

# Estimation involving a class of special arithmetical functions

## ABSTRACT

The mean value estimation of arithmetical function is closely related to many problems in number theory. Let  $f$  be an arithmetical function satisfying some conditions. Let  $[r]$  be the integral part of  $r$ . This paper proves the asymptotic expression

$S_f(y) := \sum_{n \leq y} f([y/n]) / ([y/n]^{k-1})$  ( $k \in \mathbb{N}^+$ ) by Dirichlet hyperbolic principle, and that the error term of this asymptotic formula is  $O(y)$ .

*Keywords:* Arithmetical function; Asymptotic formula; Integral part; Dirichlet hyperbolic principle.

## 1. INTRODUCTION

Generally,  $\varphi(n) = \sum_{1 \leq g \leq n, (g,n)=1} 1$  is called Euler totient function.  $\sigma(n) = \sum_{g|n} g$  is called the sum of divisors function.  $\zeta(s)$  is called the Riemann function.  $[r]$  is called the integral part of  $r$ . In 2019, Bordellès, Dai, Heyman, Pan and Shparlinski[1] proved that

$$\left( \frac{2629}{4009} \cdot \frac{1}{\zeta(2)} + o(1) \right) y \log y \leq \sum_{n \leq y} \varphi([y/n]) \leq \left( \frac{2629}{4009} \cdot \frac{1}{\zeta(2)} + \frac{1380}{4009} + o(1) \right) y \log y \quad (1)$$

for  $y \rightarrow \infty$ . In 2019, Wu[2] estimated the bound of  $\sum_{n \leq y} \varphi([y/n])$  by using the method of exponential sum and obtained that

$$\frac{4}{\pi^2} y \log y + O(y) \leq \sum_{n \leq y} \varphi([y/n]) \leq \left( \frac{1}{3} + \frac{4}{\pi^2} \right) y \log y + O(y) \quad (2)$$

for  $y \rightarrow \infty$ . In 2020, using the theory of exponential pairs and the Vinogradov method, Zhai[3] proved that

$$\sum_{n \leq y} \varphi([y/n]) = \frac{y \log y}{\zeta(2)} + O\left(y (\log y)^{2/3} (\log_2 y)^{1/3}\right) \quad (3)$$

for  $y \rightarrow \infty$ . Owing to  $\varphi(n)$  and  $\sigma(n)$  have analogous properties, it is meaningful to study the asymptotic behaviour of  $S_\sigma(y) := \sum_{n \leq y} \sigma([y/n])$ . In 2021, Zhao and Wu[4] proved that

$$\sum_{n \leq y} \sigma([y/n]) = \frac{\pi^2}{6} y \log y + O\left(y (\log y)^{2/3} (\log_2 y)^{4/3}\right) \quad (4)$$

for  $y \rightarrow \infty$ . In 2022, Ma, Wu and Zhao[5], defined arithmetical function  $z$  by Dirichlet convolution for general arithmetical function  $f$ , and obtained the asymptotic estimation results of  $\sum_{n \leq y} f([y/n])$  when  $f$  satisfies certain conditions. In 2023, Li and Ma[6] defined the

arithmetical function  $z$  by Dirichlet convolution for the general arithmetical function  $f$ , and obtained the asymptotic estimation results of  $S_f(y) := \sum_{n \leq y} f([y/n]) / ([y/n])$  when  $f$  satisfies certain conditions.

Naturally, according to the cognitive law from special to general, we consider whether we can get a more extensive conclusion. In this paper, we consider the asymptotic estimation results of  $S_f(y) := \sum_{n \leq y} f([y/n]) / ([y/n]^{k-1})$  ( $k \in \mathbb{N}^+$ ) and the asymptotic form of the remainder when the number theory function  $f$  satisfies certain conditions.

Inspired by [7], let  $t_1, t_2, t_3$  be three increasing functions defined on  $[1, \infty)$  such that when  $y \geq 1$ ,

$$1 \leq t_i(y) \leq y^{\eta_i} \quad (i=1,2,3), \quad t_3(y) \rightarrow \infty, \quad (5)$$

where  $\eta_i$  is a constant on  $(0,1)$ .

