Estimation of Reliability under Conditional Stress – Strength Setup based on Weibull Distribution

Abstract

In this research, for independent Weibull random variables X and Y with common shapes but different scale parameters, the reliability under conditional stress-strength model is estimated. The maximum likelihood estimator, asymptotic confidence interval, Bootstrap estimators, Boot-p estimators, and Bayes estimator under-squared error loss function with associated highest posterior density interval are constructed for conditional stress-strength reliability. Simulation study is conducted to estimate MSE of estimator of the conditional stress-strength reliability. The real data analysis is also carried out.

Keywords: Weibull distribution, Stress-strength reliability, Conditional Stress-strength model, Maximum likelihood estimator, Bootstrap confidence Interval, Bayes estimator, MCMC technique

1 Introduction

The capacity of a system or component to carry out a necessary task in a particular environment for a predetermined amount of time is known as reliability. In other words, reliability is the probability that the system will perform satisfactorily for intended period of time. The well-known stress - strength reliability model compares the strength and stress on a certain system and it is denoted by R, is defined as $R = P(X > Y)$ where X denotes strength of a component, which is under stress Y. In the stress-strength modeling, R is a measure of component reliability when it is subjected to random stress Y and has strength X. Therefore R and $(1 - R)$ indicate the system performance and probability of system failure respectively. For example, in the event that Y denotes a treatment group and X represents the response for the control group, R represents the treatment's impact.It is not necessary for Stress and Strength to be associated in any way because of their nature. Consequently, a number of authors used the premise that X and Y are independent variables to draw conclusions regarding $P(X < Y)$. Applications of reliability can be found in many fields, including engineering, biostatistics, quality control, economics, psychology, and medicine.

The work of Wilcoxon (1945), Mann and Whitney (1947), Birnbaum (1956), Birnbaum and McCarty (1948), Govindajulu (1967, 1968), Owen et al. (1964), Sen (1960, 1967), and others established the Stress and Strength model in a nonparametric setup. It is said that Church and Harris first used the term "stress-strength." There are several published studies that looked at different alternatives for distributions of stress and strength. The initial work by Owen et al in 1964 focused on constructing confidence limits for the probability $P(X < Y)$ assuming dependence or independence between normally distributed random variables X and Y. Subsequent research expanded the estimation of this probability for various distributions, including exponential, normal, Pareto, and even broader exponential families. For instance, Kelly et al (1976), Tong (1974), Church and Harris (1970), Beg and Singh (1979), and Tong (1977) contributed to estimating $P(X < Y)$ under different distributional assumptions. The introduction of time-dependent models by Bilikam in 1985 was a significant step. Bilikam's model considered stress and strength as continuous random processes. In this framework, X and Y were assumed to be stochastically independent but related through time-dependent parameters $\theta_1(t)$ and $\theta_2(t)$. Recently, Erylmaz S (2011) investigated stress-strength reliability within the framework of multi-state system modeling. Additionally, Tarvirdizade B and Ahmadpour M (2016) examined the estimation of stress-strength reliability $P(X > Y)$ by utilizing upper record values. This estimation was carried out for independent random variables X and Y, assuming a two-parameter bathtub-shaped lifetime distribution with identical parameters but different scale parameters.

The recent years the problem of estimating the reliability under a conditional stress-strength set-up was attempted by researchers such as Saber and Khorshidian (2020), Raid et.al (2021), and Saber et.al (2021). They studied the reliability of the system in a conditional stress–strength set-up when stress – strength variable follows the Exponential distribution, Kumaraswamy Distribution, and Generalized exponential lifetime distribution. In all these studies the authors considered the estimation of reliability using classical and Bayesian approaches. They also considered bootstrap confidence intervals. In a conditional stress-strength setup, a system's reliability is represented by $R^{|a,b}$ was introduced by Saber and Khorshidian (2020). The reliability under conditional stress-strength reliability is given by:

$$R^{|a,b} = P(X > Y | X > a, Y > b) = \begin{cases} \frac{\bar{F}_Y(b) - \int_b^\infty F_X(y) f_Y(y) dy}{\bar{F}_X(b)\bar{F}_Y(b)} & a = b \\ \frac{\bar{F}_Y(b) - \int_b^\infty F_X(y) f_Y(y) dy}{\bar{F}_X(a)\bar{F}_Y(b)} & a < b \\ \frac{\int_a^\infty F_Y(x) f_X(x) dx - F_Y(b)\bar{F}_X(a)}{\bar{F}_X(a)\bar{F}_Y(b)} & a > b \end{cases} \tag{1}$$

The Weibull distribution has been extensively studied and applied across various fields due to its versatility in modeling a wide range of phenomena, especially in reliability engineering, survival analysis, and lifetime modeling. Its high flexibility and wide range of shapes allow it to have failure rates that are increasing constant, and decreasing. As a result, it has a wide range of uses, including in hydrology, industrial engineering, weather forecasting, and insurance. Researchers such as Nelson (1982) and Meeker and Escobar (1998) extensively use the Weibull distribution in reliability engineering. Kundu and Gupta (2006) studied the estimate of $R = P(Y < X)$ where $X \sim W(\alpha, \theta_1)$ and $X \sim W(\alpha, \theta_2)$ are two independent Weibull distributions with distinct scale parameters but the same shape parameter. But this paper studies the conditional stress–strength model for Weibull distribution. The Weibull distribution with parameter (γ, θ) is denoted by $W(\gamma, \theta)$. The cumulative distribution function (cdf) and the probability density function (pdf) of this distribution are, respectively,

$$F(z) = 1 - e^{(-\gamma z^{\theta})}, z > 0, \gamma, \theta > 0 \tag{2}$$

And

$$f(z) = \gamma\theta z^{\theta-1} e^{(-\gamma z^{\theta})}, z > 0, \gamma, \theta > 0 \tag{3}$$

The remainder of this paper is structured as follows: Section 2 discusses how $R^{|a,b}$ is derived in the case of the Weibull distribution. Section 3 presents the Maximum Likelihood estimator (MLE) of reliability, $R^{|a,b}$, along with its associated asymptotic distribution and confidence interval. Section 4 considers the bootstrap method used to estimate $R^{|a,b}$ and construct dependability confidence intervals. The Bayesian technique to reliability estimate is discussed in Section 5. Sections 6 and 7 present the simulation findings and real data analysis, respectively.

2 Conditional Stress – Strength Reliability for Weibull distribution.

When the variable strength (X) and stress (Y) are independently distributed with $W(\gamma_1, \theta)$ and $W(\gamma_2, \theta)$, respectively, the conditional reliability of the stress-strength model is derived in this section. Below is a result representing the reliability in a conditional stress-strength set-up.

Result: Let X and Y be independent random variables from the Weibull distributions with parameters (γ_1, θ) and (γ_2, θ) respectively, that is $X \sim W(\gamma_1, \theta)$ and $Y \sim W(\gamma_2, \theta)$. Based on conditional stress - strength model, the reliability $R^{|a,b}$ is given by:

$$R^{|a,b} = \begin{cases} \frac{\gamma_2}{\gamma_1+\gamma_2} & a = b \\ \\ \frac{\gamma_2}{\gamma_1+\gamma_2} e^{-\gamma_1(b^\theta - a^\theta)} & a < b \\ \\ 1 - \frac{\gamma_1}{\gamma_1+\gamma_2} e^{-\gamma_2(a^\theta - b^\theta)} & a > b \end{cases} \tag{4}$$

Proof: Suppose the strength variable X follows $W(\gamma_1, \theta)$ and stress variable Y follows $W(\gamma_2, \theta)$ distributions.

