Some Convergence Theorems of Henstock-Kurzweil-Dunford-Stieltjes Integral and Henstock-Kurzweil-Pettis-Stieltjes Integral of Banach-valued Functions on $\mathbb{R}$

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Original Article

Abstract

Let X be an arbitrary Banach space. The establishment of the Henstock-Kurzweil-Dunford-Stieltjes (HKDS) Integral and Henstock-Kurzweil-Pettis-Stieltjes (HKPS) Integral of an X-valued function over $\mathbb{R}$ shows a viable and more generalized integration process utilizing the notion of dual spaces and weakly measurable functions. In this manuscript, the authors have discussed about some convergence theorems of Henstock-Kurzweil-Dunford-Stieltjes Integral and Henstock-Kurzweil-Pettis-Stieltjes Integral of X-valued functions on $\mathbb{R}$ via uniform convergence with respect to the integrand and integrator.

Keywords: HKDS integral, HKPS integral, Bounded variation, Uniform Convergence.
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1 Introduction

For an arbitrary Banach space X with its corresponding dual and second dual space, X^* and X^{**}, it is known that an X-valued function f over a closed interval $[a,b]$ is said to be Henstock-Kurzweil-Dunford-Stieltjes integrable with respect to a function $g : [a,b] \to \mathbb{R}$ of bounded variation over $[a,b]$ if:

- (i) For all $x^* \in X^*$, the function $x^* \circ f : [a,b] \to \mathbb{R}$ is HKS-integrable with respect to g on $[a,b]$.
- (ii) For each compact subinterval $E \subset [a,b]$, there exists an element $x_E^{**} \in X^{**}$ such that

$$x_E^{**} \circ x^* = (\mathbf{HKS}) \int_E x^* \circ f \; dg$$

for all $x^* \in X^*$.

For a compact subinterval $E \subset [a,b]$, the value of HKDS-integral on E is

$$(\mathbf{HKDS}) \int_E f \; dg = x_E^{**}.[11]$$

On the other hand, if $f : [a,b] \to X$ is HKDS-integrable such that $(\mathbf{HKDS}) \displaystyle\int_E f\, dg \in X$, particularly $(\mathbf{HKDS}) \displaystyle\int_E f\, dg \in e(X)$, for every compact subinterval $E \subset [a,b]$, where e is the canonical embedding of X into X^{**}, then f is called Henstock-Kurzweil-Pettis-Stieltjes integrable with respect to g and

$$(\mathbf{HKPS}) \int_E f\, dg = (\mathbf{HKDS}) \int_E f\, dg$$

is called the HKPS-integral of f over the compact subinterval $E \subset [a,b]$ with respect to g. [11]

With these integrals, this article is devoted on constructing potential convergence theorems using the notion of uniform convergence supplementing our existing knowledge on HKDS-integral and HKPS-integral. A sequence $\langle f_n \rangle_{n=1}^{\infty}$ of functions with common domain E, a function f on E and a subset A of E, we say that the sequence $\langle f_n \rangle_{n=1}^{\infty}$ converges to f uniformly on A provided that for each $\varepsilon > 0$, there is an index $N \in \mathbb{N}$ for which

$$|f - f_n| < \varepsilon \text{ on } A$$

for all $n \geq \mathbb{N}$. [6]

2 Preliminary Notes

Essential terminologies needed in directing the conceptualization of the results are discussed on this section. Throughout the rest of the paper, we consider an arbitrary Banach space X.

Definition 2.1. [9] A **compact interval** in $\mathbb{R}$ is just a closed interval of the form $[u,v]$ where $u, v \in \mathbb{R}$. This interval is said to be **non-degenerate** if $u \neq v$.

Definition 2.2. [9] Two intervals $[u,v], [y,z] \in \mathbb{R}$ are said to be **non-overlapping** if

$$(u,v) \cap (y,z) = \varnothing.$$

Definition 2.3. [9] A function $\delta : [u,v] \to \mathbb{R}^+$ is called a **gauge** on $[u,v]$.

Definition 2.4. [9] A **point-interval pair** $(t, [u,v])$ consists of a point $t \in \mathbb{R}$ and an interval $[u,v]$ in $\mathbb{R}$. Here, t is known as a **tag** of $[u,v]$.

Definition 2.5. [9] If $\{([u_k, v_k]) : k = 1, 2, \cdots, p\}$ is a finite collection of pairwise non-overlapping subintervals of $[a,b]$ such that $[a,b] = \bigcup_{k=1}^{p} [u_k, v_k]$, we say that $\{[u_k, v_k] : k = 1, 2, \cdots, p\}$ is a **division** of $[a,b]$.

Definition 2.6. [9] A **Perron partition** P of $[a,b]$ is a finite collection of point-interval pairs $\{(t_k, [u_k, v_k]) : k = 1, 2, \cdots, p\}$ where $\{[u_k, v_k \ : \ k = 1, 2, \ldots, p]\}$ is a division of $[a,b]$ and $t_k \in [u_k, v_k]$ for $k = 1, 2, \cdots, p$. Here, t_k is called a **tag** of $[u_k, v_k]$.

