{4}} + 31612f_{n+\frac{1}{2}} - 18144f_{n+\frac{3}{4}} \\ +76482f_{n+1} - 31472f_{n+\frac{5}{4}} + 49644f_{n+\frac{3}{2}} + 751f_{n+2}\Big) \tag{33}$$

$$y''_{n+2} = y''_n + \frac{1}{945}h\Big(115f_n + 1312f_{n+\frac{1}{2}} - 2048f_{n+\frac{3}{4}} \\ +3132f_{n+1} - 2048f_{n+\frac{5}{4}} + 1312f_{n+\frac{3}{2}} + 115f_{n+2}\Big) \tag{34}$$

# 3 Analysis of the Properties of the Derived Method

This section present the analysis of the basic properties of the Six-Point y-Function Hybrid Block method

## 3.1 Order of the Method

Assuming the linear operator $\mathcal{L}$ associated with the $k-step$ hybrid scheme is defined as

$$\mathcal{L}\{y(x):h\} = \sum_{j=0}^{k}[\alpha_j y_{n+j} - h^3(\beta_j f(x_{n+j}))] \tag{35}$$

where $\alpha_0$ and $\beta_0$ are not both zero and $y(x) \in C^{(n)}[a,b]$. Expanding $y_{n+j}$ and $f_{n+j}$ as Taylors series expansion gives

$$y_{n+j} = y(x_n + jh) = y(x_n) + jhy'(x_n) + \frac{(jh)^2}{2!}y''(x_n) + \cdots + \frac{(jh)^{p+3}}{(p+3)!}y^{p+3}(x_n) \tag{36}$$

$$f_{n+j} = y^{(iii)}(x_n+jh) = y^{(iii)}(x_n)+jhy(iv)(x_n)+\frac{(jh)^2}{2!}y(v)(x_n)+\frac{(jh)^3}{3!}y(vi)(x_n)+\cdots+\frac{(jh)^p}{(p)!}y^{p+3}(x_n) \tag{37}$$

Substituting and collecting like terms gives

$$\mathcal{L}\{y(x),h\} = C_0 y(x) + C_1 hy'(x_n) + C_2 h^2 y''(x_n) + \ldots C_p h^p y^p(x) \tag{38}$$

Therefore, applying the linear operator L(37) to determine the order and error constant of the main method

$$\begin{aligned} &y_{n+2} - \frac{130208}{16873}y_{n+\frac{7}{4}} + \frac{500782}{16873}y_{n+\frac{3}{2}} - \frac{678432}{16873}y_{n+\frac{5}{4}} + \frac{678432}{16873}y_{n+\frac{3}{4}} - \frac{500782}{16873}y_{n+\frac{1}{2}} \\ &+ \frac{130208}{16873}y_{n+\frac{1}{4}} - y_n = \frac{h^3}{1079872}(1025f_n + 295310f_{n+1} + 1025f_{n+2}) \end{aligned} \tag{39}$$

where $C_p$ are constants. Since $C_0 = C_1 = C_2 = \ldots = C_{p+2} = 0$, $C_{p+3} \neq 0$ is the error constant. The method is of order 8 with error constant $c_{p+3} = -\frac{420767}{33439057182720}$.

### 3.2 Consistency of the method

From equation (8), the first ($\rho$) and second ($\sigma$) characristics polynomials of the method are given as:
$\rho(r) = r^2 - \frac{130208}{16873}r^{\frac{7}{4}} + \frac{500782}{16873}r^{\frac{3}{2}} - \frac{678432}{16873}r^{\frac{5}{4}} + \frac{678432}{16873}r^{\frac{3}{4}} - \frac{500782}{16873}r^{\frac{1}{2}} + \frac{130208}{16873}r^{\frac{1}{4}} - r^0 = 0$

$\sigma(r) = \frac{1025}{1079872}r^0 + \frac{295310}{1079872}r + \frac{1025}{1079872}r^2$

Applying the following conditions, it shows that the method presented in this article is consistent

(i) The order of the method p = 8 > 1

which is obvious that condition (i) is satisfied

$(ii) \sum_{j=0}^{k} \alpha_j = 0$

*since* $\alpha_0 = 1, \alpha_1 = \frac{-130208}{16873}, \alpha_2 = \frac{500782}{16873}, \alpha_3 = \frac{-678432}{16873}, \alpha_4 = \frac{678432}{16873}, \alpha_5 = \frac{-500782}{16873}, \alpha_6 = \frac{130208}{16873}, \alpha_7 = -1$

$$\sum_{J=0}^{k} \alpha_j = 1 - \frac{130208}{16873} + \frac{500782}{16873} - \frac{678432}{16873} + \frac{678432}{16873} - \frac{500782}{16873} + \frac{130208}{16873} - 1 = 0$$

condition (ii) is satisfied

$(iii)\rho(r) = \rho^{'}(r) = 0$

$\rho(r) = r^2 - \frac{130208}{16873}r^{\frac{7}{4}} + \frac{500782}{16873}r^{\frac{3}{2}} - \frac{678432}{16873}r^{\frac{5}{4}} + \frac{678432}{16873}r^{\frac{3}{4}} - \frac{500782}{16873}r^{\frac{1}{2}} + \frac{130208}{16873}r^{\frac{1}{4}} - r^0$

$\rho(1) = 1 - \frac{130208}{16873} + \frac{500782}{16873} - \frac{678432}{16873} + \frac{678432}{16873} - \frac{500782}{16873} + \frac{130208}{16873} - 1 = 0$

$\rho'(r) = 2r - \frac{227864}{16873}r^{\frac{3}{4}} + \frac{751173}{16873}r^{\frac{1}{2}} - \frac{848040}{16873}r^{\frac{1}{4}} + \frac{508824}{16873}r^{-\frac{1}{4}} - \frac{250391}{16873}r^{-\frac{1}{2}} + \frac{32552}{16873}r^{-\frac{3}{4}}$