Case $a = b$: The reliability under conditional stress - strength set up, $R^{|a,b}$ can be derived from (1), (2), (3) as

$$R^{|a,b} = \frac{\bar{F}_Y(b) - \int_b^\infty F_X(y) f_Y(y) dy}{\bar{F}_X(b)\bar{F}_Y(b)} = \frac{\gamma_2}{\gamma_1 + \gamma_2} \tag{5}$$

Case $a < b$: The reliability under conditional stress - strength set up, $R^{|a,b}$ can be derived from (1), (2), (3) as

$$R^{|a,b} = \frac{\bar{F}_Y(b) - \int_b^\infty F_X(y) f_Y(y) dy}{\bar{F}_X(a)\bar{F}_Y(b)} = \frac{\gamma_2}{\gamma_1 + \gamma_2} e^{-\gamma_1(b^\theta - a^\theta)} \tag{6}$$

Case $a > b$: The reliability under conditional stress - strength set up, $R^{|a,b}$ can be derived from (1), (2), (3) as

$$R^{|a,b} = \frac{\int_a^\infty F_Y(x) f_X(x) dx - F_Y(b)\bar{F}_X(a)}{\bar{F}_X(a)\bar{F}_Y(b)} = \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{\lambda_1(e^{a^\theta} - e^{b^\theta})} = 1 - \frac{\gamma_1}{\gamma_1 + \gamma_2} e^{-\gamma_2(a^\theta - b^\theta)} \tag{7}$$

Combining (5), (6), and (7) together we get (4).

3 Likelihood inference of conditional reliability

This section deduces the maximum likelihood estimator (MLE) of $R^{|a,b}$ and establishes the asymptotic distribution of the MLE of $R^{|a,b}$. The confidence intervals are constructed using the asymptotic distribution of MLE of $R^{|a,b}$. Suppose that two random samples, $(X_1, X_2, ...X_n)$ and $(Y_1, Y_2, ...Y_m)$, of sizen and m, respectively, from $W(\gamma_1, \theta)$ and $W(\gamma_2, \theta)$. The likelihood function for X and Y respectively given in (8) and (9).

$$L(\gamma_1, \theta | \underline{x}) = \gamma_1^n \theta^n \left(\prod_{i=1}^n x_i^{\theta - 1} \right) e^{\sum_{i=1}^n -\gamma_1 x_i^\theta} \tag{8}$$

$$L(\gamma_2, \theta | \underline{y}) = \gamma_2^m \theta^m \left(\prod_{i=1}^m y_i^{\theta - 1} \right) e^{\sum_{i=1}^m -\gamma_2 y_i^\theta} \tag{9}$$

Then, the joint log-likelihood function is given by

$$l(\gamma_1, \gamma_2, \theta | \underline{x}, \underline{y}) = n \log \gamma_1 + m \log \gamma_2 + (n+m) \log \theta - \gamma_1 \sum_{i=1}^{n} x_i^{\theta} - \gamma_2 \sum_{i=1}^{m} y_i^{\theta} + (\theta - 1) \left(\sum_{i=1}^{n} \log x_i + \sum_{i=1}^{m} \log y_i \right) \quad (10)$$

The likelihood equations are obtained as

$$\frac{\partial l}{\partial \gamma_1} = \frac{n}{\gamma_1} - \sum_{i=1}^{n} x_i^{\theta} = 0 \quad (11)$$

$$\frac{\partial l}{\partial \gamma_2} = \frac{m}{\gamma_2} - \sum_{i=1}^{m} y_i^{\theta} = 0 \quad (12)$$

$$\frac{\partial l}{\partial \theta} = \frac{m+n}{\theta} + \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{m} \log y_i - \gamma_1 \sum_{i=1}^{n} x_i^{\theta} \log x_i - \gamma_2 \sum_{i=1}^{m} y_i^{\theta} \log y_i = 0 \quad (13)$$

The MLE of the parameters γ_1 and γ_2 are given by

$$\hat{\gamma_1} = \frac{n}{\sum_{i=1}^{n} x_i^{\hat{\theta}}} \quad (14)$$

$$\hat{\gamma_2} = \frac{m}{\sum_{i=1}^{m} y_i^{\hat{\theta}}} \quad (15)$$

And $\hat{\theta}$ is the MLE of the parameter θ which is obtained by solving non-linear equation (16)

$$h(\theta) = (n + m) \left\{ \hat{\gamma_1} \sum_{i=1}^{n} x_i^{\theta} \log x_i + \hat{\gamma_2} \sum_{i=1}^{m} y_i^{\theta} \log y_i - \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{m} \log y_i \right\}^{-1} \quad (16)$$

The solution $\hat{\theta}$ to a nonlinear equation (16) can be found through an iterative process. This process continues until the difference between consecutive values of θ_j and θ_{j+1} becomes sufficiently small that is $|\theta_j - \theta_{j+1}|$ is very small. This indicates that the iterations have reached a point where terminating them is appropriate, as they have likely converged to the solution. The MLE of $R^{|a,b}$ is obtained as

$$\hat{R^{|a,b}} = \begin{cases} \frac{\hat{\gamma_2}}{\hat{\gamma_1}+\hat{\gamma_2}} & a = b \\ \frac{\hat{\gamma_2}}{\hat{\gamma_1}+\hat{\gamma_2}} e^{-\hat{\gamma_1}(b^{\hat{\theta}} - a^{\hat{\theta}})} & a < b \\ 1 - \frac{\hat{\gamma_1}}{\hat{\gamma_1}+\hat{\gamma_2}} e^{-\hat{\gamma_2}(a^{\hat{\theta}} - b^{\hat{\theta}})} & a > b \end{cases} \quad (17)$$

Asymptotic Confidence Interval

The asymptotic distributions of $\hat{\beta} = \left(\hat{\gamma_1}, \hat{\gamma_2}, \hat{\theta}\right)$ and $R^{|a,b}$ are determined. The Fisher information matrix of $\beta = (\gamma_1, \gamma_2, \theta)$, denoted as $I(\beta) = E(JI(\beta))$, where $J(\beta) = [J_{i,j}]_{i,j=1,2,3}$ represents the observed information matrix. The information matrix $J(\beta)$ is given by:

$$J(\beta) = -\begin{pmatrix} \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\gamma_1^2} & \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\gamma_1\partial\gamma_2} & \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\gamma_1\partial\theta} \\ \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\gamma_2\partial\gamma_1} & \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\gamma_2^2} & \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\gamma_2\partial\theta} \\ \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\theta\partial\gamma_1} & \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\theta\partial\gamma_2} & \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\theta^2} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix} \tag{18}$$

and the elements of $J(\beta)$ are as follows:

$$J_{11} = -\frac{n}{\gamma_1^2} \qquad J_{22} = -\frac{m}{\gamma_2^2} \qquad J_{12} = J_{21} = 0;$$

$$J_{13} = J_{31} = -\sum_{i=1}^{n} x_i^{\theta} \log x_i;$$

$$J_{23} = J_{32} = -\sum_{i=1}^{m} y_i^{\theta} \log y_i;$$

$$J_{33} = -\frac{n+m}{\theta^2} - \gamma_1 \sum_{i=1}^{n} x_i^{\theta} (\log x_i)^2 - \gamma_2 \sum_{i=1}^{m} y_i^{\theta} (\log y_i)^2$$