Let  $f$  be an arithmetical function, define  $f = id^k * z$  ( $k \in \mathbb{N}^+$ ), and assume that  $f$  satisfies the following conditions:

$$|f(n)| \leq n^k t_1(n) \quad (n \geq 1), \quad (6)$$

$$\sum_{n \leq y} \frac{|z(n)|}{n^{k-1}} \leq y t_2(y) \quad (y \geq 1), \quad (7)$$

$$\sum_{n \leq y} \frac{z(n)}{n^{k-1}} \leq C_z y + O\left(\frac{y}{t_3(y)}\right) \quad (y \geq 1), \quad (8)$$

where  $C_z$  is a constant (can be equal to 0). Then we get the following results.

**Theorem 1.1**(i) For any constant  $B \geq 1, k \in \mathbb{N}^+$ , we have

$$S_f(y) := \sum_{n \leq y} \frac{f([y/n])}{[y/n]^{k-1}} = C_f y \log y + O_B(yE(y, h)) \quad (9)$$

for  $y \geq 3$  and  $1 \leq h \leq (\log y)^{B+1}$ , where  $C_f := \sum_{n=1}^{\infty} z(n) n^{-(k+1)}$ , and

$$E(y, h) := (\log y)^{2/3} (\log_2 y)^{1/3} t_1(y) + \frac{t_2(y/h) \log y}{h} + \frac{h \log y}{t_3(y^{1/2}/h)},$$

where the implied constant is only related to  $B$ .

(ii) If there exists a positive constant  $c_1 < 1$  such that the condition  $\left| \frac{f(p-1)}{(p-1)^{k-1}} \right| < c_1 f(1) p$  or

$\left| \frac{f(p-1)}{(p-1)^{k-1}} \right| > c_1^{-1} f(1) p > 0$  holds for infinitely many primes  $p$ , then the remainder  $yE(y, h)$  of

$S_f(y)$  is  $\Omega(y)$ .

## 2. LEMMAS

In order to prove the theorem, some lemmas are required. Define  $\psi(r) := r - [r] - (1/2)$ .

**Lemma 2.1**[8] For  $y \geq 10$ ,  $\exp\{c_2 (\log y)^{2/3}\} \leq M \leq y^{2/3}$  and  $M < M' \leq 2M$ , there are positive numbers  $c_2$  and  $c_3$  such that

$$\sum_{M \leq n < M'} \frac{1}{n} \psi(y/n) \ll e^{-c_3(\log M)^3 / (\log y)^2} \frac{(\log M)^3}{(\log y)^2}.$$

**Lemma 2.2**[4] Let  $2 \leq b_1 < b_2 \leq y$ ,  $F_y(r) := (1/r)\psi(y/r)$ . Let  $V_{F_y}[b_1, b_2]$  be the total variation of  $F_y$  on  $[b_1, b_2]$ . Then we have

$$V_{F_y}[b_1, b_2] \ll \frac{y}{b_1^2} + \frac{1}{b_1},$$

where the implied constant is absolute.

**Lemma 2.3** (i) Let  $f(n)$  satisfies the conditions (7) and (8), we can obtain

$$\sum_{n \leq y} \frac{f(n)}{n^{k-1}} = \frac{1}{2} C_f y^2 - C_z y \frac{(h - [h])^2 + [h]}{2h} - \Delta_z(y, h) + O\left(\frac{yt_2(y/h)}{h} + \frac{yh}{t_3(y/h)}\right) \quad (10)$$

for  $y \geq 2$  and  $1 \leq h \leq y^{1/3}$ , where  $C_f := \sum_{n=1}^{\infty} z(n)n^{-(k+1)}$  and

$$\Delta_z(y, h) := y \sum_{g \leq y/h} \frac{z_2(g)}{g^k} \psi(y/g). \quad (11)$$

(ii) we have

$$\sum_{n \leq y} \frac{f(n)}{n^{k-1}} = \frac{1}{2} C_f y^2 + O(y(\log y)t_2(y)) \quad (12)$$

for  $y \geq 2$ .

**Proof.** Using  $f(n) = \sum_{gm=n} z(g)m^k$  and Dirichlet hyperbolic principle, we can write

$$\sum_{n \leq y} \frac{f(n)}{n^{k-1}} = \sum_{gm \leq x} \frac{z(g)}{g^{k-1}} m = S_1 + S_2 - S_3, \quad (13)$$

where  $S_1 := \sum_{g \leq y/h} \sum_{m \leq y/g} \frac{z(g)}{g^{k-1}} m$ ,  $S_2 := \sum_{m \leq h} \sum_{g \leq y/m} \frac{z(g)}{g^{k-1}} m$ ,  $S_3 := \sum_{g \leq y/h} \sum_{m \leq h} \frac{z(g)}{g^{k-1}} m$ .

