

On Dual Hyperbolic Generalized Guglielmo Numbers

Abstract. In this research, the generalized dual hyperbolic Guglielmo numbers are introduced. Various special cases are explored (including dual hyperbolic triangular numbers, dual hyperbolic triangular-Lucas numbers, dual hyperbolic oblong numbers, and dual hyperbolic pentagonal numbers). Binet's formulas, generating functions and summation formulas for these numbers are presented. Moreover, Catalan's and Cassini's identities are provided, along with matrices associated with these sequences.

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1. Introduction

A hypercomplex system, in mathematical and geometric contexts, denotes a system that extends the principles of complex numbers. These systems exhibit intriguing algebraic properties and are often explored due to their applications in physics and engineering. We give brief on the application areas of hypercomplex number systems in physics and engineering fields as follows.

- In physics

Quantum Mechanics: Hypercomplex numbers find applications in quantum mechanics, where they are utilized to model certain quantum states and operations. The extended algebraic structure of hypercomplex systems allows for a more nuanced representation of quantum phenomena.

Relativity: In the context of relativistic physics, hypercomplex numbers can be employed to describe spacetime transformations and events. Their mathematical properties provide a suitable framework for addressing relativistic effects in a more comprehensive manner.

- Engineering:

Robotics and Control Systems: Hypercomplex numbers can be applied in robotics and control systems to represent spatial transformations and rotations efficiently. Their use allows for more concise and elegant formulations of complex robotic movements.

Signal Processing: Hypercomplex systems find applications in signal processing, where they can be employed to analyze and manipulate multidimensional signals. The extended algebraic structure aids in handling complex transformations in signal data.

Computer Graphics: In computer graphics and computer-aided design (CAD), hypercomplex numbers play a role in representing and manipulating three-dimensional objects. Their use contributes to more sophisticated and accurate modeling.

Unlike complex numbers, hypercomplex systems offer a more intricate framework for describing transformations and symmetries in higher-dimensional spaces. As Kantor discusses in [13], hypercomplex number systems are extensions of real numbers. The principal hypercomplex systems include complex numbers, hyperbolic numbers, and dual numbers. Complex numbers, characterized by a real part and an imaginary part, form the foundation of hypercomplex systems. Hyperbolic numbers, which extend the concept of complex numbers, have applications in various mathematical models. Dual numbers, on the other hand, introduce dual units and find utility in algebraic structures. Now, let's present some specific information about these hypercomplex number systems.

- Complex numbers are formed by extending the real number system with the imaginary unit, denoted as " i ", which satisfies the equation $i^2 = -1$. Complex numbers is defined by,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\}.$$

- Hyperbolic (double, split-complex) numbers, [15], Split-complex numbers, also known as hyperbolic numbers, extend the real number system with a new element j , where $j^2 = 1$. Hyperbolic numbers is defined by,

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}.$$

- Dual numbers, [9], extend the real number system by introducing a new element ε , where $\varepsilon^2 = 0$. Dual numbers is defined by,

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

- Some non-commutative examples of hypercomplex number systems are quaternions, [10]. Quaternions extend the concept of complex numbers by introducing three imaginary units, generally denoted as i , j and k . A quaternion has the form as $a_0 + ia_1 + ja_2 + ka_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$. In addition that, the quaternion units i , j , and k are satisfy specific multiplication rules. Quaternion numbers is defined by

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

- The other number systems are octonions and sedenions, see [16], [21]. The algebras \mathbb{C} (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), \mathbb{O} (octonions) and \mathbb{S} (sedenions) are real algebras achieved from the real numbers \mathbb{R} by a doubling procedure named the Cayley-Dickson Process that extended beyond the sedenions to form what are known as the 2^n -ions. In the table below, we present papers that have been published in the literature.

Table 1. Papers that have been published in the literature related to 2^n -ions.