$\rho'(1) = 2 - \frac{227864}{16873} + \frac{751173}{16873} - \frac{848040}{16873} + \frac{508824}{16873} - \frac{250391}{16873} + \frac{32552}{16873} = 0$

it follows that $\rho(1) = \rho'(1) = 0$ showing that the condition (iii) is satisfied as well

$(iv)\rho'''(r) = n!\sigma(r)$ and for the principal root r=1

$\rho(r) = r^2 - \frac{130208}{16873}r^{\frac{7}{4}} + \frac{500782}{16873}r^{\frac{3}{2}} - \frac{678432}{16873}r^{\frac{5}{4}} + \frac{678432}{16873}r^{\frac{3}{4}} - \frac{500782}{16873}r^{\frac{1}{2}} + \frac{130208}{16873}r^{\frac{1}{4}} - r^0$

and

$\sigma(r) = \frac{1025}{1079872}r^0 + \frac{295310}{1079872}r + \frac{1025}{1079872}r^2$

$\rho'''(r) = \frac{85449}{33746}r^{-\frac{5}{4}} - \frac{751173}{67492}r^{-\frac{3}{2}} + \frac{318015}{33746}r^{-\frac{7}{4}} + \frac{318015}{33746}r^{-\frac{9}{4}} - \frac{751173}{67492}r^{-\frac{5}{2}} + \frac{85449}{33746}r^{-\frac{11}{4}}$

and

$\rho'''(1) = \frac{85449}{33746} - \frac{751173}{67492} + \frac{318015}{33746} + \frac{318015}{33746} - \frac{751173}{67492} + \frac{85449}{33746} = \frac{55755}{33746}$

$\sigma(1) = \frac{1025}{1079872} + \frac{295310}{1079872} + \frac{1025}{1079872} = \frac{18585}{67492}$

Therefore

$3!\sigma(1) = 6 \times \frac{18585}{67492} = \frac{55755}{33746}$

$\rho'''(1) = 3!\sigma(1)$

That is

$\rho'''(1) = \frac{55755}{33746} = 3!\sigma(1)$

For the principal root $r = 1$; it is observed that last condition above is satisfied, hence the method is consistent. condition (iv) is satisfied.

### 3.3 Zero stability of the method

using (11)-(34) as $h \to 0$, we have

$det[\lambda A^{(0)} - A^{(0)}]$

$$= det \left[ \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right]$$

$= \lambda^8 - \lambda^7 = 0$

By solving for $\lambda$ we have

$\lambda^7 (\lambda - 1) = 0$

solving the above equation for $\lambda$, $\lambda = 1, \lambda = 0, \lambda = 0, \lambda = 0, \lambda = 0, \lambda = 0, \lambda = 0, \lambda = 0$

Hence, The method is zero-stable.

### 3.4 Convergency of the method

The necessary and sufficient condition for any method to be convergent is for it to be consistent and zero stable [20]. Thus, since it has been successfully shown above that the new method with y-function, six off step points is consistent and zero stable, thus the method is convergent.

### 3.5 Region of Absolute Stability of the method

The region of absolute stability of the method was examined via the procedure discussed in [2]. The stability matrix can be expressed as

$$M(z) = zB(I - zA)^{-1}U + V \tag{40}$$

together with the Stability function

$$p(n, z) = det(-M(z) + nI) \tag{41}$$

for the Stability properties, the method (11) – (18) was formulated as a general linear method of the form,

$$\begin{bmatrix} Y \\ --- \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} A & & U \\ --- & --- & --- \\ B & & V \end{bmatrix} \begin{bmatrix} h^3 f(u) \\ --- \\ Y_{i-1} \end{bmatrix} \tag{42}$$

where $n$ represents the roots of the first characteristics polynomial, and

$$Y_{i-1} = \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_n \end{bmatrix}, Y_{i+1} = \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+2} \end{bmatrix},$$

$$A = \begin{bmatrix} \frac{7703}{3225600} & -\frac{12695}{4644864} & \frac{79777}{29030400} & -\frac{6469}{3440640} & \frac{5731}{7257600} & -\frac{4727}{29030400} & 0 & \frac{4001}{928972800} \\ \frac{3197}{151200} & -\frac{449}{25920} & \frac{1331}{75600} & -\frac{2909}{241920} & \frac{2287}{453600} & -\frac{157}{151200} & 0 & \frac{199}{7257600} \\ \frac{22761}{358400} & -\frac{999}{28672} & \frac{2223}{51200} & -\frac{34101}{1146880} & \frac{2241}{179200} & -\frac{3699}{1433600} & 0 & \frac{783}{11468800} \\ \frac{68}{525} & -\frac{271}{5670} & \frac{1256}{14175} & -1/18 & \frac{332}{14175} & -\frac{137}{28350} & 0 & \frac{29}{226800} \\ \frac{12125}{55296} & -\frac{261875}{4644864} & \frac{62375}{387072} & -\frac{505625}{6193152} & \frac{3125}{82944} & -\frac{125}{16128} & 0 & \frac{7625}{37158912} \\ \frac{1863}{5600} & -\frac{27}{448} & \frac{729}{2800} & -\frac{891}{8960} & \frac{351}{5600} & -\frac{9}{800} & 0 & \frac{27}{89600} \\ \frac{216433}{460800} & -\frac{98441}{1658880} & \frac{1601467}{4147200} & -\frac{160867}{1474560} & \frac{55223}{518400} & -\frac{127253}{16588800} & 0 & \frac{57281}{132710400} \\ \frac{2944}{4725} & -\frac{16}{567} & \frac{256}{525} & -\frac{44}{945} & \frac{1664}{14175} & \frac{176}{4725} & 0 & \frac{23}{14175} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1347197}{928972800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{663}{89600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{206091}{11468800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7523}{226800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{656375}{12386304} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6939}{89600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{14138117}{132710400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{667}{4725} \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} y_{n+1/4} \\ y_{n+1/2} \\ y_{n+3/4} \\ y_{n+1} \\ y_{n+5/4} \\ y_{n+3/2} \\ y_{n+7/4} \\ y_{n+2} \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad f(y) = \begin{bmatrix} f_{n+1/4} \\ f_{n+1/2} \\ f_{n+3/4} \\ f_{n+1} \\ f_{n+5/4} \\ f_{n+3/2} \\ f_{n+7/4} \\ f_{n+2} \end{bmatrix}$$