Firstly, according to conditions (7), we can calculate

$$\begin{aligned} S_1 &= \sum_{g \leq x/h} \frac{z(g)}{g^{k-1}} \sum_{m \leq y/g} m \\ &= \sum_{g \leq y/h} \frac{z(g)}{g^{k-1}} \left(1 + 2 + \cdots + \left\lfloor \frac{y}{g} \right\rfloor\right) \\ &= \frac{1}{2} \sum_{g \leq y/h} \frac{z(g)}{g^{k-1}} \left(\frac{y}{g} - \psi\left(\frac{y}{g}\right) - \frac{1}{2}\right) \left(\frac{y}{g} - \psi\left(\frac{y}{g}\right) + \frac{1}{2}\right) \\ &= \frac{y^2}{2} \sum_{g \leq y/h} \frac{z(g)}{g^{k+1}} - \Delta_z(y, h) + O\left(\frac{yt_2(y/h)}{h}\right), \end{aligned} \quad (14)$$

where  $\Delta_z(y, h)$  is defined by (11), and the first term of (14) can be calculated by using condition (8).

$$\begin{aligned} \sum_{g \leq y/h} \frac{z(g)}{g^{k+1}} &= \sum_{g=1}^{\infty} \frac{z(g)}{g^{k+1}} - \sum_{g > y/h} \frac{z(g)}{g^{k+1}} \\ &= C_f - \int_{\frac{y}{h}}^{\infty} \frac{1}{u^2} d\left(C_z u + O\left(\frac{u}{t_3(u)}\right)\right) \end{aligned}$$

$$= C_f - C_z \frac{h}{y} + O\left(\frac{h}{yt_3(y/h)}\right).$$

Then,

$$S_1 = \frac{1}{2}C_f y^2 - \frac{1}{2}C_z h y - \Delta_z(y, h) + O\left(\frac{yt_2(y/h)}{h} + \frac{yh}{t_3(y/h)}\right), \quad (15)$$

where  $C_f := \sum_{n=1}^{\infty} z(n)n^{-(k+1)}$ ,  $\Delta_z(y, h)$  is defined by (11).

Secondly, we can calculate according to conditions (8).

$$\begin{aligned} S_2 &= \sum_{m \leq h} m \sum_{g \leq y/m} \frac{z(g)}{g^{k-1}} \\ &= \sum_{m \leq h} m \left( C_z \frac{y}{m} + O\left(\frac{y}{mt_3(y/m)}\right) \right) \\ &= C_z y [h] + O\left(\frac{yh}{t_3(y/h)}\right), \quad (16) \\ S_3 &= \sum_{g \leq y/h} \frac{z(g)}{g^{k-1}} \sum_{m \leq h} m \\ &= \sum_{g \leq y/h} \frac{z(g)}{g^{k-1}} \frac{[h]([h]+1)}{2} \\ &= \left( C_z \frac{y}{h} + O\left(\frac{y}{ht_3(y/h)}\right) \right) \frac{[h]([h]+1)}{2} \\ &= \frac{[h]([h]+1)}{2h} C_z y + O\left(\frac{yh}{t_3(y/h)}\right). \quad (17) \end{aligned}$$

Substituting the results of (15), (16) and (17) into (13), we can get (10).

Letting  $h=1$  in (10), we can write

$$\Delta_z(y, 1) \ll y \sum_{g \leq y} \frac{|z(g)|}{g^k} \ll y(\log y) t_2(y).$$

Then we can get (12).

**Lemma 2.4** Let  $f$  satisfies the condition (7),  $\Delta_z(y, h)$  be defined by (11). Let  $B$  be a positive constant,  $M_0 := \exp\left\{\left((B+3)/c_3\right)^{1/3} (\log y)^{2/3} (\log_2 y)^{1/3}\right\}$ , where  $c_3$  is a constant given by Lemma 2.1, then for  $y \geq 10$  and  $2 \leq h \leq \sqrt{M_0}$ , we have

$$\left| \sum_{M_0 < n \leq \sqrt{y}} \Delta_z\left(\frac{y}{n}, h\right) \right| + \left| \sum_{M_0 < n \leq \sqrt{y}} \Delta_z\left(\frac{y}{n}-1, h\right) \right| \ll \frac{yt_2(y/h)}{(\log y)^B} + \frac{yt_2(y/h) \log y}{h}. \quad (18)$$