Definition 2.7. [9] Let δ be a gauge on $[a,b]$. A Perron partition $\{(t_k, [u_k, v_k]) : k = 1, 2, \cdots, p\}$ of $[a,b]$ is **δ-fine** if $[u_k, v_k] \subset (t_k - \delta(t_k), t_k + \delta(t_k))$.

Definition 2.8. [18] A function $g : [a,b] \to \mathbb{R}$ is said to be of **bounded variation** on $[a,b]$ if $\sup \left\{ \sum_{[u_k, v_k] \in D} |g(u_k) - g(v_k)| \right\}$ is finite where the supremum is taken over all divisions $D = \{[u_k, v_k]\}$ of $[a,b]$.

Definition 2.9. [8] A normed space $(X, \|\cdot\|)$ is said to be **complete** if all Cauchy sequences in X are convergent. In this case, X is a **Banach space**.

Definition 2.10. [6] A sequence $\{f_n\}$ of real-valued functions on D is said to be **uniformly bounded on D** provided there is some $M > 0$ for which

$$|f_n| \leq M$$

on D for all $n \in \mathbb{N}$.

Definition 2.11. [8] An operator $T : V \to W$ between vector spaces V and W is a **linear operator** if for all $x, y \in V$ and scalars a,

$$T(x + y) = Tx + Ty \quad \text{and} \quad T(ax) = aTx.$$

Definition 2.12. [8] A **linear functional** is any linear operator $f : X \to K$, where X is a normed space over field K, where $K = \mathbb{R}$ or $K = \mathbb{C}$.

Definition 2.13. [8] Let X and Y be normed spaces and $T : \mathscr{D}(T) \to Y$ a linear operator, where $\mathscr{D}(T) \subset X$. The operator T is said to be **bounded** if there is a real number c such that for all $x \in \mathscr{D}(T)$,

$$\|Tx\| \leq c\|x\|.$$

Here, the smallest possible value c can take is observed on this inequality, $\frac{\|Tx\|}{\|x\|} \leq c$ where c must be at least as big as the supremum of

$$\left\{ \frac{\|Tx\|}{\|x\|} \ : \ x \in \mathscr{D}(T) \right\}.$$

This quantity is denoted by $\|T\|$ and is called the **norm of the operator T**. If $c = \|T\|$, then

$$\|Tx\| \leq \|T\|\|x\|.$$

In case of linear functionals, we have

$$|f(x)| \leq \|f\|\|x\|.$$

Definition 2.14. [8] Let X be a vector space over K. Define
$X^* = \{f : X \to K \mid f \text{ is a linear functional}\}$ and the following operations in X^*,

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad \text{and} \quad (\alpha f)(x) = \alpha f(x).$$

Then, $\langle X^*, +, \cdot \rangle$ is a vector space and is called the **algebraic dual space** of X.

Definition 2.15. [8] Let X be a vector space over field K. Define
$X^{**} = \{g : X^* \to K \mid g \circ f \text{ for all } f \in X^* \text{ where } g \text{ is a linear functional}\}$ and the following operations in X^{**},

$$(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f \quad \text{and} \quad (\alpha g) \circ f = \alpha(g \circ f).$$

Then, $\langle X^{**}, +, \cdot \rangle$ is a vector space and is called the **second algebraic dual space** of X.

3 Main Results

The main results of this study is divided into two parts. The first part provides the convergence theorems of HKDS-integral and HKPS-integral of Banach-valued functions over $\mathbb{R}$ and the second part presents the Saks-Henstock lemma for these integrals.

3.1 Some Convergence Theorems

3.1.1 Uniform Convergence with respect to Integrand

Before presenting the uniform convergence for the integrand of HKDS-integral, we have the following lemma,

Lemma 3.1. *Let $f : [a, b] \to X$ be bounded linear operator and $g : [a, b] \to \mathbb{R}$ be a function of bounded variation. If the **HKS**-integral of f with respect to g exists on $[a, b]$, then*

$$\left\|(\boldsymbol{HKS})\int_E f\ dg\right\|_X \leq \|f\| \cdot M$$

for every compact subinterval $E \subset [a, b]$, where $M = \sup\left\{\sum_{[u_k,v_k]\in D} |g(u_k) - g(v_k)|\right\}$.

