Estimation involving a class of special arithmetical functions

ABSTRACT

The mean value estimation of arithmetical function is closely related to many problems in number theory. Let f be an arithmetical function satisfying some conditions. Let $[r]$ be the integral part of r.This paper proves thattheasymptotic expression $S_f(y) := \sum_{n\le y} f([y/n])/([y/n]^{k-1})(k\in \square^{+})$ andthe error term of this asymptotic formula is $\Omega(y)$.The arithmetical function in this paper satisfies certain conditions, and the Dirichlet hyperbolic principle is used in the proof of the conclusion. With the different values of the independent variable of the function, the function value of the arithmetical function is often irregular, and the property of the mean value of the arithmetical function is more regular than that of the arithmetical function itself. Therefore, with the help of the mean value estimation results of the arithmeticalfunction, we can have a deeper understanding of the nature of the arithmetical function itself, and then provide ideas for solving more problems.

Keywords:Arithmetical function; Asymptotic formula; Integral part; Dirichlet hyperbolic principle.

1. INTRODUCTION

Generally, $\varphi(n) = \sum_{1\le g\le n,(g,n)=1} 1$ is called Euler totient function. $\sigma(n) = \sum_{g|n} g$ is called the sum of divisors function. $\zeta(s)$ is called the Riemann function. $[r]$ is called the integral part of r. In 2019, Bordellès, Dai, Heyman, Pan and Shparlinski[1] proved that

$$\left(\frac{2629}{4009}\cdot\frac{1}{\zeta(2)}+o(1)\right)y\log y \le \sum_{n\le y}\varphi([y/n]) \le \left(\frac{2629}{4009}\cdot\frac{1}{\zeta(2)}+\frac{1380}{4009}+o(1)\right)y\log y \quad (1)$$

for $y\to\infty$. In 2019, Chern[2] improved the upper and lower bounds of $\sum_{n\le y}\varphi([y/n])$ and obtained that

$$\frac{285}{416}\cdot\frac{1}{\zeta(2)}y\log y + O(y\log_2 y) \le \sum_{n\le y}\varphi\left(\left[\frac{y}{n}\right]\right) \le \left(\frac{285}{416}\cdot\frac{1}{\zeta(2)}+\frac{131}{416}\right)y\log y + O(y\log_2 y) \quad (2)$$

for $y\to\infty$.In 2019, Wu[3] estimated the bound of $\sum_{n\le y}\varphi([y/n])$ by using the method of exponential sum and obtained that

$$\frac{4}{\pi^2}y\log y + O(y) \le \sum_{n\le y}\varphi([y/n]) \le \left(\frac{1}{3}+\frac{4}{\pi^2}\right)y\log y + O(y) \quad (3)$$

for $y\to\infty$. In 2020, using the theory of exponential pairs[4] and the Vinogradov method[5], Zhai[6] proved that

$$\sum_{n\le y}\varphi([y/n]) = \frac{y\log y}{\zeta(2)} + O\left(y(\log y)^{2/3}(\log_2 y)^{1/3}\right) \quad (4)$$

for $y \to \infty$. Owing to $\varphi(n)$ and $\sigma(n)$ have analogous properties, it is meaningful to study the asymptotic behaviourof $S_\sigma(y) := \sum_{n \le y} \sigma([y/n])$. In 2021, Zhao and Wu[7] proved that

$$\sum_{n \le y} \sigma([y/n]) = \frac{\pi^2}{6} y \log y + O\left(y (\log y)^{2/3} (\log_2 y)^{4/3}\right) \tag{5}$$

for $y \to \infty$. In 2022, Ma, Wu and Zhao[8], defined arithmetical function z by Dirichlet convolution for general arithmetical function f, and obtained the asymptotic estimation results of $\sum_{n \le y} f([y/n])$ when f satisfies certain conditions. In 2023, Ma and Sun[9] used the method of three-dimensional exponential sum to obtain that

$$\sum_{n \le y} \frac{f([y/n])}{[y/n]} = y \sum_{m \ge 1} \frac{f(m)}{m^2 (m+1)} + O\left(y^{1/4+\varepsilon}\right) \tag{6}$$

for $y \to \infty$, $f \in \{\sigma, \varphi, \beta, \psi\}$ (here σ, φ, β, ψ are some given arithmetical functions with similar product structure)andany $\varepsilon > 0$ .In 2023, Li and Ma[10] defined the arithmetical function z by Dirichlet convolution for the general arithmetical function f, and obtained the asymptotic estimation results of $S_f(y) := \sum_{n \le y} f([y/n])/([y/n])$ when f satisfies certain conditions.