**Proof.** Define  $\Delta_1(y, h) := \sum_{M_0 < n \leq \sqrt{y}} \Delta_z\left(\frac{y}{n}, h\right)$ ,  $\Delta_2(y, h) := \sum_{M_0 < n \leq \sqrt{y}} \Delta_z\left(\frac{y}{n}-1, h\right)$ . By (11), we can write

$$\Delta_1(y, h) = y \sum_{M_0 < n \leq \sqrt{y}} \sum_{g \leq (y/(nh))} \frac{z(g)}{g^k n} \psi\left(\frac{y}{gn}\right)$$

$$\begin{aligned}
&= y \sum_{g \leq (y/(M_0 h))} \frac{z(g)}{g^k} \sum_{M_0 < n \leq \min\{\sqrt{y}, y/(gh)\}} \frac{1}{n} \psi\left(\frac{y}{gn}\right) \\
&= y \Delta_1^\dagger(y, h) + y \Delta_1^\#(y, h), \tag{19}
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1^\dagger(y, h) &:= \sum_{g \leq (y/(M_0 h))} \frac{z(g)}{g^k} \sum_{M_0 < n \leq (y/g)^{2/3}} \frac{1}{n} \psi\left(\frac{y}{gn}\right), \\
\Delta_1^\#(y, h) &:= \sum_{g \leq (y/(M_0 h))} \frac{z(g)}{g^k} \sum_{(y/g)^{2/3} < n \leq \min\{\sqrt{y}, y/(gh)\}} \frac{1}{n} \psi\left(\frac{y}{gn}\right).
\end{aligned}$$

Let  $M_l := 2^l M_0$ , we can deduce  $0 \leq l \leq (\log((y/g)^{2/3}/M_0))/\log 2$ .

Define  $S_l(g) := \sum_{M_l < n \leq 2M_l} \frac{1}{n} \psi\left(\frac{y}{gn}\right)$ . Since  $M_0 \leq M_l \leq (y/g)^{2/3}$ , using Lemma 2.1, we can deduce

$$S_l(g) \ll e^{-\mathcal{G}((\log M_l)^3 / (\log(y/g))^2)},$$

where  $\mathcal{G}(r) := c_3 r - \log r$ .

For  $l \geq 0$  and  $g \leq y/(M_0 h)$ , we can obtain

$$\begin{aligned}
&(\log M_l)^3 / (\log(y/g))^2 \geq (\log M_0)^3 / (\log y)^2 = ((B+3)/c_3) \log_2 y, \\
&\mathcal{G}\left(\frac{(\log M_l)^3}{(\log(y/g))^2}\right) \geq \mathcal{G}\left(\frac{(B+3)}{c_3} \log_2 y\right) \\
&= (B+3) \log_2 y - \log\left(\frac{B+3}{c_3} \log_2 y\right) \geq (B+2) \log_2 y,
\end{aligned}$$

then  $S_l(g) \ll (\log y)^{-B-2}$ .

Using condition(7), we can write

$$\begin{aligned}
\Delta_1^\dagger(y, h) &\ll \sum_{g \leq (y/(M_0 h))} \frac{|z(g)|}{g^k} \sum_{2^l M_0 \leq (y/g)^{2/3}} |S_l(g)| \\
&\ll \frac{1}{(\log y)^{B+1}} \sum_{g \leq y/h} \frac{|z(g)|}{g^k} \ll \frac{t_2(y/h)}{(\log y)^B}. \tag{20}
\end{aligned}$$

Now we bound  $\Delta_1^\#(y, h)$ . This requires the help of the following summation formula:

$$\sum_{M_1 < n \leq M_2} F(n) = \int_{M_1+1}^{M_2+1} F(u) du + \frac{1}{2}(F(M_1+1) - F(M_2+1)) + O(V_F[M_1+1, M_2+1]). \tag{21}$$

This formula has been proved in Zhao and Wu[4]. In the formula,  $F(r)$  is a bounded variation function defined on the closed interval  $[n, n+1]$ , and  $V_F[n, n+1]$  is the total variation of  $F$ . We apply (21) to

$$F_{(y/g)}(r) = \frac{1}{r} \psi\left(\frac{y/g}{r}\right), \quad M_1 = \lceil (y/g)^{2/3} \rceil, \quad M_2 = \left\lceil \min\left\{\sqrt{y}, \frac{y}{gh}\right\} \right\rceil.$$