Proof: Let $x^* \in X^*$. Since f is **HKS**-integrable with respect to g on $[a, b]$, then $(\mathbf{HKS})\int_E f\ dg$ exists. Let $\varepsilon > 0$ and a compact subinterval $E \subset [a, b]$. Since f is **HKS**-integrable, choose a gauge δ on $[a, b]$ such that for every δ-fine Perron partition P of $[a, b]$, we have

$$\left\|S(f; g; P) - (\mathbf{HKS})\int_E f\ dg\right\|_X < \varepsilon.$$

By hypothesis, $\|f\|$ exists. Let Q be a δ-fine Perron partition of $[a, b]$. Notice that,

$$\begin{aligned}
||S(f; g; Q)||_X &= \left\|\sum_{(t_k,[u_k,v_k])\in Q} f(t_k)[(g(v_k) - g(u_k)]\right\|_X \\
&\leq \sum_{(t_k,[u_k,v_k])\in Q} \|f(t_k)[(g(v_k) - g(u_k)]\|_X \\
&= \sum_{(t_k,[u_k,v_k])\in Q} \|f(t_k)\|_X \cdot |g(v_k) - g(u_k)| \\
&\leq \sum_{(t_k,[u_k,v_k])\in Q} \|f\| \cdot |g(v_k) - g(u_k)| \\
&= \|f\| \sum_{(t_k,[u_k,v_k])\in Q} |g(v_k) - g(u_k)| \\
&\leq \|f\| \cdot M.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|(\mathbf{HKS})\int_E f\ dg\right\|_X &\leq \left\|S\left(f; g; Q - (\mathbf{HKS})\int_E f\ dg\right)\right\|_X + \|S(f; g; Q)\|_X \\
&\leq \varepsilon + \|f\| \cdot M.
\end{aligned}$$

By arbitrariness of ε, the conclusion follows. □

Theorem 3.2. ***(Uniform Convergence I)*** *Let $f_n : [a, b] \to X$ and let $g : [a, b] \to \mathbb{R}$ be a function of bounded variation. Suppose that $\langle f_n\rangle_{n=1}^{\infty}$ is a sequence of bounded and **HKDS**-integrable functions with respect to g over $[a, b]$. If f_n converges uniformly to $f : [a, b] \to X$ on $[a, b]$, then f is **HKDS**-integrable with respect to g over $[a, b]$ and*

$$\lim_{n\to\infty}(\boldsymbol{HKDS})\int_E f_n\ dg = (\boldsymbol{HKDS})\int_E f\ dg$$

for all compact subinterval $E \subset [a, b]$.

Proof: Let $\varepsilon > 0$ and $x^* \in X^*$. Note that f_n converges uniformly to f on $[a,b]$. So, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, and for all $h \in [a,b]$, we have

$$\|f_n(h) - f(h)\|_X < \frac{\varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M+1)} \tag{1}$$

where $M = \sup\left\{\sum_{[u_k v_k] \in D} |g(u_k) - g(v_k)|\right\}$. If $m, n \geq N_1$ and $h \in [a,b]$, then

$$\begin{aligned}
\|f_n(h) - f_m(h)\|_X &= \|f_n(h) - f(h) + f(h) - f_m(h)\|_X \\
&\leq \|f_n(h) - f(h)\|_X + \|f(h) - f_m(h)\|_X \\
&< \frac{\varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M+1)} + \frac{\varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M+1)} \\
&= \frac{2 \cdot \varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M+1)}.
\end{aligned}$$

Consequently, for all $m, n \geq N_1$, we have

$$\|f_n - f_m\| \leq \frac{2 \cdot \varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M+1)}.$$

Now, from hypothesis, $\langle x^* \circ f_n \rangle_{n=1}^{\infty}$ is a sequence of **HKS**-integrable functions with respect to g over $[a,b]$ by Theorem 3.1.5 on [11]. Let $E \subset [a,b]$ be a compact subinterval. If $m, n \geq N_1$, then using Lemma 3.1 and by the linearity property of the integrand of **HKS**-integral, observe that

$$\begin{aligned}
&\left|(\mathbf{HKS}) \int_E x^* \circ f_n \, dg - (\mathbf{HKS}) \int_E x^* \circ f_m \, dg\right| \\
&= \left|x^* \Big(\mathbf{HKS} \int_E f_n \, dg - \mathbf{HKS} \int_E f_m \, dg\Big)\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \|x^*\|_{X^*} \cdot \left\|(\mathbf{HKS}) \int_E f_n \, dg - (\mathbf{HKS}) \int_E f_m \, dg\right\|_X \\
&= \|x^*\|_{X^*} \cdot \left\|(\mathbf{HKS}) \int_E (f_n - f_m) \, dg\right\|_X \\
&\leq \|x^*\|_{X^*} \cdot \|f_n - f_m\| \cdot M \\
&\leq \|x^*\|_{X^*} \frac{2 \cdot \varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M+1)} \cdot M \\
&= \frac{2 \cdot \varepsilon}{3} < \varepsilon.
\end{aligned}$$

Hence, $\left\langle (\mathbf{HKS}) \int_E x^* \circ f_n \, dg \right\rangle_{n=1}^{\infty}$ is Cauchy. Since X is a Banach space, $\left\langle (\mathbf{HKS}) \int_E x^* \circ f_n \, dg \right\rangle_{n=1}^{\infty}$ converges to, say $A \in X$. Thus, there is an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$\left\|(\mathbf{HKS}) \int_E x^* \circ f_n \, dg - A\right\|_X < \frac{\varepsilon}{3}.$$