The components of the Fisher information matrix are derived by computing the expected values of the observed matrix $J(\beta)$, expressed as $I(\beta) = E[J(\beta)]$. This Fisher information matrix $I(\beta)$ can be computed as

$$I(\beta) = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \tag{19}$$

Where

$$\begin{aligned}
&I_{11} = -E(J_{11}) = \frac{n}{\gamma_1^2} \\
&I_{12} = I_{21} = -E(J_{12}) = -E(J_{21}) = 0 \\
&I_{13} = J_{31} = -E(J_{13}) = -E(J_{31}) = E(\textstyle\sum_{i=1}^{n} x_i^{\theta} \log x_i) \\
&I_{22} = -E(I_{22}) = \frac{m}{\gamma_2^2} \\
&I_{23} = I_{32} = -E(J_{23}) = -E(J_{32}) = E(\textstyle\sum_{i=1}^{m} y_i^{\theta} \log y_i) \\
&I_{33} = -E(J_{33}) = \frac{n+m}{\theta^2} + \gamma_1 E\left(\textstyle\sum_{i=1}^{n} x_i^{\theta} (\log x_i)^2\right) + \gamma_2 E\left(\textstyle\sum_{i=1}^{m} y_i^{\theta} (\log y_i)^2\right)
\end{aligned}$$

As $n \to \infty$ and $m \to \infty$, then by using of multivariate central limit theorem (CLT) of $\hat{\theta}$, we have

$\hat{\theta} \rightarrow N_3\left(\theta, \Sigma\right)$ where $\hat{\theta} = \left(\hat{\lambda_1}, \hat{\lambda_2}, \hat{\beta}\right)$ and Σ is inverse of the Fisher information matrix:

$$\Sigma = \frac{1}{\det I(\beta)} \begin{pmatrix} I_{22}I_{33} - I_{32}I_{23} & I_{13}I_{32} - I_{12}I_{33} & I_{12}I_{23} - I_{13}I_{22} \\ I_{23}I_{31} - I_{21}I_{33} & I_{11}I_{33} - I_{13}I_{31} & I_{13}I_{21} - I_{11}I_{23} \\ I_{21}I_{32} - I_{22}I_{31} & I_{12}I_{31} - I_{11}I_{32} & I_{11}I_{22} - I_{12}I_{21} \end{pmatrix} \tag{20}$$

The asymptotic distribution of $R^{|a,b}$ is derived by employing the multivariate Delta approach as outlined in the following lemma.

Lemma: Consider a sequence $\{X_n\}_{n=1}^{\infty}$ of random vectors converging in distribution to $N_k(\mu, \Sigma)$ i.e. $X_n \rightarrow N_k(\mu, \Sigma)$. Let $g(x) : R_k \rightarrow R$ be a function continuous in its first partial derivatives, and let $\sigma^2 = \Delta^T \Sigma \Delta > 0$, where $\Delta = \frac{\partial g(\mu)}{\partial \mu}$. Then, $\frac{g(X_n) - g(\mu)}{\sigma} \rightarrow N(0, 1)$

Result: As sample size increases, $\frac{\hat{R}^{|a,b} - R^{a,b}}{\sigma}$ converges in distribution to a standard normal distribution. Mathematically, As $n \rightarrow \infty$ and $m \rightarrow \infty$, then

$$\frac{\hat{R}^{|a,b} - R^{a,b}}{\sigma} \xrightarrow{d} N(0, 1) \tag{21}$$

Here, σ^2 is obtained with the help of $g(\beta)\Sigma g(\beta)^T$ for different cases $a = b, a < b, a > b$.
Also $\beta = (\lambda_1, \lambda_2, \beta)$, and $g(\beta)$ denotes the derivative of $R^{|a,b}$ with respect to β. It's represented as $g(\beta) = \left(\frac{\partial R^{|a,b}}{\partial \gamma_1}, \frac{\partial R^{|a,b}}{\partial \gamma_2}, \frac{\partial R^{|a,b}}{\partial \theta}\right)$. Additionally, Σ stands for the inverse of the Fisher information matrix.

Proof: Take the partial derivatives of $R^{|a,b}$, as outlined in equation (4), with respect to γ_1, γ_2, and θ to derive equations (22), (23), and (24) correspondingly.

$$\frac{\partial R^{|a,b}}{\partial \gamma_1} = \begin{cases} -\frac{\gamma_2}{(\gamma_1+\gamma_2)^2} & a = b \\ \frac{\gamma_2}{\gamma_1+\gamma_2} e^{-\gamma_1(b^\theta - a^\theta)} \left(a^\theta - b^\theta - \frac{1}{(\gamma_1+\gamma_2)}\right) & a < b \\ -\frac{\gamma_2}{(\gamma_1+\gamma_2)^2} e^{-\gamma_2(a^\theta - b^\theta)} & a > b \end{cases} \tag{22}$$

$$\frac{\partial R^{|a,b}}{\partial \gamma_2} = \begin{cases} \frac{\gamma_1}{(\gamma_1+\gamma_2)^2} & a = b \\ \frac{\gamma_1}{(\gamma_1+\gamma_2)^2} e^{-\gamma_1(a^\theta - b^\theta)} & a < b \\ \frac{\gamma_1}{\gamma_1+\gamma_2} e^{-\gamma_2(b^\theta - a^\theta)} \left(a^\theta - b^\theta + \frac{1}{\gamma_1+\gamma_2}\right) & a > b \end{cases} \tag{23}$$

$$\frac{\partial R^{|a,b}}{\partial \theta} = \begin{cases} 0 & a = b \\ \\ \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2} e^{-\gamma_1(b^\theta - a^\theta)} \left(a^\theta \log a - b^\theta \log b\right) & a < b \\ \\ \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2} e^{-\gamma_2(a^\theta - b^\theta)} \left(a^\theta \log a - b^\theta \log b\right) & a > b \end{cases} \tag{24}$$

Since $(\hat{\beta} - \beta) \to N(\beta, \Sigma)$, Using Cramer's theorem,

$$\left(g(\hat{\beta}) - g(\beta)\right) \to N(0, g(\beta)\Sigma g(\beta)^T) \tag{25}$$

Where $g(\beta) = \frac{\partial R^{|a,b}}{\partial \beta} = \left(\frac{\partial R^{|a,b}}{\partial \gamma_1}, \frac{\partial R^{|a,b}}{\partial \gamma_2}, \frac{\partial R^{|a,b}}{\partial \theta}\right)$
The confidence interval at the $(1-\alpha)\%$ level for $R^{|a,b}$ is expressed as

$$\begin{aligned} R^{|a,b} &\in \left(\hat{R^{|a,b}} - Z_{1-\frac{\alpha}{2}}\sigma_1^2, \hat{R^{|a,b}} + Z_{1-\frac{\alpha}{2}}\sigma_1^2\right) \quad & a = b \\ R^{|a,b} &\in \left(\hat{R^{|a,b}} - Z_{1-\frac{\alpha}{2}}\sigma_2^2, \hat{R^{|a,b}} + Z_{1-\frac{\alpha}{2}}\sigma_2^2\right) \quad & a < b \\ R^{|a,b} &\in \left(\hat{R^{|a,b}} - Z_{1-\frac{\alpha}{2}}\sigma_3^2, \hat{R^{|a,b}} + Z_{1-\frac{\alpha}{2}}\sigma_3^2\right) \quad & a > b \end{aligned} \tag{26}$$

The values of $\sigma_1^2, \sigma_2^2, \sigma_3^2$ are determined by evaluating $g(\beta)\Sigma g(\beta)^T$ under various conditions namely when $a = b$, when $a < b$, and when $a > b$.