Now, putting the values of the variables A, B, U, V , M and I in equations (41) and (42), to obtained the Stability function. The stability polynomial (42) and its first derivatives (43) are then plotted in MATLAB (R2012a) environment. It should be noted that $M$ is 8 by 8 identity matrix. The region of absolute stability (RAS) of the method is displayed in the Figure 1 below;

$$f(z) = \left(\eta + \frac{67057\,z^2}{3888000} + \frac{4966\,z}{4725} - 1\right)\eta^7 \tag{43}$$

$$f'(z) = \frac{(469399\,z + 14302080)\,\eta^7}{13608000} \tag{44}$$

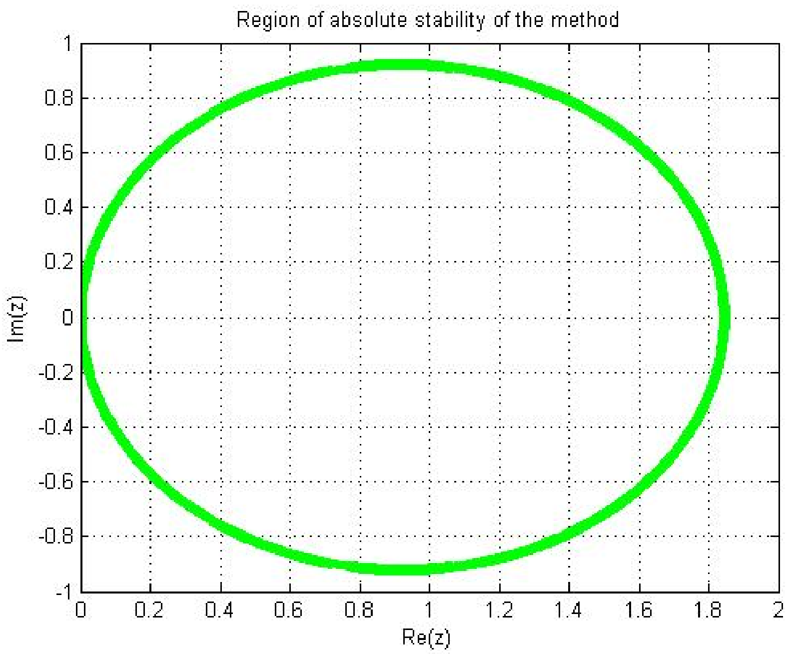


Figure 1: **Region of Absolute Stability of the new method. The figure depicts the area where the method is stable.**

The region of absolute stability of the method is P-stable, since the region consists of the complex plane outside the enclosed figure and its interval of periodicity lies between $(0, 1.83)$ which falls within the interval of periodicity for P-stability. $(0, \infty)$.

# 4 Numerical Experiments

In this session, the usability of the proposed method was tested on some numerical examples that mesured the accuracy of the method by calculating the absolute error it generated when applied to the sample problems.

## 4.1 Problem 1

The general third-order IVP of ordinary differential equation
$y''' = 3sin(x), y(0) = 1; y'(0) = 0; y''(0) = -2; h = 0.1$
Exact solution: $y(x) = 3cosx + \frac{x^2}{2} - 2$
Source: Kashkari and Algarni [12]

Table 1: Numerical Results for problem 1, k=2, p=8, h=0.1

| $x$ | Exact Solution | Computed Solution | Error in new method |
|---|---|---|---|
| 0.10 | 0.99001249583407729828669 | 0.990012495834077298743777 | 4.57087E-19 |
| 0.20 | 0.96019973352372489337259 | 0.960199733523724893573521 | 2.00931E-19 |
| 0.30 | 0.91100946737681805892693 | 0.911009467376818052888852 | 6.03807E-18 |
| 0.40 | 0.84318298200865524839558 | 0.843182982008655245939563 | 2.45601E-18 |
| 0.50 | 0.75774768567111814834885 | 0.757747685671118141235350 | 7.11350E-18 |
| 0.60 | 0.65600684472903489172286 | 0.656006844729034900154670 | 8.43181E-18 |
| 0.70 | 0.53952656185346527876758 | 0.539526561853465294538051 | 1.57704E-17 |
| 0.80 | 0.41012012804149626276225 | 0.410120128041496317611556 | 5.48493E-17 |
| 0.90 | 0.26982990481199336945415 | 0.269829904811993457496799 | 8.80426E-17 |
| 1.00 | 0.12090691760441915220281 | 0.120906917604419317412088 | 1.65209E-16 |