Put $N = \max\{N_1, N_2\}$. Observe that $x^* \circ f_N$ is (**HKS**)-integrable with respect to g on $[a,b]$, so we can select a gauge δ such that for any δ-fine Perron partition P of $[a,b]$, we have

$$\left|S(x^* \circ f_N; g; P) - (\mathbf{HKS}) \int_E x^* \circ f_N \, dg\right| < \frac{\varepsilon}{3}.$$

Note that from (1), we have

$$\begin{aligned}
&\left|S(x^* \circ f; g; P) - S(x^* \circ f_N; g; P)\right| \\
&= \left| \sum_{(t_k,[u_k,v_k])\in P} x^*(f(t_k))(g(v_k) - g(u_k)) - \sum_{(t_k,[u_k,v_k])\in P} x^*(f_N(t_k))(g(v_k) - g(u_k)) \right| \\
&= \sum_{(t_k,[u_k,v_k])\in P} \left| x^*(f(t_k))[g(v_k) - g(u_k)] - x^*(f_N(t_k))[g(v_k) - g(u_k)] \right| \\
&\leq \sum_{(t_k,[u_k,v_k])\in P} \left| x^*(f(t_k)) - x^*(f_N(t_k)) \right| \cdot \left| g(v_k) - g(u_k) \right| \\
&\leq \sum_{(t_k,[u_k,v_k])\in P} \|x^*\|_{X^*} \cdot \|f(t_k) - f_N(t_k)\|_X \cdot \left| g(v_k) - g(u_k) \right| \\
&\leq \|x^*\|_{X^*} \cdot \frac{\varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M+1)} \cdot M \\
&= \frac{\varepsilon}{3}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|S(x^* \circ f; g; P) - A\right| &= \Big|S(x^* \circ f; g; P) - S(x^* \circ f_N; g; P) + S(x^* \circ f_N; g : P) \\
&\quad - (\mathbf{HKS}) \int_E x^* \circ f_N \; dg + (\mathbf{HKS}) \int_E x^* \circ f_N \; dg - A\Big| \\
&\leq \left|S(x^* \circ f; g; P) - S(x^* \circ f_N; g; P)\right| + \left|S(x^* \circ f_N; g; P) - (\mathbf{HKS}) \int_E x^* \circ f_N \; dg\right| \\
&+ \left|(\mathbf{HKS}) \int_E x^* \circ f_N \; dg - A\right| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

This exhibits the **HKS**-integrability of $x^* \circ f$ with respect to g on $[a, b]$. So,

$$\lim_{n\to\infty} (\mathbf{HKS}) \int_E x^* \circ f_n \; dg = A = (\mathbf{HKS}) \int_E x^* \circ f \; dg.$$

By Theorem 3.1.5 on [11], f is **HKDS**-integrable with respect to g on $[a, b]$ for all $x^* \in X^*$. Now, for each $n \in \mathbb{N}$, put

$$x^{**}_{n,E}(x^*) = (\mathbf{HKS}) \int_E x^* \circ f_n \; dg.$$

Also,

$$x^{**}_E(x^*) = (\mathbf{HKS}) \int_E x^* \circ f \; dg.$$

This means that for all $x^* \in X^*$ and $n \in \mathbb{N}$, $x^{**}_{n,E}$ converges to x^{**}_E in X^{**}. Hence,

$$x^{**}_{n,E} = \lim_{n\to\infty} (\mathbf{HKDS}) \int_E f_n \; dg = (\mathbf{HKDS}) \int_E f \; dg = x^{**}_E.$$

□

For HKPS-integral, we have a similar convergence theorem,

Theorem 3.3. ***(Uniform Convergence I)*** *Let $f_n : [a, b] \to X$ and let $g : [a, b] \to \mathbb{R}$ be a function of bounded variation. Suppose that $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of bounded and* ***HKPS****-integrable functions* *with respect to g over $[a,b]$. If f_n converges uniformly to $f:[a,b]\to X$ on $[a,b]$, then f is **HKPS**-integrable with respect to g over $[a,b]$ and*

$$\lim_{n\to\infty}(\boldsymbol{HKPS})\int_E f_n\,dg=(\boldsymbol{HKPS})\int_E f\,dg$$

for all compact subinterval $E\subset[a,b]$.

Proof: Let $E\subset[a,b]$ be a compact subinterval. The assumption implies that f_n is **HKDS**-integrable with respect to g over $[a,b]$ such that $(\mathbf{HKDS})\int_E f_n\,dg\in e(X)$ for all $n\in\mathbb{N}$. For each $n\in\mathbb{N}$, take $t_n\in X$ such that

$$e(t_n)=(\mathbf{HKDS})\int_E f_n\,dg.$$

By Theorem 3.2, f is **HKDS**-integrable with respect to g over $[a,b]$ and

$$\lim_{n\to\infty}(\mathbf{HKDS})\int_E f_n\,dg=(\mathbf{HKDS})\int_E f\,dg$$

which implies $\lim_{n\to\infty}e(t_n)=(\mathbf{HKDS})\int_E f\,dg\in e(X)$. This indicates that f is **HKPS**-integrable with respect to g on $[a.b]$. Consequently, put $t\in X$ such that

$$e(t)=(\mathbf{HKDS})\int_E f\,dg$$

and the equality follows. □

3.1.2 Uniform Convergence with respect to Integrator

Let's proceed to the uniform convergence with respect to the integrator and it needs the following lemmas,