Naturally, according to the cognitive law from special to general, we consider whether we can get a more extensive conclusion. In this paper, we consider the asymptotic estimation results of $S_f(y) := \sum_{n \le y} f([y/n])/\left([y/n]^{k-1}\right)\left(k \in \mathbb{Z}^+\right)$ and the asymptotic form of the remainder when the number theory function f satisfies certain conditions.

Inspired by [11], let t_1, t_2, t_3 be three increasing functions defined on $[1,\infty)$ such that when $y \ge 1$,

$$1 \le t_i(y) \ll y^{\eta_i} \quad (i = 1,2,3), \quad t_3(y) \to \infty, \tag{7}$$

where η_i is a constant on $(0,1)$.

Let f be an arithmetical function, define $f = id^k * z\left(k \in \mathbb{Z}^+\right)$, and assume that f satisfies the following conditions:

$$|f(n)| \ll n^k t_1(n) \quad (n \ge 1), \tag{8}$$

$$\sum_{n \le y} \frac{|z(n)|}{n^{k-1}} \ll y t_2(y) \quad (y \ge 1), \tag{9}$$

$$\sum_{n \le y} \frac{z(n)}{n^{k-1}} \ll C_z y + O\left(\frac{y}{t_3(y)}\right) \quad (y \ge 1), \tag{10}$$

where C_z is a constant (can be equal to 0).Then we get the following results.

Theorem 1.1(i)For any constant $B \ge 1$, $k \in \mathbb{Z}^+$,we have

$$S_f(y) := \sum_{n \le y} \frac{f([y/n])}{[y/n]^{k-1}} = C_f y \log y + O_B\left(y E(y,h)\right) \tag{11}$$

for $y \ge 3$ and $1 \le h \le (\log y)^{B+1}$, where $C_f := \sum_{n=1}^{\infty} z(n) n^{-(k+1)}$, and

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$$E(y,h) := (\log y)^{2/3} (\log_2 y)^{1/3} t_1(y) + \frac{t_2(y/h)\log y}{h} + \frac{h\log y}{t_3(y^{1/2}/h)},$$

where the implied constant is only related to B.

(ii)If there exists a positive constant $c_1 < 1$ such that the condition $\left|\frac{f(p-1)}{(p-1)^{k-1}}\right| < c_1 f(1) p$ or $\left|\frac{f(p-1)}{(p-1)^{k-1}}\right| > c_1^{-1} f(1) p > 0$ holds for infinitely many primes p, then the remainder $yE(y,h)$ of $S_f(y)$ is $\Omega(y)$.

2. LEMMAS

In order to prove the theorem, some lemmas are required. Define $\psi(r) := r - [r] - (1/2)$.

Lemma 2.1[12]For $y \ge 10$, $\exp\left\{c_2 (\log y)^{2/3}\right\} \le M \le y^{2/3}$ and $M < M' \le 2M$, there are positive numbers c_2 and c_3 such that

$$\sum_{M \le n < M'} \frac{1}{n} \psi(y/n) \ll e^{-c_3 (\log M)^3/(\log y)^2} \frac{(\log M)^3}{(\log y)^2}.$$

Lemma 2.2[7]Let $2 \le b_1 < b_2 \le y$, $F_y(r) := (1/r)\psi(y/r)$. Let $V_{F_y}[b_1, b_2]$be the total variation of F_y on $[b_1, b_2]$. Then we have

$$V_{F_y}[b_1, b_2] \ll \frac{y}{b_1^2} + \frac{1}{b_1},$$

where the implied constant is absolute.

Lemma 2.3 (i)Let $f(n)$ satisfies the conditions (9)and (10), we can obtain

$$\sum_{n \le y} \frac{f(n)}{n^{k-1}} = \frac{1}{2} C_f y^2 - C_z y \frac{(h-[h])^2 + [h]}{2h} - \Delta_z(y,h) + O\left(\frac{y t_2(y/h)}{h} + \frac{yh}{t_3(y/h)}\right) \tag{12}$$

for $y \ge 2$ and $1 \le h \le y^{1/3}$, where $C_f := \sum_{n=1}^{\infty} z(n) n^{-(k+1)}$ and

$$\Delta_z(y,h) := y \sum_{g \le y/h} \frac{z_2(g)}{g^k} \psi(y/g). \tag{13}$$

(ii)we have

$$\sum_{n \le y} \frac{f(n)}{n^{k-1}} = \frac{1}{2} C_f y^2 + O\big(y(\log y) t_2(y)\big) \tag{14}$$

for $y \ge 2$.