4 Bootstrap Approach for $R^{|a,b}$

This section examines the confidence interval for $R^{|a,b}$ using the parametric bootstrap method. The process of generating parametric bootstrap samples for $R^{|a,b}$, as suggested by Efron and Tibshirani in 1993, is detailed here:

- Calculate $\hat{\gamma_1}, \hat{\gamma_2}, \hat{\theta}, \hat{R^{|a,b}}$, which represent the maximum likelihood estimators (MLEs) of $\gamma_1, \gamma_2, \theta, R^{|a,b}$, respectively, using the samples $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ...Y_m)$.
- Generate independent bootstrap samples $X^* = (X_1^*, X_2^*, ..., X_n^*)$ drawn from $W(\gamma_1, \theta)$ and $Y^* = (Y_1^*, Y_2^*, ...Y_m^*)$ drawn from $W(\gamma_2, \theta)$. Using the bootstrap data, calculate bootstrap estimations for the parameters, denoted as $\gamma_1^*, \gamma_2^*, \theta^*$, and $R^{*|a,b}$.
- Iterate the previously mentioned process B times to generate a series of bootstrap samples for $R^{|a,b}$, denoted as $R_1^{*|a,b}, R_2^{*|a,b}, ...R_B^{*|a,b}$.

With the obtained bootstrap samples of R, the $100(1-\alpha\%)$ percentile bootstrap confidence interval for $R^{|a,b}$ is constructed and presented as follows:

$$\left(\hat{R}^{*|a,b}_{\left(\frac{\alpha}{2}\right)}, \quad \hat{R}^{*|a,b}_{\left(1-\frac{\alpha}{2}\right)}\right) \tag{27}$$

Where $\hat{R}^{*|a,b}_{(\alpha)}$ is the quantile of order γ .

5 Bayesian Inference on conditional reliability

This section related to Bayes estimation of reliability within a conditional stress-strength framework. To achieve this, conjugate prior distributions for the parameters γ_1, γ_2, and θ are specified as $Gamma(a_1, b_1)$, $Gamma(a_2, b_2)$, and $Gamma(a_3, b_3)$, correspondingly. The joint prior density is given by:

$$\pi\left(\gamma_1, \gamma_2, \theta\right) = \frac{b_1^{a_1}\gamma_1^{a_1-1}e^{-b_1\gamma_1}}{\Gamma\left(a_1\right)}\frac{b_2^{a_2}\gamma_2^{a_2-1}e^{-b_2\gamma_2}}{\Gamma\left(a_2\right)}\frac{b_3^{a_3}\gamma_3^{a_3-1}e^{-b_3\gamma_3}}{\Gamma\left(a_3\right)} \tag{28}$$

Hence, under the assumption of independence among γ_1, γ_2, and θ, the joint posterior density ofγ_1, γ_2, and θ is expressed as:

$$\pi\left(\gamma_1, \gamma_2\theta|\underline{x}, \underline{y}\right) = \frac{L\left(\gamma_1, \gamma_2\theta|\underline{x}, \underline{y}\right)\pi(\gamma_1)\pi(\gamma_2)\pi(\theta)}{\int_0^\infty\int_0^\infty\int_0^\infty L\left(\gamma_1, \gamma_2\theta|\underline{x}, \underline{y}\right)\pi(\gamma_1)\pi(\gamma_2)\pi(\theta)d\theta d\gamma_1 d\gamma_2} \tag{29}$$

Now (39) can be obtained using (8), (8) and (38)

$$\pi\left(\gamma_1, \gamma_2, \theta|\underline{x}, \underline{y}\right) = \frac{\kappa}{\int_0^\infty\int_0^\infty\int_0^\infty \kappa d\theta d\gamma_1 d\gamma_2} \tag{30}$$

$$\kappa = \frac{\gamma_1^{n+a_1-1}b_1^{a_1}e^{-\gamma_1(b_1+\sum_{i=1}^n x_i^\theta)}}{\Gamma(a_1)}\frac{\gamma_2^{m+a_2-1}b_2^{a_2}e^{-\gamma_2(b_2+\sum_{i=1}^m y_i^\theta)}}{\Gamma(a_2)}\frac{\theta^{n+m+a_3-1}b_3^{a_3}e^{-\theta b_3}}{\Gamma(a_3)}\prod_{i=1}^n x_i^{\theta-1}\prod_{i=1}^m y_i^{\theta-1} \tag{31}$$

The structure of the posterior density doesn't yield a direct explicit Bayes estimator for the model parameters. Thus, the conditional distribution of γ_1, γ_2, and θ can be obtained using the Gibbs sampling technique as follows:

$$\left(\gamma_1|\gamma_2, \theta, \underline{x}, \underline{y}\right) \sim \textit{Gamma}\left(n + a_1, b_1 + \sum_{i=1}^n x_i^\theta\right) \tag{32}$$

$$\left(\gamma_2|\gamma_1, \theta, \underline{x}, \underline{y}\right) \sim \textit{Gamma}\left(m + a_2, b_2 + \sum_{i=1}^m y_i^\theta\right) \tag{33}$$

$$\pi\left(\theta|\gamma_1, \gamma_2, \underline{x}, \underline{y}\right) \propto \theta^{n+m+a_3-1}e^{-b_3\theta}\prod_{i=1}^n x_i^{\theta-1}\prod_{i=1}^m y_i^{\theta-1} \tag{34}$$

Use gamma distributions to generate random numbers for γ_1 and γ_2. Then, utilizing the "Metropolis-Hastings method," generate random values for θ with a distribution proportional to $N(\theta^{t-1}, k_\theta V_\theta)$, where θ^{t-1} represents the current value of θ, k_θ is the scaling factor, and V_θ is the variance-covariance matrix. Thus, Gibbs sampling is as follows:

1. Set t = 1 and begin with an initial guess of $\theta^{(0)} = \hat{\theta}$.

2. From *Gamma* $\left(n + a_1, b_1 + \sum_{i=1}^{n} x_i^{\theta}\right)$ distribution, generate a random value $\gamma_1^{(t)}$.

3. From *Gamma* $\left(m + a_2, b_2 + \sum_{i=1}^{m} y_i^{\theta}\right)$ distribution, generate a random value $\gamma_2^{(t)}$.

4. Generate $\theta^{(t)}$ from (34) $\pi\left(\theta|\gamma_1, \gamma_2, \underline{x}, \underline{y}\right)$ using the "Metropolis - Hastings method" with a proportional distribution as normal distribution.

5. Determine $R^{|a,b(t)}$

6. Assign t to t+1

7. Execute steps 2 to 6 repeatedly for a total of N iterations.

The approximate posterior mean and posterior variance of $R^{|a,b}$ are as follows:

$$\hat{E}\left(R^{|a,b}|\underline{x}, \underline{y}\right) = \frac{1}{N-M} \sum_{t=M+1}^{N} R^{|a,b(t)} \tag{35}$$

and

$$\hat{V}\left(R^{|a,b}|\underline{x}, \underline{y}\right) = \frac{1}{N-M} \sum_{t=M+1}^{N} \left(R^{|a,b(t)} - \hat{E}\left(R^{|a,b}|\underline{x}, \underline{y}\right)\right)^2 \tag{36}$$

Where M is the burn - in period (that is, a number of iterations before the stationary distribution is achieved).