By Lemma 2.2, we can write

$$V_{F_{(y/g)}}[M_1+1, M_2+1] \ll \frac{y/g}{(M_1+1)^2} + \frac{1}{(M_1+1)}$$

$$= \frac{y/g}{\left((y/g)^{2/3} - \{(y/g)^{2/3}\} + 1\right)^2} + \frac{1}{\left((y/g)^{2/3} - \{(y/g)^{2/3}\} + 1\right)}$$

$$\square \left(\frac{y}{g}\right)^{-1/3}.$$

Letting  $v = y/(dn)$ , we can get  $\sum_{(y/g)^{2/3} < n \leq \min\{y^{1/2}, \frac{y}{gh}\}} \frac{1}{n} \psi\left(\frac{y}{gn}\right) \square h^{-1} + \left(\frac{y}{g}\right)^{-1/3} \square h^{-1}$ .

Thus,

$$\Delta_1^\#(y, h) \square h^{-1} \sum_{g \leq y/h} \frac{|z(g)|}{g^k} \square \frac{t_2(y/h) \log y}{h}. \quad (22)$$

From(19), (20)and (22), we get

$$|\Delta_1(y, h)| \square \frac{yt_2(y/h)}{(\log y)^B} + \frac{yt_2(y/h) \log y}{h}.$$

Using the same proof idea, we can get that  $|\Delta_2(y, h)|$  has the same bound.

### 3. THE PROOF PROCESS OF THEOREM 1.1

#### 3.1 Proof of Theorem 1.1 (i)

Letting  $g = \left\lfloor \frac{y}{n} \right\rfloor$ , we can derive  $\frac{y}{n} - 1 < g \leq \frac{y}{n}$  and  $\frac{y}{g+1} < n \leq \frac{y}{g}$ . Let  $f(0) = 0$ , we have

$$S_f(y) = \sum_{g \leq y} \frac{f(g)}{g^{k-1}} \sum_{\substack{\frac{y}{g+1} < n \leq \frac{y}{g} \\ g+1 < n \leq \frac{y}{g}}} 1$$

$$= \sum_{gn \leq y} \frac{f(g)}{g^{k-1}} - \sum_{gn \leq y, g \geq 2} \frac{f(g-1)}{(g-1)^{k-1}}$$

$$= \sum_{gn \leq y, g \geq 2} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{n \leq y} f(1)$$

$$= S_1(y, f) + S_2(y, f) - S_3(y, f), \quad (23)$$

where

$$S_1(y, f) := \sum_{gn \leq y, 2 \leq g \leq y^{1/2}} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{n \leq y} f(1),$$

$$S_2(y, f) := \sum_{gn \leq y, g \geq 2, n \leq y^{1/2}} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{n \leq y} f(1),$$

$$S_3(y, f) := \sum_{n \leq y^{1/2}, 2 \leq g \leq y^{1/2}} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{n \leq y} f(1).$$

Firstly, we estimate  $S_1(y, f)$ . We can write

$$S_1(y, f) = \sum_{2 \leq g \leq y^{1/2}} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) \left[ \frac{y}{g} \right] + [y] f(1)$$

$$= y \sum_{2 \leq g \leq y^{1/2}} \frac{1}{g} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + [y]f(1) + O \left( \sum_{2 \leq g \leq y^{1/2}} \left| \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right| \right). \quad (24)$$

Using Lemma 2.3 (ii) and integration by parts, we can write the sum of the first term of (24) as

$$\begin{aligned} & \sum_{2 \leq g \leq y^{1/2}} \frac{1}{g} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) \\ &= \sum_{2 \leq g \leq y^{1/2}} \frac{f(g)}{g^k} - \sum_{1 \leq g \leq y^{1/2}-1} \frac{f(g)}{g^{k-1}(g+1)} \\ &= \sum_{2 \leq g \leq y^{1/2}} \frac{f(g)}{g^k(g+1)} + \frac{f(\lceil y^{1/2} \rceil)}{\lceil y^{1/2} \rceil^{k-1}(\lceil y^{1/2} \rceil+1)} - \frac{f(1)}{2} \\ &= \sum_{2 \leq g \leq y^{1/2}} \frac{f(g)}{g^{k+1}} - \sum_{2 \leq g \leq y^{1/2}} \frac{f(g)}{g^{k+1}(g+1)} + \frac{f(\lceil y^{1/2} \rceil)}{\lceil y^{1/2} \rceil^{k-1}(\lceil y^{1/2} \rceil+1)} - \frac{f(1)}{2} \\ &= \int_{2-}^{y^{1/2}} u^{-2} d \left( \frac{1}{2} C_f u^2 + O(u t_2(u) \log u) \right) + O(t_1(y^{1/2})) + O(1) \\ &= \frac{1}{2} C_f \log y + O(t_1(y^{1/2})), \quad (25) \end{aligned}$$