Proof. Using $f(n) = \sum_{gm=n} z(g) m^k$ and Dirichlet hyperbolic principle, we can write

$$\sum_{n \le y} \frac{f(n)}{n^{k-1}} = \sum_{gm \le x} \frac{z(g)}{g^{k-1}} m = S_1 + S_2 - S_3, \tag{15}$$

where $S_1 := \sum_{g \le y/h} \sum_{m \le y/g} \frac{z(g)}{g^{k-1}} m$, $S_2 := \sum_{m \le h} \sum_{g \le y/m} \frac{z(g)}{g^{k-1}} m$, $S_3 := \sum_{g \le y/h} \sum_{m \le h} \frac{z(g)}{g^{k-1}} m$.

Firstly, according to conditions (9), we can calculate

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$$S_1=\sum_{g\le x/h}\frac{z(g)}{g^{k-1}}\sum_{m\le y/g}m$$
$$=\sum_{g\le y/h}\frac{z(g)}{g^{k-1}}\left(1+2+\cdots+\left[\frac{y}{g}\right]\right)$$
$$=\frac{1}{2}\sum_{g\le y/h}\frac{z(g)}{g^{k-1}}\left(\frac{y}{g}-\psi\left(\frac{y}{g}\right)-\frac{1}{2}\right)\left(\frac{y}{g}-\psi\left(\frac{y}{g}\right)+\frac{1}{2}\right)$$
$$=\frac{y^2}{2}\sum_{g\le y/h}\frac{z(g)}{g^{k+1}}-\Delta_z(y,h)+O\left(\frac{yt_2(y/h)}{h}\right), \tag{16}$$

where $\Delta_z(y,h)$ is defined by (13), and the first term of (16)can be calculated by usingthecondition (10).

$$\sum_{g\le y/h}\frac{z(g)}{g^{k+1}}=\sum_{g=1}^{\infty}\frac{z(g)}{g^{k+1}}-\sum_{g>y/h}\frac{z(g)}{g^{k+1}}$$
$$=C_f-\int_{\frac{y}{h}}^{\infty}\frac{1}{u^{-2}}\mathrm{d}\left(C_zu+O\left(\frac{u}{t_3(u)}\right)\right)$$
$$=C_f-C_z\frac{h}{y}+O\left(\frac{h}{yt_3(y/h)}\right).$$

Then,

$$S_1=\frac{1}{2}C_fy^2-\frac{1}{2}C_zhy-\Delta_z(y,h)+O\left(\frac{yt_2(y/h)}{h}+\frac{yh}{t_3(y/h)}\right), \tag{17}$$

where $C_f:=\sum_{n=1}^{\infty}z(n)n^{-(k+1)}$, $\Delta_z(y,h)$ is defined by (13).

Secondly, we can calculate according to thecondition (10).

$$S_2=\sum_{m\le h}m\sum_{g\le y/m}\frac{z(g)}{g^{k-1}}$$
$$=\sum_{m\le h}m\left(C_z\frac{y}{m}+O\left(\frac{y}{mt_3(y/m)}\right)\right)$$
$$=C_zy[h]+O\left(\frac{yh}{t_3(y/h)}\right), \tag{18}$$

$$S_3=\sum_{g\le y/h}\frac{z(g)}{g^{k-1}}\sum_{m\le h}m$$
$$=\sum_{g\le y/h}\frac{z(g)}{g^{k-1}}\frac{[h]([h]+1)}{2}$$
$$=\left(C_z\frac{y}{h}+O\left(\frac{y}{ht_3(y/h)}\right)\right)\frac{[h]([h]+1)}{2}$$
$$=\frac{[h]([h]+1)}{2h}C_zy+O\left(\frac{yh}{t_3(y/h)}\right). \tag{19}$$

Substituting the results of (17), (18) and (19) into (15), we can get (12).
Letting $h=1$ in (12), we can write

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$$\Delta_z(y,1) \ll y\sum_{g\le y}\frac{|z(g)|}{g^k} \ll y(\log y)t_2(y).$$

Then we can get (14).