Based on the $R^{|a,b(t)}$ value, the Chen and Shao (1998) approach can be utilised to generate a $100(1-\gamma)\%$ HPD credible interval. The HPD credible interval is given as follows:

$$\left(R^{|a,b}_{[N\frac{\gamma}{2}]}, \qquad R^{|a,b}_{[N(1-\frac{\gamma}{2})]}\right) \tag{37}$$

where $R^{|a,b}_{[N\frac{\gamma}{2}]}$ and $R^{|a,b}_{[N(1-\frac{\gamma}{2})]}$ are the $\left[N\frac{\gamma}{2}\right]^{\text{th}}$ smallest integer and the $\left[N(1-\frac{\gamma}{2})\right]^{\text{th}}$ smallest integer of $R^{|a,b(t)}, t = M+1, M+2, ..., N$, respectively.

6 Simulation Study

In this section, a Monte Carlo simulation analysis is carried out to evaluate the performance of the point estimators and confidence intervals constructed in this research. Confidence intervals are compared using length and coverage probability, whereas point estimators are evaluated using mean square error.All predicted values have been rounded to five digits after 5000 iterations. The usage of abbreviations in the tables is as follows: *MLE: Maximum Likelihood Estimation Method, Bootstrap: Bootstrap Estimation, Bayesian: Bayesian Approach, MSE: Mean Square Error, ACI: Asymptotic Confidence interval, BPCI: Boot - percentile Confidence Interval, HPDCI: HPD credible interval* and *CP: Coverage Probability*. The tables below present the findings of the simulation study.

Table 1: ***Point Estimation:*** MLE, Bootstrap and Bayesian estimators and corresponding MSE for $R^{|1,3}$

$\gamma_1 = 0.1, \gamma_2 = 5, \theta = 1.2, R^{|1,3} = 0.74563$						
(n, m)	**MLE**		**Bootstrap**		**Bayesian**	
	$\hat{R}^{|13}$	MSE	$\hat{R}^{|13}$	MSE	$\hat{R}^{|13}$	MSE
(5, 8)	0.78488	0.01866	0.84987	0.01935	0.78249	0.001358
(8, 5)	0.74916	0.01747	0.62715	0.03286	0.66303	0.006823
(8, 8)	0.932	0.0157	0.60499	0.03728	0.72966	0.000255
(10, 15)	0.83281	0.01318	0.76163	0.00726	0.76702	0.000458
(15, 10)	0.77663	0.01328	0.72186	0.0065	0.66469	0.006552
(15, 15)	0.86497	0.01242	0.8147	0.00773	0.74278	0.000008
(25, 25)	0.84231	0.01152	0.86501	0.01543	0.70308	0.001811
(35, 25)	0.83229	0.01107	0.62369	0.01826	0.67225	0.005385
(25, 35)	0.8672	0.01106	0.74235	0.00295	0.79114	0.002071
(55, 35)	0.88557	0.01084	0.67302	0.00695	0.65436	0.008330
(35, 55)	0.80697	0.01062	0.91114	0.02781	0.81035	0.004188
(55, 55)	0.86641	0.01034	0.81503	0.00568	0.75927	0.000186

Table 2: ***Interval Estimation:*** MLE, Bootstrap and Bayesian estimators and corresponding CP for $R^{|1,3}$

$\gamma_1 = 0.1, \gamma_2 = 5, \theta = 1.2, R^{|1,3} = 0.74563$						
(n, m)	**MLE**		**Bootstrap**		**Bayesian**	
	ACI	CP	BPCI	CP	HPDCI	CP
(5, 8)	(0.59658, 0.97318)	0.98	(0.64627, 0.97089)	1	(0.5718, 0.92795)	1
(8, 5)	(0.64161, 0.85671)	0.97	(0.35054, 0.87169)	0.96	(0.38581, 0.86223)	1
(8, 8)	(0.6847, 0.9993)	1	(0.32022, 0.83683)	0.99	(0.50684, 0.88681)	1
(10, 15)	(0.68157, 0.88406)	0.99	(0.56917, 0.89642)	1	(0.59379, 0.89036)	0.98
(15, 10)	(0.63924, 0.91402)	1	(0.55164, 0.85493)	1	(0.45146, 0.81974)	1
(15, 15)	(0.65962, 0.93433)	1	(0.69796, 0.91083)	0.98	(0.57578, 0.86352)	1
(25, 25)	(0.68038, 0.90424)	1	(0.78818, 0.92677)	1	(0.54429, 0.81457)	0.99
(35, 25)	(0.72354, 0.92104)	0.99	(0.50469, 0.82535)	1	(0.50736, 0.78325)	1
(25, 35)	(0.69659, 0.93781)	1	(0.61722, 0.83735)	1	(0.66769, 0.87162)	1
(55, 35)	(0.7027, 0.91844)	1	(0.59106, 0.84571)	1	(0.45568, 0.75704)	1
(35, 55)	(0.72903, 0.88492)	1	(0.86778, 0.94599)	1	(0.66245, 0.87915)	1
(55, 55)	(0.7128, 0.92002)	1	(0.75187, 0.86913)	1	(0.57017, 0.83662)	1