Theorem 3.4. *Let $g,g_n:[a,b]\to\mathbb{R}$ and $\langle g_n\rangle_{n=1}^{\infty}$ be a sequence of functions such that g_n converges uniformly to g and g_n is uniformly bounded. If g_n is a bounded variation on $[a,b]$ for all $n\in\mathbb{N}$, then g is also a bounded variation on $[a,b]$.*

Proof: Let $g:[a,b]\to\mathbb{R}$ and $\langle g_n\rangle_{n=1}^{\infty}$ be a sequence of functions on $[a,b]$. By assumption, for each $n\in\mathbb{N}$, g_n is a bounded variation on $[a,b]$. This means that for each $n\in\mathbb{N}$,

$$\sup_n\left\{\sum_{[u_k,v_k]\in D}|g_n(u_k)-g_n(v_k)|\right\}<\infty.$$

This implies that $\sum_{[u_k,v_k]\in D}|g_n(u_k)-g_n(v_k)|<\infty$ for each $n\in\mathbb{N}$. Let S be a division of $[a,b]$ and let $M>0$ such that $|g_n|\le M$. Note that

$$\sum_{[u,v]\in S}|g(u)-g(v)|=\sum_{[u_k,v_k]\in S}\left|\lim_{n\to\infty}g_n(u_k)-\lim_{n\to\infty}g_n(v_k)\right|$$

since g_n converges uniformly to g on $[a,b]$. Now,

$$\begin{aligned}
\sum_{[u,v]\in S} |g(u)-g(v)| &= \sum_{[u_k,v_k]\in S} \left|\lim_{n\to\infty} g_n(u_k) - \lim_{n\to\infty} g_n(v_k)\right| \\
&= \sum_{[u_k,v_k]\in S} \left(\lim_{n\to\infty} |g_n(u)-g_n(v)|\right) \\
&= \lim_{n\to\infty} \left(\sum_{[u_k,v_k]\in S} |g_n(u_k)-g_n(v_k)| \right) \\
&\leq \lim_{n\to\infty} \left(\sum_{[u_k,v_k]\in S} |g_n(u_k)| + |g_n(v_k)| \right) \\
&= \lim_{n\to\infty} \left(\sum_{[u_k,v_k]\in S} |g_n(u_k)| \right) + \lim_{n\to\infty} \left(\sum_{[u_k,v_k]\in S} |g_n(v_k)| \right) \\
&\leq \lim_{n\to\infty} \sum_{[u_k,v_k]\in S} M + \lim_{n\to\infty} \sum_{[u_k,v_k]\in S} M \\
&= \lim_{n\to\infty} \sum_{[u_k,v_k]\in S} 2M = k\cdot 2M < \infty
\end{aligned}$$

where k is the number of subintervals on S. Fix $M_o = k \cdot 2M$. Then $\sum_{[u,v]\in S} |g(u)-g(v)| \leq M_o$. Taking the supremum, we have

$$\sup \left\{ \sum_{[u,v]\in S} |g(u)-g(v)| \right\} \leq M_o < \infty.$$

Therefore, g is a bounded variation on $[a,b]$. □

Theorem 3.5. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function and $\langle g_n \rangle_{n=1}^{\infty}$ be a sequence of functions that are of bounded variation. If g_n converges uniformly to g and* $\sup\{|D| \ : \ D$ *is a division of* $[a,b]\}$ *is finite. Then the sequence*

$$\left\langle (\boldsymbol{HKS}) \int_{[a,b]} x^*(f) \ \ dg_n \right\rangle_{n=1}^{\infty}$$

is Cauchy for all $x^* \in X^*$.

Proof: Let $x^* \in X^*$. Note that Proposition 3.3.2 on [11] states that $x^* \circ f$ is continuous on $[a,b]$. Using Lemma 3.3.1 on [11], $x^* \circ f$ is **HKS**-integrable with respect to g_n on $[a,b]$ for all $x^* \in X^*$. Fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, there exists a gauge δ_n such that

$$\left| S(x^* \circ f; g_n; P_n) - (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \ dg_n \right| < \frac{\varepsilon}{4}$$

for every δ_n-fine Perron partition P_n of $[a,b]$. Put $\delta = \inf\{\delta_n \ : \ n \in \mathbb{N}\}$. Let P be a δ-fine Perron partition of $[a,b]$. Then P is a δ_n-fine Perron partition of $[a,b]$ for all $n \in \mathbb{N}$. This implies

$$\left| S(x^* \circ f; g_n; P) - (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \ dg_n \right| < \frac{\varepsilon}{4}.$$