Lemma 2.4 Let f satisfies the condition (9), $\Delta_z(y,h)$ be defined by (13). Let B be a positive constant, $M_0 := \exp\left\{((B+3)/c_3)^{1/3}(\log y)^{2/3}(\log_2 y)^{1/3}\right\}$, where c_3 is a constant given by Lemma2.1, then for $y \ge 10$ and $2 \le h \le \sqrt{M_0}$, we have

$$\left|\sum_{M_0<n\le\sqrt{y}}\Delta_z\left(\frac{y}{n},h\right)\right| + \left|\sum_{M_0<n\le\sqrt{y}}\Delta_z\left(\frac{y}{n}-1,h\right)\right| \ll \frac{yt_2(y/h)}{(\log y)^B} + \frac{yt_2(y/h)\log y}{h}. \tag{20}$$

Proof. Define $\Delta_1(y,h) := \sum_{M_0<n\le\sqrt{y}}\Delta_z\left(\frac{y}{n},h\right)$, $\Delta_2(y,h) := \sum_{M_0<n\le\sqrt{y}}\Delta_z\left(\frac{y}{n}-1,h\right)$. By (13), we can write

$$\Delta_1(y,h) = y\sum_{M_0<n\le\sqrt{y}}\ \sum_{g\le(y/(nh))}\frac{z(g)}{g^k n}\psi\left(\frac{y}{gn}\right)$$

$$= y\sum_{g\le(y/(M_0h))}\frac{z(g)}{g^k}\sum_{M_0<n\le\min\{\sqrt{y},\,y/(gh)\}}\frac{1}{n}\psi\left(\frac{y}{gn}\right)$$

$$= y\Delta_1^{\dagger}(y,h) + y\Delta_1^{\#}(y,h), \tag{21}$$

where

$$\Delta_1^{\dagger}(y,h) := \sum_{g\le(y/(M_0h))}\frac{z(g)}{g^k}\sum_{M_0<n\le(y/g)^{2/3}}\frac{1}{n}\psi\left(\frac{y}{gn}\right),$$

$$\Delta_1^{\#}(y,h) := \sum_{g\le(y/(M_0h))}\frac{z(g)}{g^k}\sum_{(y/g)^{2/3}<n\le\min\{\sqrt{y},\,y/(gh)\}}\frac{1}{n}\psi\left(\frac{y}{gn}\right).$$

Let $M_l := 2^l M_0$, we can deduce $0 \le l \le \left(\log\left((y/g)^{2/3}/M_0\right)\right)/\log 2$.

Define $S_l(g) := \sum_{M_l<n\le 2M_l}\frac{1}{n}\psi\left(\frac{y}{gn}\right)$. Since $M_0 \le M_l \le (y/g)^{2/3}$, using Lemma 2.1, we can deduce

$$S_l(g) \ll e^{-\vartheta\left((\log M_l)^3/(\log(y/g))^2\right)},$$

where $\vartheta(r) := c_3 r - \log r$.

For $l \ge 0$ and $g \le y/(M_0h)$, we can obtain

$$(\log M_l)^3/(\log(y/g))^2 \ge (\log M_0)^3/(\log y)^2 = ((B+3)/c_3)\log_2 y,$$

$$\vartheta\left(\frac{(\log M_l)^3}{(\log(y/g))^2}\right) \ge \vartheta\left(((B+3)/c_3)\log_2 y\right)$$

$$= (B+3)\log_2 y - \log\left(\frac{B+3}{c_3}\log_2 y\right) \ge (B+2)\log_2 y,$$

then $S_l(g) \ll (\log y)^{-B-2}$.

Using thecondition(9), we can write

$$\Delta_1^{\dagger}(y,h) \ll \sum_{g\le(y/(M_0h))}\frac{|z(g)|}{g^k}\sum_{2^l M_0\le(y/g)^{2/3}}|S_l(g)|$$

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$$\ll \frac{1}{(\log y)^{B+1}} \sum_{g \le y/h} \frac{|z(g)|}{g^k} \ll \frac{t_2(y/h)}{(\log y)^B}. \tag{22}$$

Now we bound $\Delta_1^{\#}(y,h)$. This requires the help of the following summation formula:

$$\sum_{M_1 < n \le M_2} F(n) = \int_{M_1+1}^{M_2+1} F(u)\mathrm{d}u + \frac{1}{2}\left(F(M_1+1) - F(M_2+1)\right) + O\left(V_F\left[M_1+1, M_2+1\right]\right). \tag{23}$$