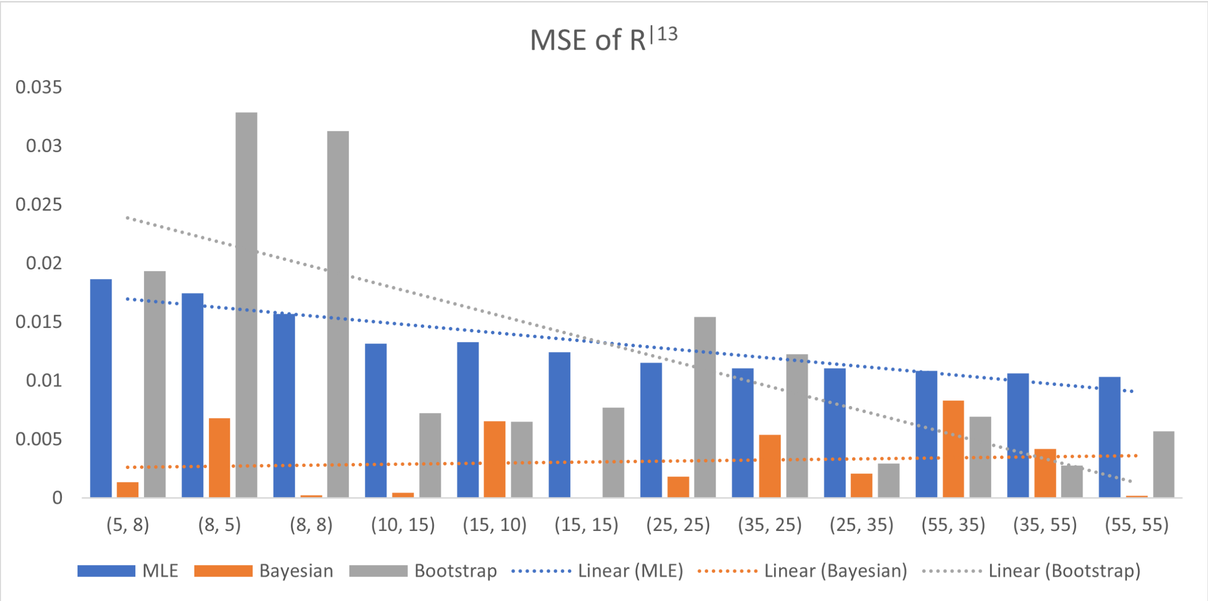

Figure 1: Plot of mean square error against sample size for $R^{|1,3}$

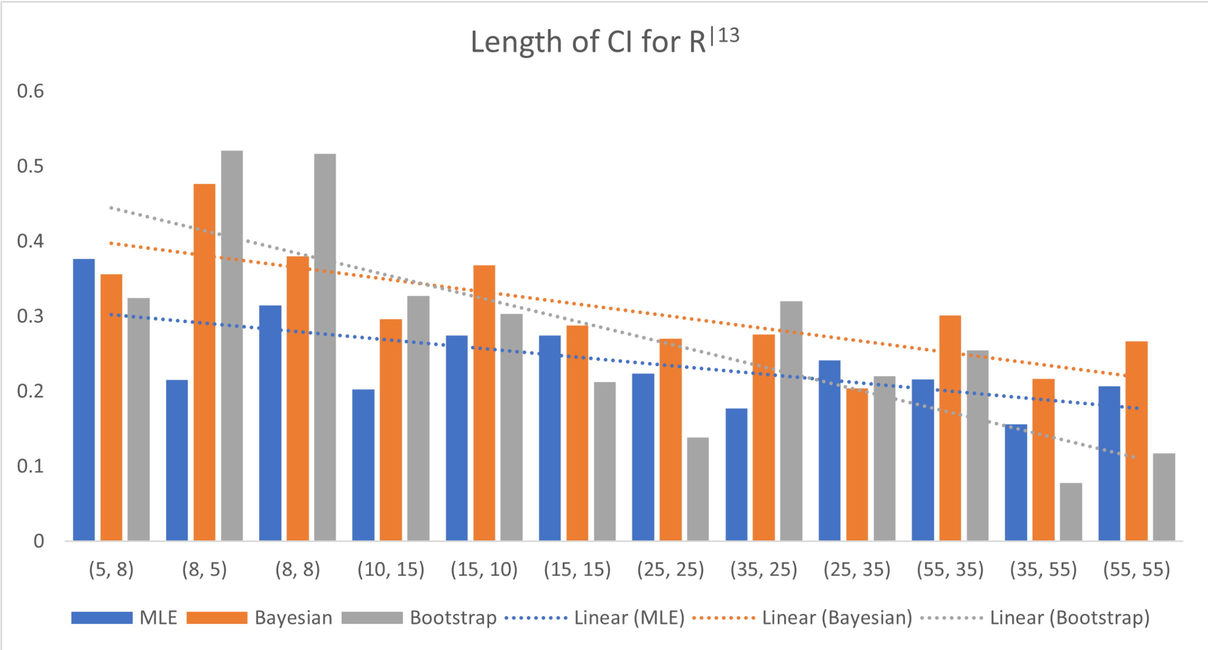

Figure 2: Plot of length of confidence interval against sample size $R^{|1,3}$

Table 3: ***Point Estimation:*** MLE, Bootstrap and Bayesian estimators and corresponding MSE for $R^{|1,1}$

$\gamma_1 = 0.3, \gamma_2 = 2, \theta = 1.2, R^{|1,1} = 0.86957$						
(n, m)	**MLE**		**Bootstrap**		**Bayesian**	
	$\hat{R}^{|11}$	MSE	$\hat{R}^{|11}$	MSE	$\hat{R}^{|11}$	MSE
(5, 8)	0.8523	0.00695	0.86153	0.00839	0.723	0.00148
(8, 5)	0.87171	0.00639	0.89322	0.00646	0.75364	0.00344
(8, 8)	0.86785	0.0052	0.84782	0.00752	0.75601	0.00289
(10, 15)	0.93748	0.00365	0.88731	0.00313	0.72389	0.00222
(15, 10)	0.89505	0.00318	0.80344	0.00999	0.82335	0.00214
(15, 15)	0.86349	0.00272	0.96697	0.00988	0.75172	0.00389
(25, 25)	0.83223	0.00165	0.85897	0.00198	0.82459	0.00202
(35, 25)	0.91685	0.00123	0.87983	0.00144	0.82519	0.00197
(25, 35)	0.86522	0.00135	0.85666	0.00171	0.75399	0.00336
(55, 35)	0.89775	0.00081	0.88067	0.00099	0.8017	0.00461
(35, 55)	0.89498	0.00089	0.87082	0.00095	0.73925	0.00198
(55, 55)	0.87144	0.00071	0.86173	0.00084	0.83581	0.00114

Table 4: ***Interval Estimation:*** MLE, Bootstrap and Bayesian estimators and corresponding CP for $R^{|1,1}$

$\gamma_1 = 0.3, \gamma_2 = 2, \theta = 1.2, R^{|1,1} = 0.86957$						
(n, m)	**MLE**		**Bootstrap**		**Bayesian**	
	ACI	CP	BPCI	CP	HPDCI	CP
(5, 8)	(0.67248, 0.93201)	0.97	(0.6354, 0.9839)	1	(0.48316, 0.89854)	1
(8, 5)	(0.73797, 0.95205)	0.98	(0.71134, 0.99094)	1	(0.65171, 0.98848)	1
(8, 8)	(0.73088, 0.948)	0.96	(0.64733, 0.97376)	1	(0.6671, 0.98207)	1
(10, 15)	(0.68575, 0.99102)	1	(0.76676, 0.97313)	1	(0.55023, 0.96188)	0.98
(15, 10)	(0.64826, 0.94185)	1	(0.64299, 0.93703)	1	(0.62091, 0.93895)	1
(15, 15)	(0.76507, 0.96192)	1	(0.82189, 0.99371)	1	(0.64202, 0.94291)	1
(25, 25)	(0.71168, 0.95277)	1	(0.76833, 0.93231)	1	(0.7613, 0.9882)	0.99
(35, 25)	(0.78455, 0.95916)	1	(0.80533, 0.94524)	1	(0.7287, 0.99056)	1
(25, 35)	(0.77448, 0.95596)	1	(0.77062, 0.92582)	1	(0.6724, 0.97246)	1
(55, 35)	(0.75121, 0.94429)	0.99	(0.81887, 0.93183)	1	(0.64883, 0.90622)	1
(35, 55)	(0.82626, 0.96371)	0.98	(0.80376, 0.92388)	1	(0.52835, 0.88924)	1
(55, 55)	(0.81896, 0.92391)	1	(0.8047, 0.91143)	1	(0.71634, 0.91979)	1