Authors and Title of the paper↓	Papers↓
Biss, D.K., Dugger, D., Isaksen, D.C., Large annihilators in Cayley-Dickson algebras	[3]
Hamilton, W.R., Elements of Quaternions	[10]
Imaeda, K., Sedenions: algebra and analysis	[12]
Moreno, G., The zero divisors of the Cayley-Dickson algebras over the real numbers	[14]
Göcen, M., Soykan, Y., Horadam 2^k -Ions	[11]
Soykan, Y., Tribonacci and Tribonacci-Lucas Sedenions	[16]
On higher order Fibonacci hyper complex numbers	[6]

A dual hyperbolic number is a type of hypercomplex number, specifically a member of the hyperbolic number system. A dual hyperbolic number is defined by

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

$\mathbb{H}_{\mathbb{D}}$, the set of all dual hyperbolic numbers, are generally denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The $\{1, j, \varepsilon, \varepsilon j\}$ is linear independent and $\mathbb{H}_{\mathbb{D}} = sp\{1, j, \varepsilon, \varepsilon j\}$ so that $\{1, j, \varepsilon, \varepsilon j\}$ is a basis of $\mathbb{H}_{\mathbb{D}}$. For more detail, see [2].

The next properties holds for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j, \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0, j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon, \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0, j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

We claim that p and q be two dual hyperbolic numbers that $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ and then we can write the product of p and q as

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and we can write the sum dual hyperbolic numbers p and q as componentwise.

The dual hyperbolic numbers form a commutative ring, real vector space and an algebra. $\mathbb{H}_{\mathbb{D}}$ is not field since every dual hyperbolic numbers doesn't have an inverse. For more detail about dual hyperbolic numbers, see [2].

Next, we give some properties about generalized Guglielmo numbers.

A generalized Guglielmo sequence $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2)\}_{n \geq 0}$ is defined by the third-order recurrence relations

$$(1.1) \quad W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3}; \quad W_0, W_1, W_2 \quad (n \geq 3)$$

with the initial values W_0, W_1, W_2 not all being zero. The sequence $\{W_n\}_{n \geq 0}$ can be given to negative subscripts by defining

$$W_{-n} = 3W_{-(n-1)} - 3W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, \dots$. Thus, recurrence (1.1) is true for all integer n .

In the Table 2 we give the first few generalized Guglielmo numbers with positive subscript and negative subscript

Table 2. A few generalized Guglielmo numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$3W_0 - 3W_1 + W_2$
2	W_2	$6W_0 - 8W_1 + 3W_2$
3	$W_0 - 3W_1 + 3W_2$	$10W_0 - 15W_1 + 6W_2$
4	$3W_0 - 8W_1 + 6W_2$	$15W_0 - 24W_1 + 10W_2$
5	$6W_0 - 15W_1 + 10W_2$	$21W_0 - 35W_1 + 15W_2$
6	$10W_0 - 24W_1 + 15W_2$	$28W_0 - 48W_1 + 21W_2$

If we take $W_0 = 0, W_1 = 1, W_2 = 3$ then $\{T_n\}$ is the Triangular sequence, if we take $W_0 = 3, W_1 = 3, W_2 = 3$ then $\{H_n\}$ is the triangular-Lucas sequence, if we take $W_0 = 0, W_1 = 2, W_2 = 6$ then $\{O_n\}$ is the oblong sequence and if we take $W_0 = 0, W_1 = 1, W_2 = 5$ then $\{p_n\}$ is the pentagonal sequence. In other words, triangular sequence $\{T_n\}_{n \geq 0}$, triangular-Lucas sequence $\{H_n\}_{n \geq 0}$, oblong sequence $\{O_n\}_{n \geq 0}$ and pentagonal sequence $\{p_n\}_{n \geq 0}$ are given by the third-order recurrence relations

$$(1.2) \quad T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, \quad T_0 = 0, T_1 = 1, T_2 = 3,$$

$$(1.3) \quad H_n = 3H_{n-1} - 3H_{n-2} + H_{n-3}, \quad H_0 = 3, H_1 = 3, H_2 = 3,$$

$$(1.4) \quad O_n = 3O_{n-1} - 3O_{n-2} + O_{n-3}, \quad O_0 = 0, O_1 = 2, O_2 = 6,$$

$$(1.5) \quad p_n = 3p_{n-1} - 3p_{n-2} + p_{n-3}, \quad p_0 = 0, p_1 = 1, p_2 = 5.$$

In addition that the sequences given above can be extended to negative subscripts by defining,

$$\begin{aligned} T_{-n} &= 3T_{-(n-1)} - 3T_{-(n-2)} + T_{-(n-3)}, \\ H_{-n} &= 3H_{-(n-1)} - 3H_{-(n-2)} + H_{-(n-3)}, \\ O_{-n} &= 3O_{-(n-1)} - 3O_{-(n-2)} + O_{-(n-3)}, \\ p_{-n} &= 3p_{-(n-1)} - 3p_{-(n-2)} + p_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, \dots$ respectively. As a result, recurrences (1.2)-(1.5) are true for all integer n .