This formula has been proved in Zhao and Wu[7]. In the formula, $F(r)$ is a bounded variation function defined on the closed interval $[n, n+1]$, and $V_F[n, n+1]$ is the total variation of F. We apply (23) to

$$F_{(y/g)}(r) = \frac{1}{r}\psi\left(\frac{y/g}{r}\right), \; M_1 = \left[(y/g)^{2/3}\right], \; M_2 = \left[\min\left\{\sqrt{y}, \frac{y}{gh}\right\}\right].$$

By Lemma 2.2 , we can write

$$\begin{aligned} V_{F_{(y/g)}}\left[M_1+1, M_2+1\right] &\ll \frac{y/g}{(M_1+1)^2} + \frac{1}{(M_1+1)} \\ &= \frac{y/g}{\left((y/g)^{2/3} - \left\{(y/g)^{2/3}\right\} + 1\right)^2} + \frac{1}{\left((y/g)^{2/3} - \left\{(y/g)^{2/3}\right\} + 1\right)} \\ &\ll \left(\frac{y}{g}\right)^{-1/3}. \end{aligned}$$

Letting $v = y/(dn)$, we can get $\displaystyle\sum_{(y/g)^{2/3} < n \le \min\left\{y^{1/2}, \frac{y}{gh}\right\}} \frac{1}{n}\psi\left(\frac{y}{gn}\right) \ll h^{-1} + \left(\frac{y}{g}\right)^{-1/3} \ll h^{-1}$.

Thus,

$$\Delta_1^{\#}(y,h) \ll h^{-1} \sum_{g \le y/h} \frac{|z(g)|}{g^k} \ll \frac{t_2(y/h)\log y}{h}. \tag{24}$$

From(21), (22)and (24), we get

$$\left|\Delta_1(y,h)\right| \ll \frac{y t_2(y/h)}{(\log y)^B} + \frac{y t_2(y/h)\log y}{h}.$$

Using the same proof idea, we can get that $\left|\Delta_2(y,h)\right|$ has the same bound.

3. THE PROOF PROCESS OF THEOREM 1.1

3.1ProofofTheorem 1.1 (i)

Letting $g = \left[\frac{y}{n}\right]$, we can derive $\frac{y}{n} - 1 < g \le \frac{y}{n}$ and $\frac{y}{g+1} < n \le \frac{y}{g}$. Let $f(0) = 0$, we have

$$\begin{aligned} S_f(y) &= \sum_{g \le y} \frac{f(g)}{g^{k-1}} \sum_{\frac{y}{g+1} < n \le \frac{y}{g}} 1 \\ &= \sum_{gn \le y} \frac{f(g)}{g^{k-1}} - \sum_{gn \le y, g \ge 2} \frac{f(g-1)}{(g-1)^{k-1}} \\ &= \sum_{gn \le y, g \ge 2} \left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}}\right) + \sum_{n \le y} f(1) \\ &= S_1(y,f) + S_2(y,f) - S_3(y,f), \end{aligned} \tag{25}$$

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where

$$S_1(y,f) := \sum_{gn\le y, 2\le g\le y^{1/2}} \left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{n\le y} f(1),$$

$$S_2(y,f) := \sum_{gn\le y, g\ge 2, n\le y^{1/2}} \left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{n\le y} f(1),$$

$$S_3(y,f) := \sum_{n\le y^{1/2}, 2\le g\le y^{1/2}} \left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{n\le y} f(1).$$

Firstly, we estimate $S_1(y,f)$. We can write

$$S_1(y,f) = \sum_{2\le g\le y^{1/2}} \left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) \left[\frac{y}{g} \right] + [y] f(1)$$

$$= y \sum_{2\le g\le y^{1/2}} \frac{1}{g} \left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + [y] f(1) + O\left(\sum_{2\le g\le y^{1/2}} \left| \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right| \right). \qquad (26)$$