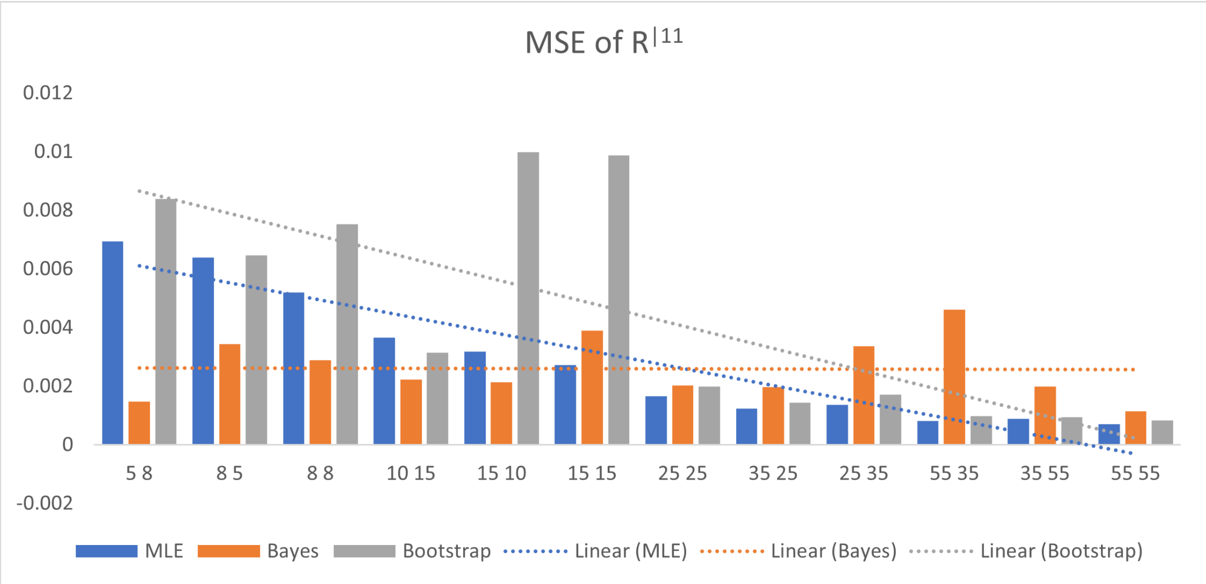

Figure 3: Plot of mean square error against sample size for $R^{|1,1}$

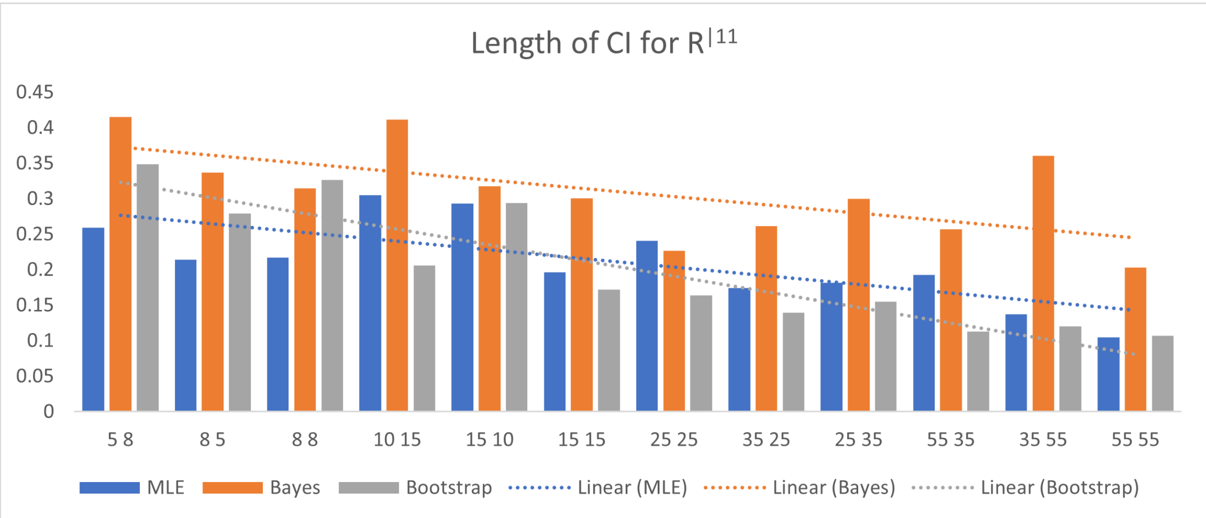

Figure 4: Plot of length of confidence interval against sample size $R^{|1,1}$

Table 5: ***Point Estimation:*** MLE, Bootstrap and Bayesian estimators and corresponding MSE for $R^{|3,1}$

$\gamma_1 = 0.09, \gamma_2 = 0.6, \theta = 1.2, R^{|3,1} = 0.97476$						
(n, m)	**MLE**		**Bootstrap**		**Bayesian**	
	$\hat{R}^{|31}$	MSE	$\hat{R}^{|31}$	MSE	$\hat{R}^{|31}$	MSE
(5, 8)	0.9687	0.00343	0.85812	0.00231	0.95885	0.000250
(8, 5)	0.98851	0.00329	0.98167	0.00059	0.83825	0.001860
(8, 8)	0.96421	0.00244	0.95606	0.00211	0.89779	0.001920
(10, 15)	0.96505	0.00179	0.96942	0.00065	0.9518	0.000530
(15, 10)	0.94228	0.00198	0.98704	0.0003	0.9489	0.000670
(15, 15)	0.97674	0.0016	0.95394	0.00123	0.97322	0.000240
(25, 25)	0.97896	0.00134	0.98662	0.00027	0.92629	0.000230
(35, 25)	0.97079	0.00136	0.99086	0.00031	0.98447	0.000940
(25, 35)	0.97354	0.00119	0.98117	0.00017	0.95824	0.000270
(55, 35)	0.87062	0.00118	0.9747	0.0002	0.9537	0.000440
(35, 55)	0.95289	0.00105	0.94802	0.00101	0.99417	0.000380
(55, 55)	0.96175	0.00101	0.96488	0.00025	0.948	0.000720

Table 6: ***Interval Estimation:*** MLE, Bootstrap and Bayesian estimators and corresponding CP for $R^{|3,1}$

$\gamma_1 = 0.09, \gamma_2 = 0.6, \theta = 1.2, R^{|3,1} = 0.97476$						
(n, m)	**MLE**		**Bootstrap**		**Bayesian**	
	ACI	CP	BPCI	CP	HPDCI	CP
(5, 8)	(0.90464, 0.99999)	0.97	(0.81483, 0.99195)	1	(0.8977, 0.98992)	0.98
(8, 5)	(0.86704, 0.99996)	0.98	(0.91424,1)	1	(0.86695, 0.9926)	1
(8, 8)	(0.91003, 0.99999)	1	(0.85141, 0.99988)	1	(0.86843, 0.99659)	0.96
(10, 15)	(0.92433, 0.99979)	1	(0.9053, 0.99905)	1	(0.89114, 0.98423)	1
(15, 10)	(0.8685, 0.99735)	1	(0.95501, 0.9996)	1	(0.86078, 0.98171)	1
(15, 15)	(0.94874, 0.9974)	0.99	(0.88348, 0.99366)	1	(0.93811, 0.98922)	1
(25, 25)	(0.96476, 0.99316)	1	(0.95689, 0.99937)	1	(0.94116, 0.99994)	1
(35, 25)	(0.89482, 0.99677)	1	(0.9731, 0.99918)	1	(0.94379, 0.99314)	0.99
(25, 35)	(0.95388, 0.9932)	1	(0.95416, 0.99748)	1	(0.92852, 0.98775)	1
(55, 35)	(0.86689, 0.99435)	1	(0.94198, 0.99566)	1	(0.92313, 0.98316)	1
(35, 55)	(0.92614, 0.99765)	1	(0.91049, 0.97787)	1	(0.89903, 0.99878)	1
(55, 55)	(0.93624, 0.98726)	1	(0.9383, 0.98585)	1	(0.91228, 0.98095)	1