Since $x^* \circ f$ is continuous on $[a,b]$, $x^* \circ f$ is bounded in $[a,b]$. This implies an existence of $K > 0$ such that $|x^*(f(h))| \leq K$ for all $h \in [a,b]$. Now, put

$$W = \sup\{|D| \ : \ D \text{ is a division of } [a,b]\}.$$

Since g_n converges uniformly to g on $[a,b]$, there is $N \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$ and $h \in [a,b]$, we have

$$|g_n(h) - g(h)| \leq \frac{\varepsilon}{16(K+1)(W+1)}.$$

By Lemma 3.4, g is a function of bounded variation on $[a,b]$. Also, $g_n - g$ is a function of bounded variation on $[a,b]$. Let D be a division of $[a,b]$. We will now find a bound for $\sum_{[u_k,v_k]\in D} |(g_n-g)(u_k) - (g_n-g)(v_k)|$,

$$\begin{aligned}
&\sum_{[u_k,v_k]\in D} |(g_n-g)(u_k) - (g_n-g)(v_k)| \\
&\quad \leq \sum_{[u_k,v_k]\in D} |(g_n-g)(u_k)| + \sum_{[u_k,v_k]\in D} |(g_n-g)(v_k)| \\
&\quad = \sum_{[u_k,v_k]\in D} |g_n(u_k) - g(u_k)| + \sum_{[u_k,v_k]\in D} |g_n(v_k) - g(v_k)| \\
&\quad \leq \sum_{[u_k,v_k]\in D} \frac{\varepsilon}{8(K+1)(W+1)} \\
&\quad = |D| \cdot \frac{\varepsilon}{8(K+1)(W+1)} \\
&\quad \leq W \cdot \frac{\varepsilon}{8(K+1)(W+1)} = \frac{\varepsilon}{8(K+1)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&|S(x^* \circ f; g_n; P) - S(x^* \circ f; g; P)| \\
&\quad = |S(x^* \circ f; g_n - g; P)| \\
&\quad = \left| \sum_{(t_k,[u_k,v_k])\in P} x^*(f(t_k))[(g_n-g)(v_k) - (g_n-g)(u_k)] \right| \\
&\quad \leq \sum_{(t_k,[u_k,v_k])\in P} |x^*(f(t_k))[(g_n-g)(v_k) - (g_n-g)(u_k)]| \\
&\quad \leq \sum_{(t_k,[u_k,v_k])\in P} |x^*(f(t_k))| \cdot |[(g_n-g)(v_k) - (g_n-g)(u_k)]| \\
&\quad \leq \sum_{(t_k,[u_k,v_k])\in P} K \cdot |(g_n-g)(v_k) - (g_n-g)(u_k)| \\
&\quad = K \cdot \sum_{(t_k,[u_k,v_k])\in P} |(g_n-g)(v_k) - (g_n-g)(u_k)| \\
&\quad \leq K \cdot \frac{\varepsilon}{8(K+1)} = \frac{\varepsilon}{8}.
\end{aligned}$$

So, if $m, n \geq N$, then

$$\begin{aligned}
&|S(x^* \circ f; g_n; P) - S(x^* \circ f; g_m; P)| \\
&\quad \leq |S(x^* \circ f; g_n; P) - S(x^* \circ f; g; P)| + |S(x^* \circ f; g; P) - S(x^* \circ f; g_m; P)| \\
&\quad = |S(x^* \circ f; g_n - g; P)| + |S(x^* \circ f; g_m - g; P)| \\
&\quad < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} \\
&\quad = \frac{\varepsilon}{4}.
\end{aligned}$$

Therefore, for all $m, n \geq N$,

$$\begin{aligned}
&\left|(\mathbf{HKS}) \int_{[a,b]} x^* \circ f \, dg_n - (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \, dg_m\right| \\
&= \left|(\mathbf{HKS}) \int_{[a,b]} x^* \circ f \, dg_n - S(x^* \circ f; g_n; P)\right. \\
&\quad + S(x^* \circ f; g_n; P) - S(x^* \circ f; g; P) \\
&\quad\quad + S(x^* \circ f; g; P - S(x^* \circ f; g_m; P) \\
&\quad\quad\quad \left. + S(x^* \circ f; g_m; P) - (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \, dg_m\right| \\
&\leq \left|(\mathbf{HKS}) \int_{[a,b]} x^* \circ f \, dg_n - S(x^* \circ f; g_n; P\right| \\
&\quad + |S(x^* \circ f; g_n; P) - S(x^* \circ f; g; P)| \\
&\quad\quad + |S(x^* \circ f; g; P - S(x^* \circ f; g_m; P)| \\
&\quad\quad\quad + \left|S(x^* \circ f; g_m; P) - (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \, dg_m\right| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&= \varepsilon
\end{aligned}$$

which implies that the sequence $\left\langle (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \;\; dg_n \right\rangle_{n=1}^{\infty}$ is Cauchy. □

Theorem 3.6. ***(Uniform Convergence II)*** *Let $f : [a, b] \to X$ be a continuous function on $[a, b]$ and let $\langle g_n \rangle_{n=1}^{\infty}$ be a sequence of functions on $[a, b]$ that are bounded variation. Suppose that g_n converges uniformly to g on $[a, b]$, then f is **HKDS**-integrable with respect to g on $[a, b]$ and*

$$\lim_{n \to \infty} (\boldsymbol{HKDS}) \int_E f \, dg_n = (\boldsymbol{HKDS}) \int_E f \, dg$$

for all compact subinterval $E \subset [a, b]$.