Using Lemma2.3 (ii) and integration by parts, we can write the sum of the first term of (26)as

$$\sum_{2\le g\le y^{1/2}} \frac{1}{g} \left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right)$$

$$= \sum_{2\le g\le y^{1/2}} \frac{f(g)}{g^k} - \sum_{1\le g\le y^{1/2}-1} \frac{f(g)}{g^{k-1}(g+1)}$$

$$= \sum_{2\le g\le y^{1/2}} \frac{f(g)}{g^k(g+1)} + \frac{f\left(\left[y^{1/2}\right]\right)}{\left[y^{1/2}\right]^{k-1}\left(\left[y^{1/2}\right]+1\right)} - \frac{f(1)}{2}$$

$$= \sum_{2\le g\le y^{1/2}} \frac{f(g)}{g^{k+1}} - \sum_{2\le g\le y^{1/2}} \frac{f(g)}{g^{k+1}(g+1)} + \frac{f\left(\left[y^{1/2}\right]\right)}{\left[y^{1/2}\right]^{k-1}\left(\left[y^{1/2}\right]+1\right)} - \frac{f(1)}{2}$$

$$= \int_{2-}^{y^{1/2}} u^{-2} \mathrm{d}\left(\frac{1}{2} C_f u^2 + O\left(u t_2(u) \log u\right) \right) + O\left(t_1\left(y^{1/2}\right)\right) + O(1)$$

$$= \frac{1}{2} C_f \log y + O\left(t_1\left(y^{1/2}\right)\right), \qquad (27)$$

and the sum of the third term as

$$\sum_{2\le g\le y^{1/2}} \left| \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right| \ll \sum_{2\le g\le y^{1/2}} \left| \frac{f(g)}{g^{k-1}} \right| \ll \sum_{2\le g\le y^{1/2}} g t_1(g) \ll y t_1(y). \qquad (28)$$

Inserting (27) and (28)into (26), we get

$$S_1(y,f) = \frac{1}{2} C_f y \log y + O\left(y t_1(y)\right). \qquad (29)$$

Then, we estimate $S_2(y,f)$. We write

$$S_2(y,f) = S_2^{\dagger}(y,f) + S_2^{\sharp}(y,f), \qquad (30)$$

where

$$S_2^{\dagger}(y,f) := \sum_{gn\le y, g\ge 2, n\le M_0} \left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{n\le M_0} f(1),$$

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$$S_2^{\#}(y,f) := \sum_{gn\le y,\, g\ge 2,\, M_0<n\le y^{1/2}} \left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{M_0<n\le y^{1/2}} f(1).$$

According to the condition (8), we can get

$$S_2^{\dagger}(y,f) = \sum_{n\le M_0} \frac{f([y/n])}{[y/n]^{k-1}} \ll \sum_{n\le M_0} \frac{y}{n} t_1(y) \ll y(\log y)^{2/3}(\log_2 y)^{1/3} t_1(y). \tag{31}$$

On the other hand, by Lemma 2.3(i), we can write

$$\sum_{g\le y} \frac{f(g)}{g^{k-1}} - \sum_{g\le y-1} \frac{f(g)}{g^{k-1}} = \frac{1}{2} C_f \left(y^2 - (y-1)^2 \right) - \frac{(h-[h])^2 + [h]}{2h} C_z$$

$$-\Delta_z(y,h) + \Delta_z(y-1,h) + O\left(\frac{yh}{t_3(y/h)} + \frac{y t_2(y/h)}{h} \right)$$

$$= C_f y - \Delta_z(y,h) + \Delta_z(y-1,h) + O\left(\frac{yh}{t_3(y/h)} + \frac{y t_2(y/h)}{h} \right),$$

where $\Delta_z(y,h)$ is defined by (13).

Thus, according to $1 \le h \le (\log y)^{B+1}$, we can write

$$S_2^{\#}(y,f) = \sum_{M_0<n\le y^{1/2}} \sum_{2\le g\le y/n} \left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{M_0<n\le y^{1/2}} f(1)$$

$$= \sum_{M_0<n\le y^{1/2}} \left\{ C_f \frac{y}{n} - \Delta_z\left(\frac{y}{n},h\right) + \Delta_z\left(\frac{y}{n}-1,h\right) + O\left(\frac{yh}{n t_3(\sqrt{y}/h)} + \frac{y t_2(y/h)}{nh} \right) \right\}$$

$$= \frac{1}{2} C_f y \log y - \Delta_1(y,h) + \Delta_2(y,h)$$

$$+ O\left(y(\log y)^{2/3}(\log_2 y)^{1/3} + \frac{yh\log y}{t_3(y^{1/2}/h)} + \frac{y t_2(y/h)\log y}{h} \right), \tag{32}$$

where

$$\Delta_1(y,h) := \sum_{M_0<n\le y^{1/2}} \Delta_z\left(\frac{y}{n},h\right) \ll \frac{y t_2(y/h)\log y}{h},$$

$$\Delta_2(y,h) := \sum_{M_0<n\le y^{1/2}} \Delta_z\left(\frac{y}{n}-1,h\right) \ll \frac{y t_2(y/h)\log y}{h}.$$