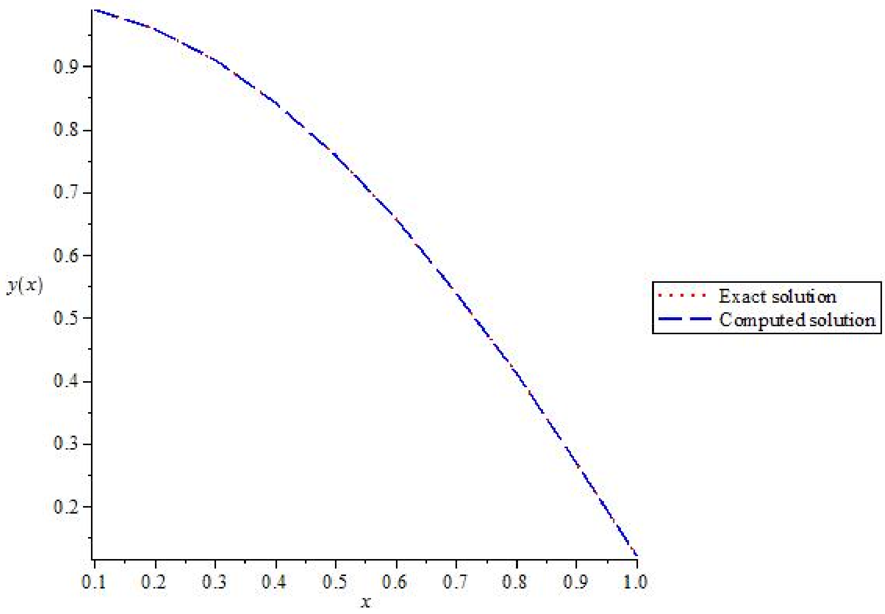


Figure 2: Comparison of Exact solution and Computed solution for problem 1

Table 2: Comparison of the results with other existing methods for problem 1.

| $x$ | Error in New Method | Error in [16] | Error in [12] |
|---|---|---|---|
| 0.10 | 4.57087E-19 | 3.774758E-15 | 4.1078E-15 |
| 0.20 | 2.00931E-19 | 2.220446E-16 | 1.6875E-14 |
| 0.30 | 6.03807E-18 | 1.776357E-15 | 5.0848E-14 |
| 0.40 | 2.45601E-18 | 1.665335E-15 | 1.1779E-13 |
| 0.50 | 7.11350E-18 | 4.218847E-15 | 2.4081E-13 |
| 0.60 | 8.43181E-18 | 1.110223E-15 | 4.3709E-13 |
| 0.70 | 1.57704E-17 | 7.105427E-15 | 7.3708E-13 |
| 0.80 | 5.48493E-17 | 7.771561E-15 | 1.1662E-12 |
| 0.90 | 8.80426E-17 | 9.714451E-15 | 1.7587E-12 |
| 1.00 | 1.65209E-16 | .1.507128E-14 | 2.5466E-12 |

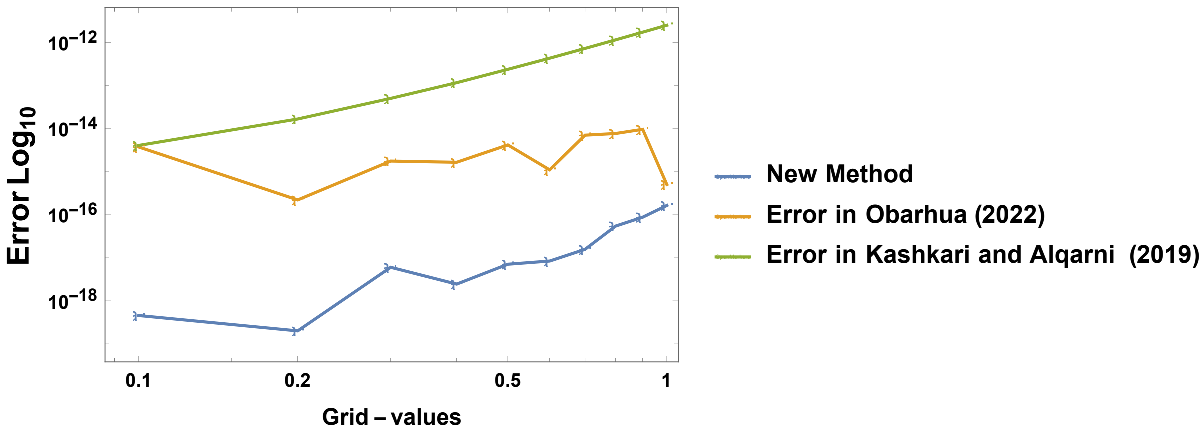


Figure 3: Comparison of absolute errors of the proposed method on problem 1 as compared with Obarhua[16] and Kashkari[12]

## 4.2 Problem 2

The second sample problem considered in this work is

$y'''(x) = x - 4y'(x); y(0) = 0; y'(0) = 0; y''(0) = 1; h = 0.1$

Exact Solution: $= y(x) = \frac{3}{16}\left(1 - cos(x) + \frac{x^2}{8}\right)$.

Source: Obarhua [16]

Table 3: Numerical Results for problem 2, h=0.1 for problem 2.

| $x$ | Exact Solution | Computed Solution | Error in new method |
|---|---|---|---|
| 0.10 | 0.00498751665476719416421300 | 0.00498751665476718485 36 | 9.31061E-18 |
| 0.20 | 0.01980106362445904697527 60 | 0.01980106362445899656 0 | 5.04152E-17 |
| 0.30 | 0.04399957220443531926732 20 | 0.04399957220443511585632 74 | 2.03410E-16 |
| 0.40 | 0.07686749199740648357735 90 | 0.07686749199740597848444 54 | 5.05092E-16 |
| 0.50 | 0.11744331764972380298732 4 | 0.11744331764972273497884 6 | 1.06800E-15 |
| 0.60 | 0.16455792103562370419280 5 | 0.16455792103562175082053 6 | 1.95337E-15 |
| 0.70 | 0.21688116070620482400936 0 | 0.21688116070620155966879 1 | 3.26434E-15 |
| 0.80 | 0.27297491043149163616358 2 | 0.27297491043148657638884 5 | 5.05977E-15 |
| 0.90 | 0.33135039275495382287187 6 | 0.33135039275494642420608 8 | 7.39866E-15 |
| 1.00 | 0.39052753185258919756204 4 | 0.39052753185257888887810 5 | 1.03086E-14 |