Proof: Let $x^* \in X^*$. Using Lemma 3.5, the sequence $\left\langle (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \;\; dg_n \right\rangle_{n=1}^{\infty}$ is Cauchy. Consequently, this sequence converges, so we can fix $\lim_{n\to\infty} (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \;\; dg_n = K$. It remains to show that $K = (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \;\; dg$. Being convergent implies the existence of $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left|(\mathbf{HKS}) \int_{[a,b]} x^* \circ f \;\; dg_n - A\right| < \frac{\varepsilon}{3}.$$

Specifically,

$$\left|(\mathbf{HKS})\int_{[a,b]} x^*\circ f \ \ dg_N - A\right| < \frac{\varepsilon}{3}. \tag{2}$$

Since $x^*\circ f$ is **HKS**-integrable with respect to g_N on $[a,b]$, we can choose a gauge δ on $[a,b]$ such that

$$\left|S(f;g_N;P) - (\mathbf{HKS})\int_{[a,b]} x^*\circ f \ \ dg_N\right| \tag{3}$$

for any δ-fine Perron partition P on $[a,b]$. Furthermore, using the part of the proof of Lemma 3.5, we can have,

$$|S(x^*\circ f;g;P) - S(x^*\circ f;g_N;P)| < \frac{\varepsilon}{3}. \tag{4}$$

Hence, by (1),(2), and (3), we have,

$$\begin{aligned}
|S(x^*\circ f;g;P) - K| &= \Bigg|S(x^*\circ f;g;P) - S(x^*\circ f;g_N;P) \\
&\quad + S(x^*\circ f;g_N;P) - (\mathbf{HKS})\int_{[a,b]} x^*\circ f\ dg \\
&\quad + (\mathbf{HKS})\int_{[a,b]} x^*\circ f\ dg - K\Bigg| \\
&\leq |S(x^*\circ f;g;P) - S(x^*\circ f;g_N;P)| \\
&\quad + \left|S(x^*\circ f;g_N;P) - (\mathbf{HKS})\int_{[a,b]} x^*\circ f\ dg\right| \\
&\quad + \left|(\mathbf{HKS})\int_{[a,b]} x^*\circ f\ dg - K\right| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Thus, $K = (\mathbf{HKS})\int_{[a,b]} x^*\circ f \ \ dg$ indicating that $x^*\circ f$ is **HKS**-integrable with respect to g on $[a,b]$. So,

$$\lim_{n\to\infty}(\mathbf{HKS})\int_{[a,b]} x^*\circ f\ dg_n = (\mathbf{HKS})\int_{[a,b]} x^*\circ f\ dg.$$

To this end, by Theorem 3.1.5 on [11], f is **HKDS**-integrable with respect to g and g_n on $[a,b]$. Now, let a compact subinterval $E\subset[a,b]$. For each $n\in\mathbb{N}$, put

$$x^{**}_{n,E}(x^*) = (\mathbf{HKS})\int_E x^*\circ f\ dg_n.$$

Also,

$$x^{**}_E(x^*) = (\mathbf{HKS})\int_E x^*\circ f\ dg.$$

This means that for all $x^*\in X^*$ and $n\in\mathbb{N}$, $x^{**}_{n,E}$ converges to x^{**}_E in X^{**}. Finally,

$$x^{**}_{n,E} = \lim_{n\to\infty}(\mathbf{HKDS})\int_E f\ dg_n = (\mathbf{HKDS})\int_E f\ dg = x^{**}_E.$$

□

On Pettis type integral, we have the following uniform convergence with respect to the integrator.

Theorem 3.7. *Let $f : [a, b] \to X$ be a continuous function on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ such that $(\boldsymbol{HKDS}) \int_E f\ dg \in e(X)$ and let $\langle g_n \rangle_{n=1}^{\infty}$ be a sequence of functions on $[a, b]$ that are of bounded variation such that $(\boldsymbol{HKDS}) \int_E f\ dg_n \in e(X)$. Suppose that g_n converges uniformly to g on $[a, b]$, then*

$$\lim_{n\to\infty} (\boldsymbol{HKPS}) \int_E f\ dg_n = (\boldsymbol{HKPS}) \int_E f\ dg$$

for all compact subinterval $E \subset [a, b]$.