Substituting (31) and (32)into (30), we can obtain

$$S_2(y,f) = \frac{1}{2} C_f y \log y + O(yE(y,h)), \tag{33}$$

where

$$C_f := \sum_{n=1}^{\infty} z(n) n^{-(k+1)}, \quad E(y,h) := (\log y)^{2/3}(\log_2 y)^{1/3} t_1(y) + \frac{t_2(y/h)\log y}{h} + \frac{h\log y}{t_3(y^{1/2}/h)}.$$

Finally, usingcondition (8) to estimate $S_3(y,f)$, we can get

$$S_3(y,f) = \frac{f([y^{1/2}])}{[y^{1/2}]^{k-1}} [y^{1/2}] \ll y t_1(y). \tag{34}$$

From (25), (29), (33) and (34), we get (11).

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3.2ProofofTheorem 1.1 (ii)

Let the remainder $yE(y,h)$ of $S_f(y)$ be $R_f(y)$, define $R_f^*(y) := \max\{|R_f(y)|, |R_f(y-1)|\}$.

Suppose that $\left|\frac{f(p-1)}{(p-1)^{k-1}}\right| < c_1 f(1) p$ holds for infinitely many primes p, then for every prime p there is

$$\sum_{g|p}\left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}}\right) + \sum_{n\le p} f(1) = S_f(p) - S_f(p-1)$$
$$= C_f p\log p - C_f(p-1)\log(p-1) + R_f(p) - R_f(p-1)$$
$$\le 2R_f^*(p) + O(\log p).$$

By using assumption $\left|\frac{f(p-1)}{(p-1)^{k-1}}\right| < c_1 f(1) p$ and condition (10), it can be obtained that for each prime p, there is

$$\sum_{g|p}\left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}}\right) + \sum_{n\le p} f(1) = \frac{f(p)}{p^{k-1}} - \frac{f(p-1)}{(p-1)^{k-1}} + \sum_{n\le p} f(1)$$
$$= z(1)p + \frac{z(p)}{p^{k-1}} f(1) - \frac{f(p-1)}{(p-1)^{k-1}} + \sum_{n\le p} f(1)$$
$$\ge (1-c_1) f(1) p + O\left(\frac{p}{t_3(p)}\right)$$
$$\ge \frac{1}{2}(1-c_1) f(1) p.$$

So for each prime p, we have $R_f^*(p) > \frac{1}{5}(1-c_1) f(1) p$.

Suppose that $\left|\frac{f(p-1)}{(p-1)^{k-1}}\right| > c_1^{-1} f(1) p > 0$ holds for infinitely many primes p, then for every prime p there is

$$\sum_{g|p}\left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}}\right) + \sum_{n\le p} f(1) \ge -2R_f^*(p) + O(\log p).$$

By using assumption $\left|\frac{f(p-1)}{(p-1)^{k-1}}\right| > c_1^{-1} f(1) p > 0$ and condition (10), it can be obtained that for each prime p, there is

$$\sum_{g|p}\left(\frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}}\right) + \sum_{n\le p} f(1) \le -\frac{1}{2}\left(c_1^{-1} - 1\right) f(1) p.$$

So for each prime p ,we have $R_f^*(p) > \frac{1}{5}\left(c_1^{-1} - 1\right) f(1) p > 0$.

In summary, the remainder of $S_f(y)$ is $\Omega(y)$.

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As applications, if $k=1$ in the theorem of this paper, the theorem in Ma, Wu and Zhao[8] can be obtained. If $k=2$ in the theorem of this paper, the theorem in Li and Ma[10] can be obtained.

4. CONCLUSION

In this paper, the asymptotic formula of $S_f(y) := \sum_{n \le y} f([y/n]) / ([y/n]^{k-1}) (k \in \mathbb{N}^{+})$ and the asymptotic form of the remainder are proved by using Dirichlet hyperbolic principle, which provides the results for the mean value estimation of more arithmetical functions, and also provides the idea for calculating the mean value estimation results of arithmetical functions.

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