We can enumerate several essential properties of generalized Guglielmo numbers that are required.

- Binet formula of generalized Guglielmo sequence can be calculated using its characteristic equation given as

$$x^3 - 3x^2 + 3x - 1 = (x - 1)^3 = 0.$$

where the roots of above equation are

$$\alpha = \beta = \gamma = 1.$$

Using these roots and the recurrence relation of $\{W_n\}$, we can write the Binet's formula can be written as

$$(1.6) \quad W_n = A_1 + A_2n + A_3n^2$$

where A_1, A_2 and A_3 are given below

$$(1.7) \quad \begin{aligned} A_1 &= W_0, \\ A_2 &= \frac{1}{2}(-W_2 + 4W_1 - 3W_0), \\ A_3 &= \frac{1}{2}(W_2 - 2W_1 + W_0). \end{aligned}$$

Binet's formula of triangular, triangular-Lucas, oblong and pentagonal sequences can be written as

$$\begin{aligned} T_n &= \frac{n(n+1)}{2}, \\ H_n &= 3, \\ O_n &= n(n+1), \\ p_n &= \frac{1}{2}n(3n-1). \end{aligned}$$

- After then we can write the generating function of generalized Guglielmo numbers and the Cassini identity for generalized Guglielmo numbers, respectively, as

$$(1.8) \quad \sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 3W_0)x + (W_2 - 3W_1 + 3W_0)x^2}{1 - 3x + 3x^2 - x^3}.$$

$$(1.9) \quad W_{n+1}W_{n-1} - W_n^2 = -\frac{1}{2}(A + Bn + Cn^2)$$

where A , B and C can be written as

$$\begin{aligned} A &= 2W_0^2 + 6W_1^2 - 6W_0W_1 - 2W_1W_2, \\ B &= -3W_0^2 - 8W_1^2 - W_2^2 + 10W_0W_1 - 4W_0W_2 + 6W_1W_2, \\ C &= W_0^2 + 4W_1^2 + W_2^2 - 4W_0W_1 + 2W_0W_2 - 4W_1W_2. \end{aligned}$$

For more details about generalized Guglielmo numbers, see [17]

Now, we give some information on published papers related to hyperbolic and dual hyperbolic numbers in literature.

- Cockle [8] studied the hyperbolic numbers with complex coefficients.
- Cheng and Thompson [5] introduced dual numbers with complex coefficients.
- Akar, Yüce and Şahin [2] presented the dual hyperbolic numbers.

Next, we give some information related to dual hyperbolic sequences presented in literature.

- Soykan, Gümüş, Göcen [18] presented dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ ($n \geq 2$) with the initial values V_0, V_1 not all being zero.

- Cihan, Azak, Güngör, Tosun, [1] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$\begin{aligned} DHF_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ DHL_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3} \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Soykan, Taşdemir and Okumuş [19] studied dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = a$, $J_1 = b$.

- Bród, Liana, Włoch [4] studied dual hyperbolic generalized balancing numbers as

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$.

Next section, we present the dual hyperbolic generalized Guglielmo numbers and give some properties of these numbers.

2. Dual Hyperbolic Generalized Guglielmo Numbers and their Generating Functions and Binet's Formulas

In this section, we define dual hyperbolic generalized Guglielmo numbers then using this definition, we present generating functions and Binet's formula of dual hyperbolic generalized Guglielmo numbers.

We now investigate dual hyperbolic generalized Guglielmo numbers over $\mathbb{H}_{\mathbb{D}}$. The n th dual hyperbolic generalized Guglielmo number is

$$(2.1) \quad \widehat{W}_n = W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}.$$

with the initial values $\widehat{W}_0, \widehat{W}_1, \widehat{W}_2$. (2.1) can be written to negative subscripts by defining,

$$(2.2) \quad \widehat{W}_{-n} = W_{-n} + jW_{-n+1} + \varepsilon W_{-n+2} + j\varepsilon W_{-n+3}$$

so identity (2.1) holds for all integers n .