Proof: Let E be a compact subinterval of $[a, b]$. By hypothesis, $(\mathbf{HKDS}) \int_E f\ dg \in e(X)$ implies f being **HKPS**-integrable with respect to g on $[a, b]$. In a similar manner, for each $n \in \mathbb{N}$, $(\mathbf{HKDS}) \int_E f\ dg_n \in e(X)$ implying that f is **HKPS**-integrable with respect to g_n on $[a, b]$. Fix $u, u_n \in X$ such that

$$e(u) = (\mathbf{HKDS}) \int_E f\ dg \quad \text{and} \quad e(u_n) = (\mathbf{HKDS}) \int_E f\ dg_n.$$

Observe that by Theorem 3.6,

$$\lim_{n\to\infty} (\mathbf{HKDS}) \int_E f\ dg_n = (\mathbf{HKDS}) \int_E f\ dg.$$

This indicates that

$$\lim_{n\to\infty} e(u_n) = e(u).$$

That is, the sequence $\langle e(u_n) \rangle_{n=1}^{\infty}$ in $e(X)$ converges to $e(u)$. Consequently, the claimed equality follows by definition of **HKPS** integral. □

4 Conclusion

Let X be a Banach space. Given a sequence of Banach-valued functions $\langle f_n \rangle_{n=1}^{\infty}$ on $\mathbb{R}$, the presentation of convergence theorems for HKDS integral and HKPS integral using the notion of uniform convergence with respect to the integrand and integrator provide sufficient conditions for a Banach-valued function f on $\mathbb{R}$ to be integrable with respect to this sequence. This is vital especially on predicting the integral values of such functions efficiently and systematically.

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6 Competing Interests

The authors declare that they have no competing interests.

References

[1] Bartle, R.G., & Sherbert, D.R. (2000). *Introduction to real analysis.* John Wiley and Sons. Inc.

[2] Bruckner, A.M., Bruckner, J.B., & Thomson, B.S. (2001). *Elementary Real Analysis.* Prentice Hall.

[3] Cao, S.S. (1992). The Henstock integral for Banach-valued functions. *Southeast Asian Bulletin, Vol.1*, 35-40.

[4] Conway, J.B. (1990). A course in functional analysis. *Springer, Vol.2.*

[5] Di Piazza, L., Marraffa, V., & Musial, K. (2016). Variational Henstock integrability of Banach space valued functions. *Mathematica Bohemica*, 287-296.

[6] Fitzpatrick, P., & Royden, H.L. (2010). *Real analysis, 4th edition.* Prentice Hall.

[7] Johnsonbaugh, R.F., & Pfaffenberger, W.E. (2010). *Foundations of Mathematical Analysis.* Dover Publications.

[8] Kreyszig, E. (1989). *Introductory Functional Analysis with Application.* John Wiley and Sons.

[9] Lee, T.Y. (2011). Henstock-Kurzweil integration on Euclidean spaces. *World Scientific, Vol.12*, 1-20.

[10] Lim, J.S., Yoon, J.H., & Eun, G.S. (1998). On Henstock-Stieltjes integral. *KangweonKyungki Math, Vol.1*, 87-96.

[11] Mangubat, D.P., & Flores, G.B.C. (2024). *On the Henstock-Kurzweil-Dunford-Stieltjes Integral and Henstock-Kurzweil-Pettis-Stieltjes Integral of Banach-valued Functions on* $\mathbb{R}$. (article to be published)

[12] McLeod, R. (1980). *The Generalized Riemann Integral.* Mathematical Association of America.

[13] Min, M.Z., & Yee, L.P. *An overview of classical integration theory.*

[14] Munkres, J.R. (2000). *Topology.*

[15] Omayan, D.O. (2022). Some Fundamental Properties of Variational Kurzweil Henstock Stieltjes Integral on a Compact Interval on $\mathbb{R}^n$. *Asian Research Journal of Mathematics, Vol.18*, 69-81.

[16] Pettis, B. (1938). On integration in vector spaces. *American Mathematical Society*, 277-304.

[17] Schwabik, S., & Guoju, Y. (2005). Topics in Banach Space Integration. *Series in Real Analysis, Vol.10.*

[18] Tikare, S.A., & Chaudhary, M.S. (2010). Henstock-Stieltjes Integral for Banach space-valued functions. *Bulletin of Kerala Mathematics Association, Vol.6.*

[19] Viro, O.Y., Ivanov, O.A., Netsvetaev, N., & Kharlamov, V.M. (2008). *Elementary Topology Problem Textbook.* American Mathematical Society Journal.

[20] Olaleru, J. O., & Okeke, G. A. (2012). Strong Convergence Theorems for Asymptotically Pseudocontractive Mappings in the Intermediate Sense. Journal of Advances in Mathematics and Computer Science, 2(3), 151–162. https://doi.org/10.9734/BJMCS/2012/1569

[21] Oke, A. S.,& Kayode, D. J. (2017). Some Theorems on Fixed Points Set of Asymptotically Demicontractive Mappings in the Intermediate Sense. Asian Research Journal of Mathematics, 6(4), 1–7. https://doi.org/10.9734/ARJOM/2017/36218

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