and the sum of the third term as

$$\sum_{2 \leq g \leq y^{1/2}} \left| \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right| \ll \sum_{2 \leq g \leq y^{1/2}} \left| \frac{f(g)}{g^{k-1}} \right| \ll \sum_{2 \leq g \leq y^{1/2}} g t_1(g) \ll y t_1(y). \quad (26)$$

Inserting (25) and (26) into (24), we get

$$S_1(y, f) = \frac{1}{2} C_f y \log y + O(y t_1(y)). \quad (27)$$

Then, we estimate  $S_2(y, f)$ . We write

$$S_2(y, f) = S_2^\dagger(y, f) + S_2^\#(y, f), \quad (28)$$

where

$$\begin{aligned} S_2^\dagger(y, f) &:= \sum_{gn \leq y, g \geq 2, n \leq M_0} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{n \leq M_0} f(1), \\ S_2^\#(y, f) &:= \sum_{gn \leq y, g \geq 2, M_0 < n \leq y^{1/2}} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{M_0 < n \leq y^{1/2}} f(1). \end{aligned}$$

According to the condition (6), we can get

$$S_2^\dagger(y, f) = \sum_{n \leq M_0} \frac{f(\lceil y/n \rceil)}{\lceil y/n \rceil^{k-1}} \ll \sum_{n \leq M_0} \frac{y}{n} t_1(y) \ll y (\log y)^{2/3} (\log_2 y)^{1/3} t_1(y). \quad (29)$$

On the other hand, by Lemma 2.3(i), we can write

$$\begin{aligned} \sum_{g \leq y} \frac{f(g)}{g^{k-1}} - \sum_{g \leq y-1} \frac{f(g)}{g^{k-1}} &= \frac{1}{2} C_f (y^2 - (y-1)^2) - \frac{(h - [h])^2 + [h]}{2h} C_z \\ &- \Delta_z(y, h) + \Delta_z(y-1, h) + O \left( \frac{yh}{t_3(y/h)} + \frac{y t_2(y/h)}{h} \right) \end{aligned}$$

$$= C_f y - \Delta_z(y, h) + \Delta_z(y-1, h) + O\left(\frac{yh}{t_3(y/h)} + \frac{yt_2(y/h)}{h}\right),$$

where  $\Delta_z(y, h)$  is defined by (11).

Thus, according to  $1 \leq h \leq (\log y)^{B+1}$ , we can write

$$\begin{aligned} S_2^\#(y, f) &= \sum_{M_0 < n \leq y^{1/2}} \sum_{2 \leq g \leq y/n} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{M_0 < n \leq y^{1/2}} f(1) \\ &= \sum_{M_0 < n \leq y^{1/2}} \left\{ C_f \frac{y}{n} - \Delta_z\left(\frac{y}{n}, h\right) + \Delta_z\left(\frac{y}{n} - 1, h\right) + O\left(\frac{yh}{nt_3(\sqrt{y}/h)} + \frac{yt_2(y/h)}{nh}\right) \right\} \\ &= \frac{1}{2} C_f y \log y - \Delta_1(y, h) + \Delta_2(y, h) \\ &\quad + O\left(y(\log y)^{2/3} (\log_2 y)^{1/3} + \frac{yh \log y}{t_3(y^{1/2}/h)} + \frac{yt_2(y/h) \log y}{h}\right), \end{aligned} \quad (30)$$

where

$$\begin{aligned} \Delta_1(y, h) &:= \sum_{M_0 < n \leq y^{1/2}} \Delta_z\left(\frac{y}{n}, h\right) \square \frac{yt_2(y/h) \log y}{h}, \\ \Delta_2(y, h) &:= \sum_{M_0 < n \leq y^{1/2}} \Delta_z\left(\frac{y}{n} - 1, h\right) \square \frac{yt_2(y/h) \log y}{h}. \end{aligned}$$