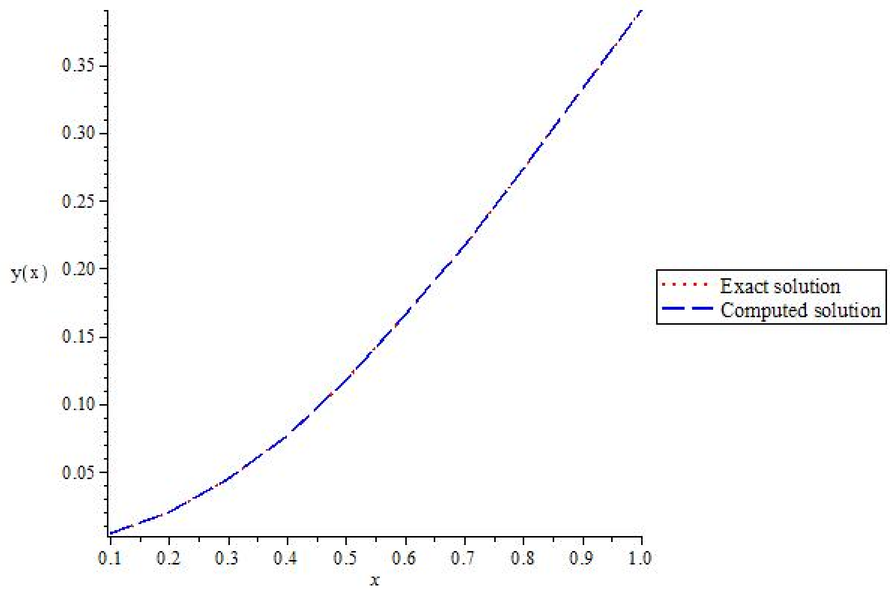


Figure 4: Comparison of Exact solution and Computed solution for problem 2

Table 4: Comparison of the results of the developed method with other existing method for problem 2.

| $x$ | Error in New Method | Error in [17] | Error in [15] |
|---|---|---|---|
| 0.10 | 9.31061E-18 | 6.44E-16 | 2.14E-12 |
| 0.20 | 5.04152E-17 | 2.19E-14 | 3.79E-14 |
| 0.30 | 2.03410E-16 | 2.35E-13 | 1.32E-12 |
| 0.40 | 5.05092E-16 | 1.68E-15 | 4.81E-12 |
| 0.50 | 1.06800E-15 | 5.10E-14 | 5.24E-13 |
| 0.60 | 1.95337E-15 | 3.44E-13 | 8.52E-14 |
| 0.70 | 3.26434E-15 | 2.85E-15 | 4.69E-11 |
| 0.80 | 5.05977E-15 | 9.50E-14 | 7.70E-12 |
| 0.90 | 7.39866E-15 | 1.24E-13 | 1.66E-12 |
| 1.00 | 1.03086E-14 | 1.10E-14 | 1.75E-10 |

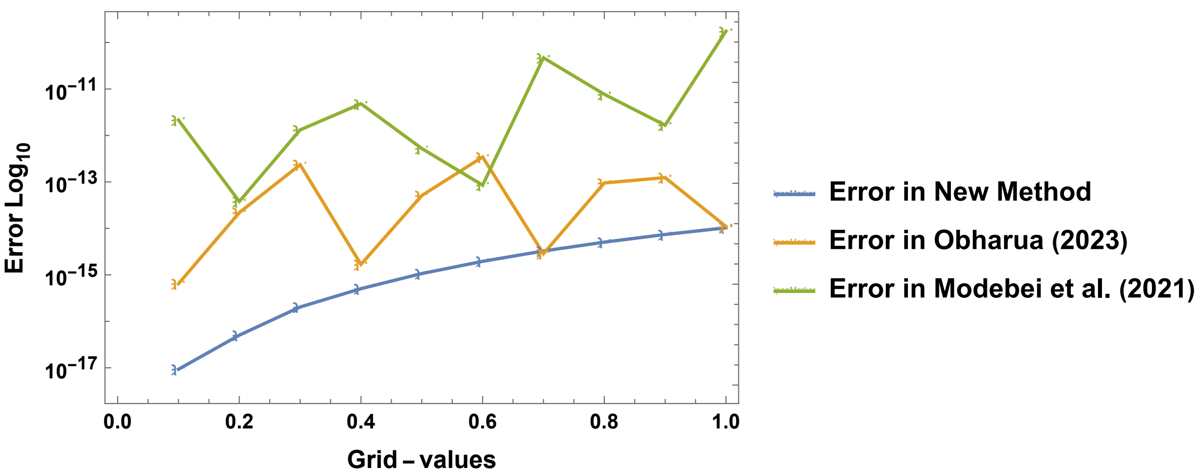


Figure 5: Comparison of absolute errors of the proposed method on problem 2 as compared with Obarhua[17] and MOdebei[15]

### 4.3 Problem 3

The third sample problem considered in this work is

$y''' = y'(x)\left(2xy''(x) + y'(x)\right), y(0) = 1; y'(0) = \frac{1}{2}; y''(0) = 0; h = 0.01$

Exact Solution: $= y(x) = 1 + \frac{1}{2}log\left(\frac{2+x}{2-x}\right)$.