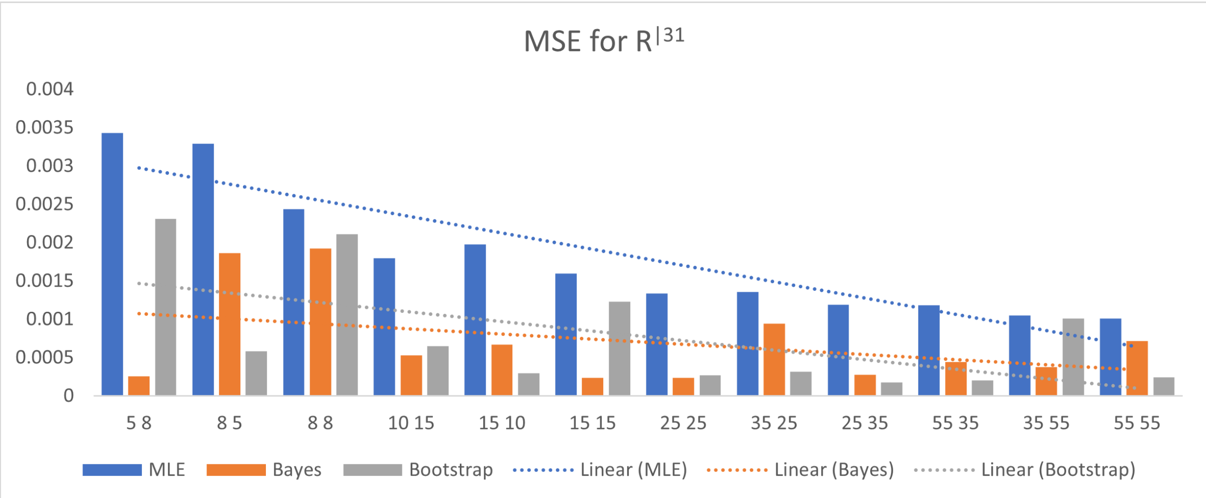

Figure 5: Plot of mean square error against sample size for $R^{|3,1}$

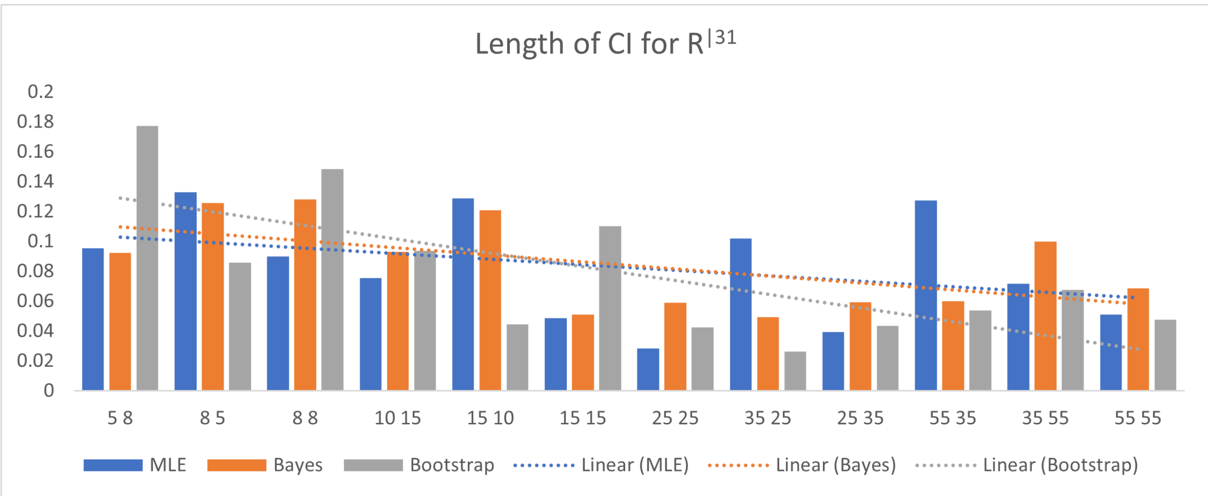

Figure 6: Plot of length of confidence interval against sample size $R^{|3,1}$

7 Data Analysis

This section is about real data analysis. Badar and Priest (1982) published the first data on fibre strength (in GPA). The data provide the strength values measured in GPA for single carbon Fibre and impregnated 1000-carbon Fibre tows. Single fibres were tension tested at gauge lengths of 1, 10, 20, and 50mm. Impregnated tows of 1000 Fibre were tested at gauge lengths of 20, 50, 150, and 300. The two data sets presented here are for single fibres tested under tension at gauge lengths of 10 mm (Data I) and 20 mm (Data II), with sample sizes of n = 63 and m = 69, respectively.

Data I: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977 2.996 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377 3.408 3.435 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020

Data II: 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770 2.773 2.800 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585..

After fitting the Weibull model to the above data sets and tested the model's validity using the Kolmogorov-Smirnov (K-S) test for each data set. The K-S distances for data sets I and II were found to be 0.056128 and 0.087616, with corresponding p-values of 0.9816 and 0.7188. The results show that the Weibull distribution fits the data sets better. Based on the parameters $\hat{\gamma_1}$ = 0.001787 , $\hat{\gamma_2}$ = 0.006032 and $\hat{\theta}$ = 5.2619, the $R^{|a,b}$ under the maximum likelihood estimation, Bootstrap estimation, and Bayes estimation are estimated as $\hat{R}_{mle}^{|1.9,1.3} = 0.80387$, $\hat{R}_{boot}^{|1.9,1.3} = 0.80316$ and $\hat{R}_{Bayes}^{|1.9,1.3} = 0.74735$ respectively. The asymptotic confidence interval (ACI), Boot P confidence interval (BPCI) and HPD credible interval (HPDCI) are also constructed and the intervals are as follows: ACI = (0.74416, 0.86357), BPCI = (0.73191, 0.86254) and HPDCI = (0.67788, 0.81274). It can be observed that the outcomes of the MLE and Bootstrap methods are equivalent in terms of point estimation and that the confidence intervals for Bayes estimation and Bootstrap estimation are similar in terms of length.

8 Conclusions

In the field of reliability research, the conditional stress-strength model stands out as an innovative extension of the stress-strength model. This particular investigation aims to estimate the reliability under conditional stress-strength scenarios, assuming the stress and strength parameters follows Weibull distributions with same shape parameters but distinct scale parameters. The study employs maximum likelihood estimation, bootstrap estimation and Bayesian approaches to determine the distribution parameters and the conditional reliability. Confidence intervals for conditional reliability are constructed through the MLE and bootstrap method, while HPD credible intervals are also established using Bayesian approach. To illustrate, a simulation study is conducted, revealing that with an increase in sample size, the lengths of the intervals tend to decrease. Moreover, the Mean Squared Error (MSE) of the Bayesian estimates frequently proves to be lower than that of the Maximum Likelihood Estimator (MLE) as well as bootstrap. It is also observed that, the MSE of the MLE decreases as the sample size grows, demonstrating the consistency of this estimator. A real data analysis is also carried out as an application of the same.

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