Now, we define some special cases of dual hyperbolic generalized Guglielmo numbers. The n th dual hyperbolic triangular numbers, the n th dual hyperbolic triangular-Lucas numbers, the n th dual hyperbolic oblong numbers and the n th dual hyperbolic pentagonal numbers, respectively, are given as

the n th dual hyperbolic triangular numbers is given $\widehat{T}_n = T_n + jT_{n+1} + \varepsilon T_{n+2} + j\varepsilon T_{n+3}$, with the initial values

$$\begin{aligned} \widehat{T}_0 &= T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3, \\ \widehat{T}_1 &= T_1 + jT_2 + \varepsilon T_3 + j\varepsilon T_4, \\ \widehat{T}_2 &= T_2 + jT_3 + \varepsilon T_4 + j\varepsilon T_5, \end{aligned}$$

the n th dual hyperbolic triangular-Lucas numbers is given $\widehat{H}_n = H_n + jH_{n+1} + \varepsilon H_{n+2} + j\varepsilon H_{n+3}$ with the initial values

$$\begin{aligned} \widehat{H}_0 &= H_0 + jH_1 + \varepsilon H_2 + j\varepsilon H_3, \\ \widehat{H}_1 &= H_1 + jH_2 + \varepsilon H_3 + j\varepsilon H_4, \\ \widehat{H}_2 &= H_2 + jH_3 + \varepsilon H_4 + j\varepsilon H_5, \end{aligned}$$

the n th dual hyperbolic oblong numbers is given $\widehat{O}_n = O_n + jO_{n+1} + \varepsilon O_{n+2} + j\varepsilon O_{n+3}$ with the initial values

$$\begin{aligned} \widehat{O}_0 &= O_0 + jO_1 + \varepsilon O_2 + j\varepsilon O_3, \\ \widehat{O}_1 &= O_1 + jO_2 + \varepsilon O_3 + j\varepsilon O_4, \\ \widehat{O}_2 &= O_2 + jO_3 + \varepsilon O_4 + j\varepsilon O_5, \end{aligned}$$

the n th dual hyperbolic pentagonal numbers is given $\widehat{p}_n = p_n + jp_{n+1} + \varepsilon p_{n+2} + j\varepsilon p_{n+3}$ with the initial values

$$\begin{aligned}\widehat{p}_0 &= p_0 + jp_1 + \varepsilon p_2 + j\varepsilon p_3, \\ \widehat{p}_1 &= p_1 + jp_2 + \varepsilon p_3 + j\varepsilon p_4, \\ \widehat{p}_2 &= p_2 + jp_3 + \varepsilon p_4 + j\varepsilon p_5.\end{aligned}$$

Note that, for dual hyperbolic triangular numbers (by using $W_n = T_n$, $T_0 = 0$, $T_1 = 1$, $T_2 = 3$) we get

$$\begin{aligned}\widehat{T}_0 &= j + 3\varepsilon + 6j\varepsilon, \\ \widehat{T}_1 &= 1 + 3j + 6\varepsilon + 10j\varepsilon, \\ \widehat{T}_2 &= 3 + 6j + 10\varepsilon + 15j\varepsilon,\end{aligned}$$

for dual hyperbolic triangular-Lucas numbers (by using $W_n = H_n$, $H_0 = 3$, $H_1 = 3$, $H_2 = 3$) we obtain

$$\begin{aligned}\widehat{H}_0 &= 3 + 3j + 3\varepsilon + 3j\varepsilon, \\ \widehat{H}_1 &= 3 + 3j + 3\varepsilon + 3j\varepsilon, \\ \widehat{H}_2 &= 3 + 3j + 3\varepsilon + 3j\varepsilon,\end{aligned}$$

for dual hyperbolic oblong numbers (by using $W_n = O_n$, $O_0 = 0$, $O_1 = 2$, $O_2 = 6$) we get

$$\begin{aligned}\widehat{O}_0 &= 2j + 6\varepsilon + 12j\varepsilon, \\ \widehat{O}_1 &= 2 + 6j + 12\varepsilon + 20j\varepsilon, \\ \widehat{O}_2 &= 6 + 12j + 20\varepsilon + 30j\varepsilon,\end{aligned}$$

and for dual hyperbolic pentagonal numbers (by using $W_n = p_n$, $p_0 = 0$, $p_1 = 1$, $p_2 = 5$) we obtain

$$\begin{aligned}\widehat{p}_0 &= j + 5\varepsilon + 12j\varepsilon, \\ \widehat{p}_1 &= 1 + 5j + 12\varepsilon + 22j\varepsilon, \\ \widehat{p}_2 &= 5 + 12j + 22\varepsilon + 35j\varepsilon,\end{aligned}$$