Source: Adoghe et.al. [7]

Table 5: Numerical Results for h=0.1 for problem 3.

| $x$ | Exact Solution | Computed Solution | Error in new method |
|---|---|---|---|
| 0.10 | 1.00500004166729167782760 | 1.00500004166718750558046 | 1.041722E-13 |
| 0.20 | 1.01000033335333476201588 | 1.01000033335000071431156 | 3.334047E-12 |
| 0.30 | 1.01500112515189941275441 | 1.01500112513803908609349 | 1.386032E-11 |
| 0.40 | 1.02000266730684958071704 | 1.02000266725326083446070 | 5.358874E-11 |
| 0.50 | 1.02500521028733068820901 | 1.02500521016165550841686 | 1.256751E-10 |
| 0.60 | 1.03000900486312647432594 | 1.03000900458177074643435 | 2.813557E-10 |
| 0.70 | 1.03501430218024176087116 | 1.03501430166055985418211 | 5.196819E-10 |
| 0.80 | 1.04002135383676821291189 | 1.04002135291527270021578 | 9.214955E-10 |
| 0.90 | 1.04503041195909056561051 | 1.04503041047730755162854 | 1.481783E-09 |
| 1.00 | 1.05004172927849126824578 | 1.05004172696707828343083 | 2.311412E-09 |

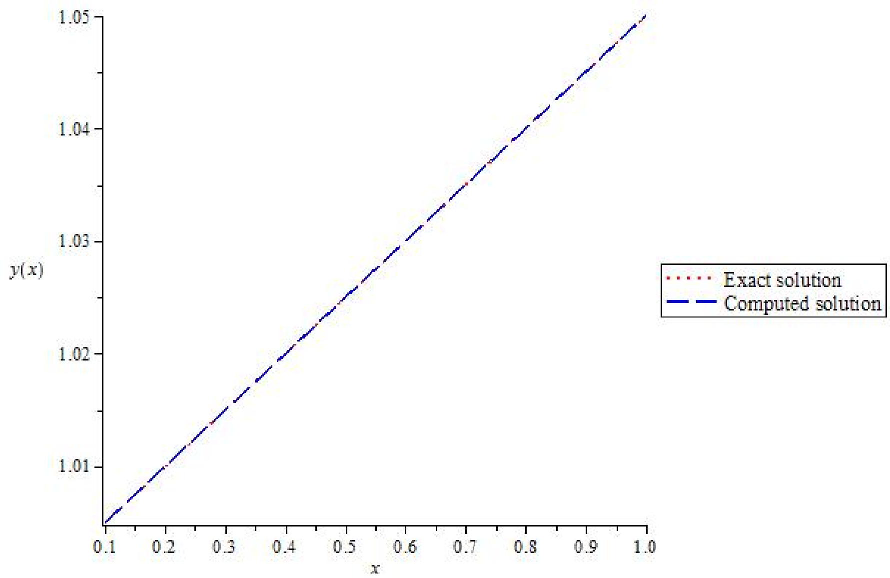


Figure 6: Comparison of Exact solution and Computed solution for problem 3

Table 6: Comparison of the errors in new method and with other methods for problem 3.

| $x$ | Error in New method | Error in [16] | Error in [7] |
|---|---|---|---|
| 0.10 | 1.041722E-13 | 9.058754E-11 | 1.93148E-08 |
| 0.20 | 3.334047E-12 | 8.052263E-10 | 5.60825E-07 |
| 0.30 | 1.386032E-11 | 3.004335E-10 | 3.75510E-06 |
| 0.40 | 5.358874E-11 | 7.869677E-09 | 1.34028E-05 |
| 0.50 | 1.256751E-10 | 1.700601E-09 | 3.25906E-05 |
| 0.60 | 2.813557E-10 | 3.258627E-09 | 5.81649E-05 |
| 0.70 | 5.196819E-10 | 5.755972E-08 | 7.15239E-05 |
| 0.80 | 9.214955E-10 | 9.595413E-08 | 2.56483E-05 |
| 0.90 | 1.481783E-09 | 1.533198E-08 | 1.70915E-05 |
| 1.00 | 2.311412E-09 | 2.373818E-07 | 6.70643E-04 |

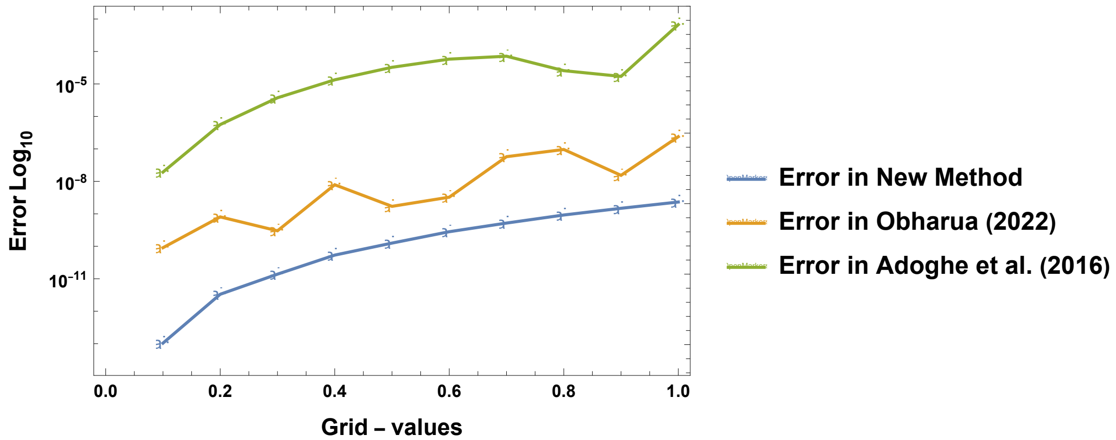


Figure 7: Comparison of absolute errors of the proposed method on problem 3 as compared with Obarhua[16] and Adoghe[7]

## 4.4 Problem 4

The last sample problem considered in this work is

$y''' = e^x, y(0) = 1; y'(0) = -1; y''(0) = 3; h = 0.1$

Exact Solution: $= y(x) = 2 + 2x^2 + e^x$.