Substituting (29) and (30) into (28), we can obtain

$$S_2(y, f) = \frac{1}{2} C_f y \log y + O(yE(y, h)), \quad (31)$$

where

$$C_f := \sum_{n=1}^{\infty} z(n) n^{-(k+1)}, \quad E(y, h) := (\log y)^{2/3} (\log_2 y)^{1/3} t_1(y) + \frac{t_2(y/h) \log y}{h} + \frac{h \log y}{t_3(y^{1/2}/h)}.$$

Finally, using condition (6) to estimate  $S_3(y, f)$ , we can get

$$S_3(y, f) = \frac{f(\lceil y^{1/2} \rceil)}{\lceil y^{1/2} \rceil^{k-1}} \lceil y^{1/2} \rceil \square yt_1(y). \quad (32)$$

From (23), (27), (31) and (32), we get (9).

### 3.2 Proof of Theorem 1.1 (ii)

Let the remainder  $yE(y, h)$  of  $S_f(y)$  be  $R_f(y)$ , define  $R_f^*(y) := \max\{|R_f(y)|, |R_f(y-1)|\}$ .

Suppose that  $\left| \frac{f(p-1)}{(p-1)^{k-1}} \right| < c_1 f(1) p$  holds for infinitely many primes  $p$ , then for every prime

$p$  there is

$$\begin{aligned} \sum_{g|p} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{n \leq p} f(1) &= S_f(p) - S_f(p-1) \\ &= C_f p \log p - C_f (p-1) \log(p-1) + R_f(p) - R_f(p-1) \\ &\leq 2R_f^*(p) + O(\log p). \end{aligned}$$

By using assumption  $\left| \frac{f(p-1)}{(p-1)^{k-1}} \right| < c_1 f(1)p$  and condition (8), it can be obtained that for each prime  $p$ , there is

$$\begin{aligned} \sum_{g|p} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{n \leq p} f(1) &= \frac{f(p)}{p^{k-1}} - \frac{f(p-1)}{(p-1)^{k-1}} + \sum_{n \leq p} f(1) \\ &= z(1)p + \frac{z(p)}{p^{k-1}} f(1) - \frac{f(p-1)}{(p-1)^{k-1}} + \sum_{n \leq p} f(1) \\ &\geq (1-c_1)f(1)p + O\left(\frac{p}{t_3(p)}\right) \\ &\geq \frac{1}{2}(1-c_1)f(1)p. \end{aligned}$$

So for each prime  $p$ , we have  $R_f^*(p) > \frac{1}{5}(1-c_1)f(1)p$ .

Suppose that  $\left| \frac{f(p-1)}{(p-1)^{k-1}} \right| > c_1^{-1}f(1)p > 0$  holds for infinitely many primes  $p$ , then for every prime  $p$  there is

$$\sum_{g|p} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{n \leq p} f(1) \geq -2R_f^*(p) + O(\log p).$$

By using assumption  $\left| \frac{f(p-1)}{(p-1)^{k-1}} \right| > c_1^{-1}f(1)p > 0$  and condition (8), it can be obtained that for each prime  $p$ , there is

$$\sum_{g|p} \left( \frac{f(g)}{g^{k-1}} - \frac{f(g-1)}{(g-1)^{k-1}} \right) + \sum_{n \leq p} f(1) \leq -\frac{1}{2}(c_1^{-1}-1)f(1)p.$$

So for each prime  $p$ , we have  $R_f^*(p) > \frac{1}{5}(c_1^{-1}-1)f(1)p > 0$ .

In summary, the remainder of  $S_f(y)$  is  $\Omega(y)$ .

As applications, if  $k=1$  in the theorem of this paper, the theorem in Ma, Wu and Zhao[5] can be obtained. If  $k=2$  in the in the theorem of this paper, the theorems in Li and Ma[6] can be obtained.

#### 4. CONCLUSION

In this paper, the asymptotic formula of  $S_f(y) := \sum_{n \leq y} f([y/n]) / ([y/n]^{k-1})$  ( $k \in \mathbb{N}^+$ ) and the

asymptotic form of the remainder are proved by using Dirichlet hyperbolic principle, which provides the results for the mean value estimation of more arithmetical functions, and also provides the idea for calculating the mean value estimation results of arithmetical functions.

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