So, using (2.1), we can write the following identity for non negative integers n ,

$$(2.3) \quad \widehat{W}_n = 3\widehat{W}_{n-1} - 3\widehat{W}_{n-2} + \widehat{W}_{n-3}.$$

and the sequence $\{\widehat{W}_n\}_{n \geq 0}$ can be given as

$$\widehat{W}_{-n} = 3\widehat{W}_{-(n-1)} - 3\widehat{W}_{-(n-2)} + \widehat{W}_{-(n-3)},$$

for $n = 1, 2, 3, \dots$ by using (2.2). As a result., recurrence (2.3) holds for all integer n .

In the Table 3, we present the first few dual hyperbolic generalized Guglielmo numbers with positive subscript and negative subscript.

Table 3. A few dual hyperbolic generalized Guglielmo numbers

n	\widehat{W}_n	\widehat{W}_{-n}
0	\widehat{W}_0	\widehat{W}_0
1	\widehat{W}_1	$3\widehat{W}_0 - 3\widehat{W}_1 + \widehat{W}_2$
2	\widehat{W}_2	$6\widehat{W}_0 - 8\widehat{W}_1 + 3\widehat{W}_2$
3	$\widehat{W}_0 - 3\widehat{W}_1 + 3\widehat{W}_2$	$10\widehat{W}_0 - 15\widehat{W}_1 + 6\widehat{W}_2$
4	$3\widehat{W}_0 - 8\widehat{W}_1 + 6\widehat{W}_2$	$15\widehat{W}_0 - 24\widehat{W}_1 + 10\widehat{W}_2$
5	$6\widehat{W}_0 - 15\widehat{W}_1 + 10\widehat{W}_2$	$21\widehat{W}_0 - 35\widehat{W}_1 + 15\widehat{W}_2$
6	$10\widehat{W}_0 - 24\widehat{W}_1 + 15\widehat{W}_2$	$28\widehat{W}_0 - 48\widehat{W}_1 + 21\widehat{W}_2$

Note that

$$\widehat{W}_0 = W_0 + jW_1 + \varepsilon W_2 + j\varepsilon W_3,$$

$$\widehat{W}_1 = W_1 + jW_2 + \varepsilon W_3 + j\varepsilon W_4,$$

$$\widehat{W}_2 = W_2 + jW_3 + \varepsilon W_4 + j\varepsilon W_5.$$

A few dual hyperbolic triangular numbers, dual hyperbolic triangular-Lucas numbers, dual hyperbolic oblong numbers and dual hyperbolic pentagonal numbers with positive subscript and negative subscript are given in the following Table 4, Table 5, Table 6 and Table 7.

Table 4. Dual hyperbolic triangular numbers

n	\widehat{T}_n	\widehat{T}_{-n}
0	$j + 3\varepsilon + 6j\varepsilon$	
1	$1 + 3j + 6\varepsilon + 10j\varepsilon$	$\varepsilon + 3j\varepsilon$
2	$3 + 6j + 10\varepsilon + 15j\varepsilon$	$1 + j\varepsilon$
3	$6 + 10j + 15\varepsilon + 21j\varepsilon$	$3 + j$
4	$10 + 15j + 21\varepsilon + 28j\varepsilon$	$6 + 3j + \varepsilon$
5	$15 + 21j + 28\varepsilon + 36j\varepsilon$	$10 + 6j + 3\varepsilon + j\varepsilon$