Source: Olabode [18]

Table 7: Numerical Results for h=0.1 for problem 4

| $x$ | Exact Solution | Computed Solution | Error in new method |
|---|---|---|---|
| 0.10 | 0.91482908192435237518829 | 0.914829081924352375024855 | 1.63435E-19 |
| 0.20 | 0.85859724183983016607893 | 0.858597241839830165218337 | 8.60593E-19 |
| 0.30 | 0.83014119242399689601626 | 0.830141192423996893477418 | 2.53884E-18 |
| 0.40 | 0.82817530235872968217515 | 0.828175302358729676568749 | 5.60640E-18 |
| 0.50 | 0.85127872929987185315135 | 0.851278729299871842530213 | 1.06211E-17 |
| 0.60 | 0.89788119960949102512463 | 0.897881199609491007042979 | 1.80816E-17 |
| 0.70 | 0.96624729252952347837545 | 0.966247292529523449706122 | 2.86693E-17 |
| 0.80 | 1.05445907150753239542046 | 1.05445907150753235242731 | 4.29931E-17 |
| 0.90 | 1.16039688884305033619987 | 1.16039688884305027431451 | 6.18853E-17 |
| 1.00 | 1.28171817154095476463971 | 1.28171817154095467854993 | 8.60897E-17 |

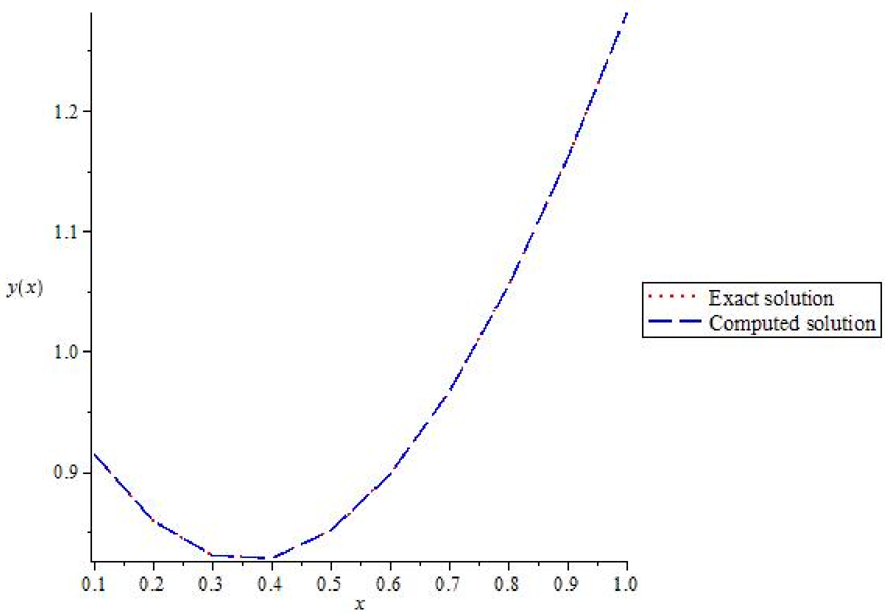


Figure 8: Comparison of Exact solution and Computed solution for problem 4

Table 8: Comparison of the result of the developed method with other methods for test problem 4.

| $x$ | Error in New Method | Error in [7] | Error in [18] |
|---|---|---|---|
| 0.10 | 1.63435E-19 | 0.000 | 7.56477E-11 |
| 0.20 | 8.60593E-19 | 2.8644E-14 | 2.60171E-10 |
| 0.30 | 2.53884E-18 | 1.6720E-12 | 5.76003E-10 |
| 0.40 | 5.60640E-18 | 2.9932E-11 | 8.41271E-10 |
| 0.50 | 1.06211E-17 | 3.1673E-11 | 1.00013E-09 |
| 0.60 | 1.80816E-17 | 9.1890E-11 | 1.09051E-09 |
| 0.70 | 2.86693E-17 | 8.9834E-11 | 1.07048E-09 |
| 0.80 | 4.29931E-17 | 1.9682E-10 | 1.49247E-09 |
| 0.90 | 6.18853E-17 | 2.1110E-10 | 3.15695E-09 |
| 1.00 | 8.60897E-17 | 4.9310E-10 | 4.45905E-09 |

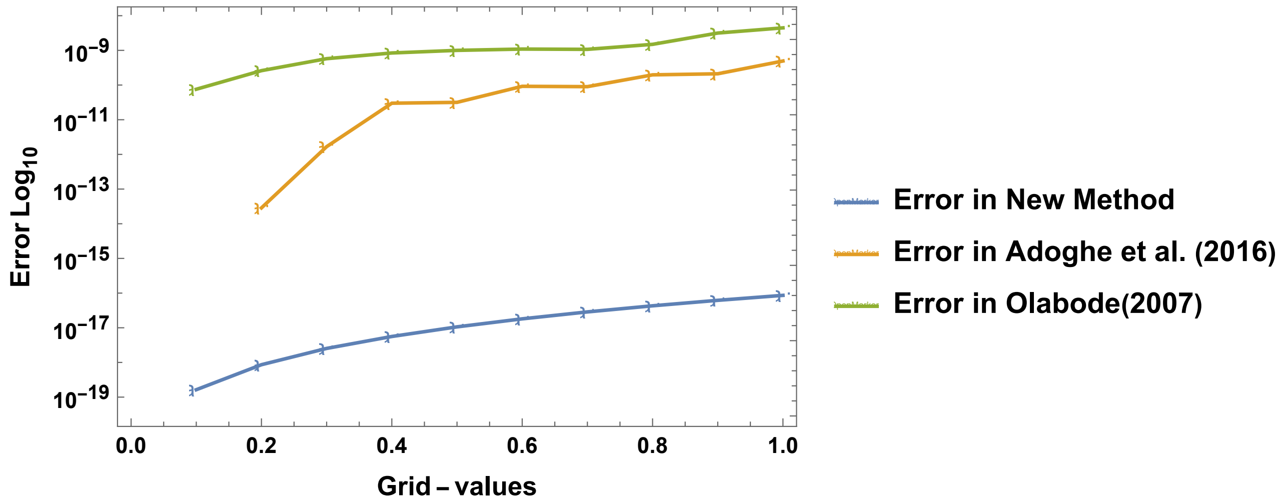


Figure 9: Comparision of absolute errors of the proposed method on problem 4 as compared with Adoghe[7] and Olabode[18]