Table 5. Dual hyperbolic triangular-Lucas numbers

n	\widehat{H}_n	\widehat{H}_{-n}
0	$3 + 3j + 3\varepsilon + 3j\varepsilon$	
1	$3 + 3j + 3\varepsilon + 3j\varepsilon$	$3 + 3j + 3\varepsilon + 3j\varepsilon$
2	$3 + 3j + 3\varepsilon + 3j\varepsilon$	$3 + 3j + 3\varepsilon + 3j\varepsilon$
3	$3 + 3j + 3\varepsilon + 3j\varepsilon$	$3 + 3j + 3\varepsilon + 3j\varepsilon$
4	$3 + 3j + 3\varepsilon + 3j\varepsilon$	$3 + 3j + 3\varepsilon + 3j\varepsilon$
5	$3 + 3j + 3\varepsilon + 3j\varepsilon$	$3 + 3j + 3\varepsilon + 3j\varepsilon$

Table 6. Dual hyperbolic oblong numbers

n	\widehat{O}_n	\widehat{O}_{-n}
0	$2j + 6\varepsilon + 12j\varepsilon$	
1	$2 + 6j + 12\varepsilon + 20j\varepsilon$	$2\varepsilon + 6j\varepsilon$
2	$6 + 12j + 20\varepsilon + 30j\varepsilon$	$2 + 2\varepsilon j$
3	$12 + 20j + 30\varepsilon + 42j\varepsilon$	$6 + 2j$
4	$20 + 30j + 42\varepsilon + 56j\varepsilon$	$12 + 6j + 2\varepsilon$
5	$30 + 42j + 56\varepsilon + 72j\varepsilon$	$20 + 12j + 6\varepsilon + 2j\varepsilon$

Table 7. Dual hyperbolic pentagonal numbers

n	\widehat{P}_n	\widehat{P}_{-n}
0	$j + 5\varepsilon + 12j\varepsilon$	
1	$1 + 5j + 12\varepsilon + 22j\varepsilon$	$2 + \varepsilon + 5j\varepsilon$
2	$5 + 12j + 22\varepsilon + 35j\varepsilon$	$7 + 2j + j\varepsilon$
3	$12 + 22j + 35\varepsilon + 51j\varepsilon$	$15 + 7j + 2\varepsilon$
4	$22 + 35j + 51\varepsilon + 70j\varepsilon$	$26 + 15j + 7\varepsilon + 2j\varepsilon$
5	$35 + 51j + 70\varepsilon + 92j\varepsilon$	$40 + 26j + 15\varepsilon + 7j\varepsilon$

Now, we will give some expressions that we will use in the rest of the paper and then we define Binet's formula for the dual hyperbolic generalized Guglielmo numbers.

$$(2.4) \quad \widehat{\alpha} = 1 + j + \varepsilon + j\varepsilon,$$

$$(2.5) \quad \widehat{\beta} = j + 2\varepsilon + 3j\varepsilon,$$

$$(2.6) \quad \widehat{\gamma} = j + 4\varepsilon + 9j\varepsilon.$$

Note that using above equalities we can write the following identities:

$$\begin{aligned} \widehat{\alpha}^2 &= 2 + 2j + 4\varepsilon + 4j\varepsilon, \\ \widehat{\beta}^2 &= 1 + 6\varepsilon + 4j\varepsilon, \\ \widehat{\gamma}^2 &= 1 + 18\varepsilon + 8j\varepsilon, \\ \widehat{\alpha}\widehat{\beta} &= 1 + j + 6\varepsilon + 6j\varepsilon, \\ \widehat{\alpha}\widehat{\gamma} &= 1 + j + 14\varepsilon + 14j\varepsilon, \\ \widehat{\beta}\widehat{\gamma} &= 1 + 6j\varepsilon + 12\varepsilon, \end{aligned}$$

THEOREM 1. (*Binet's Formula*) Let n be any integer then the Binet's formula of dual hyperbolic generalized Guglielmo number is

$$(2.7) \quad \widehat{W}_n = (\widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3) + (\widehat{a}A_2 + 2\widehat{\beta}A_3)n + \widehat{a}A_3n^2.$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$ are given in (2.4)-(2.6).