## 5 Conclusion

In this study, an order eight, six points y-function hybrid block method that solve initial value problems of third order ordinary differential equations without the need to reduce to system of first order ordinary differential equations was developed. The collocation and interpolation points were selected to collect the hybrid points at y-values. The method was zero stable, consistent, and convergent, p-stable as shown by the region of absolute stability. The accuracy and the usability of the developed method was tested by applying it to solve four numerical examples and when compared to [7],[12],[15],[16],[17],[18] was found to be efficient as it gives a minimal error, hence has higher accuracy for handling the direct solution of third-order intial value problems of ordinary differential equation.

The results of these comparisons show that the method is superior and can be used to numerically solve third order initial value problems involving ordinary differential equations.

## References


[1] Awoyemi, D.O.and Idowu, O.M.(2005): A Class of Hybrid Collocation Methods for third-order ordinary differential equations.International Journal of Computer Mathematics, 82, 1287-1293.

[2] Lambert J.D (1973): Computational method in Ordinary Differential Equation in ODEs, John Wiley and Sons, New York.

[3] Fatula, S.O. (1988). Numerical methods for IVPs in ordinary differential equtions. Academic Press Inc. Harcourt Brace Jovanovich Publishers, New York.

[4] Lambert J.D. (1991): Numerical Methods for Ordinary Differential Systems: The Initial Value Problems. New York:John Wiley, pp 6-7.

[5] Awoyemi, D.O. (2000): A class of continuous method for general second order initial value problems in ordinary differential equations. International Journal of Computer Mathematics, 72, 29-37.

[6] Awoyemi, D.O., Kayode, S.J. and Adoghe, L.O. (2014): A Five-Step P-Stable Method for the Numerical Integration of Third Order Ordinary Differential Equations. American Journal of Computational Mathematics, 4, 119-126. http://dx.doi.org/10.4236/ajcm.2014.43011.

[7] Adoghe L. O, Ogunware B. G and Omole E. O. (2016): A family of symmetric implicit higher order methods for the solution of third order initial value problems in ordinary differential equations, Journal of Theoretical Mathematics and Applications, 6(3): 67-84.

[8] Allogmany R, Ismail F. (2020): Implicit Three-Point Block Numerical Algorithm for Solving Third Order Initial Value Problem Directly With Applications Mathematics. 8(10):1771

[9] Bankole and Ezekiel (2020): A Class of Irrational Linear Multi-step Block Method for the Direct Numerical Solution of Third Order Ordinary Differential Equations, Turkish Journal of Analysis and Number Theory, 2020, Vol. 8, No. 2, 21-27 DOI:10.12691/tjant-8-2-1

[10] Duromola M.K and Momoh A. L. (2019):Hybrid numerical method with block extension for direct solution of third order ordinary differential equations American Journal of Computational Mathematics 9(02),68

[11] Duromola M. K, Momoh A. L and Adeleke J. M. (2022):One-step Hybrid Block Method for Directly Solving Fifth-order Initial Value Problems of Ordinary Differential Equations Asian Research Journal of Mathematics 18(1), 53-64

[12] Kashkari KSH, Alqarni S. (2019): Optimization of Two-Step Block Method with Three Hybrid Points for Solving Third Order Initial Value Problems, Journal of Nonlinear Science and Application. 12:450-469.

[13] Kayode, S.J. (2011): A Class of One Point Zero-Stable Continuous Hybrid Methods for Direct Solution Of Second-Order Differential Equations. African Journal of Mathematics and Computer Science Research.

[14] Kuboye J.O., Adebayo T.E., Ajibade F.D., Mmaduakor C.O. and Abimbola R.O.(2021):A Single Step Block Numerical Integrator for Direct Solution of Initial Value Problems of Third Order Ordinary Differential Equations. Kasu Journal of Mathematical Sciences (KJMS) http://www.journal.kasu.edu.ng/index.php/kjms

[15] Modebei M.I , Olaiya,O.O. and Onyekonwu,A.C,(2021): A 3-step fourth derivatives method for numerical integration of third order ordinary differential equations Int.J. Math. Ana Opt.; Theory and Applications.

[16] Obarhua F. O. (2022): A Predictor-Corrector Hybrid Method for Numerical Approximation of Third-order Initial Value Problems, International Journal of Scientific and Engineering Research, 13(3), Pp. 177-199.

[17] Obarhua F. O. (2023): Three-step Four-point Optimized Hybrid Block Method for Direct Solution of General Third Order Differential Equations, Asian Research Journal of Mathematics, 19(6), Pp. 25-44, Article no.ARJOM.98197.

[18] Olabode B.T.(2007): Some linear multi-step methods for special and general third order initial value problems of ordinary differential equations Ph.D Thesis, the Department of Mathematical Sciences Federal University of Technology Akure.

[19] Olabode B. T.(2009):An accurate Scheme by Block Method for Third Order Ordinary Differential Equations, Pacific Journal of Science and Technology, 10(1),136-142.

[20] Henrici P.(1962): Discrete variable method in ordinary differential equations. John Wiley and Sons, New York.