Proof. Using Binet's formula of the generalized Guglielmo numbers given below

$$W_n = A_1 + A_2n + A_3n^2$$

where A_1, A_2, A_3 are given in (1.7) we get

$$\begin{aligned} \widehat{W}_n &= W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}, \\ &= A_1 + A_2n + A_3n^2 + (A_1 + A_2(n+1) + A_3(n+1)^2)j + (A_1 + A_2(n+2) + A_3(n+2)^2)\varepsilon \\ &\quad + (A_1 + A_2(n+3) + A_3(n+3)^2)j\varepsilon. \\ &= (\widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3) + (\widehat{a}A_2 + 2\widehat{\beta}A_3)n + \widehat{a}A_3n^2. \end{aligned}$$

This proves (2.7). \square

In particular, for any integer n , the Binet's Formula of n th dual hyperbolic triangular numbers, Lucas-triangular numbers, oblong numbers and pentagonal numbers, respectively, provided by

$$\begin{aligned}\widehat{T}_n &= \frac{1}{2}((\beta + \gamma) + (\alpha + 2\beta)n + \alpha n^2), \\ \widehat{H}_n &= 3\alpha, \\ \widehat{O}_n &= (\beta + \gamma) + (\alpha + 2\beta)n + \alpha n^2, \\ \widehat{p}_n &= \frac{1}{2}((-\beta + 3\gamma) + (6\beta - \alpha)n + 3\alpha n^2).\end{aligned}$$

In the following Theorem, we present generating function of the dual hyperbolic generalized Guglielmo numbers.

THEOREM 2. *The generating function for the dual hyperbolic generalized Guglielmo numbers is*

$$(2.8) \quad f_{\widehat{W}_n}(x) = \frac{\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0)x^2}{(1 - 3x + 3x^2 - x^3)}.$$

Proof. We assume that $f_{\widehat{W}_n}(x)$ is the generating function of the dual hyperbolic generalized Guglielmo numbers and then we can write

$$f_{\widehat{W}_n}(x) = \sum_{n=0}^{\infty} \widehat{W}_n x^n$$

Then, using the definition of the dual hyperbolic generalized Guglielmo numbers, and subtracting $x f_{\widehat{W}_n}(x)$, $x^2 f_{\widehat{W}_n}(x)$ and $x^3 f_{\widehat{W}_n}(x)$ from $f_{\widehat{W}_n}(x)$, we get

$$\begin{aligned}(1 - 3x + 3x^2 - x^3)f_{\widehat{W}_n}(x) &= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 3x \sum_{n=0}^{\infty} \widehat{W}_n x^n + 3x^2 \sum_{n=0}^{\infty} \widehat{W}_n x^n - x^3 \sum_{n=0}^{\infty} \widehat{W}_n x^n, \\ &= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 3 \sum_{n=0}^{\infty} \widehat{W}_n x^{n+1} + 3 \sum_{n=0}^{\infty} \widehat{W}_n x^{n+2} - \sum_{n=0}^{\infty} \widehat{W}_n x^{n+3}, \\ &= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 3 \sum_{n=1}^{\infty} \widehat{W}_{n-1} x^n + 3 \sum_{n=2}^{\infty} \widehat{W}_{n-2} x^n - \sum_{n=3}^{\infty} \widehat{W}_{n-3} x^n, \\ &= (\widehat{W}_0 + \widehat{W}_1 x + \widehat{W}_2 x^2) - 3(\widehat{W}_0 x + \widehat{W}_1 x^2) + 3\widehat{W}_0 x^2 \\ &\quad + \sum_{n=3}^{\infty} (\widehat{W}_n - 3\widehat{W}_{n-1} + 3\widehat{W}_{n-2} - \widehat{W}_{n-3}) x^n, \\ &= \widehat{W}_0 + \widehat{W}_1 x + \widehat{W}_2 x^2 - 3\widehat{W}_0 x - 3\widehat{W}_1 x^2 + 3\widehat{W}_0 x^2, \\ &= \widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0)x^2.\end{aligned}$$

Note that, using the recurrence relation $\widehat{W}_n = 3\widehat{W}_{n-1} - 3\widehat{W}_{n-2} + \widehat{W}_{n-3}$ and rearranging above equation, the (2.8) has been obtained. \square

Now we can write the generating functions of the dual hyperbolic triangular, triangular-Lucas, oblong and pentagonal numbers as

$$\begin{aligned} f_{\widehat{T}_n}(x) &= \frac{(j + 3\varepsilon + 6j\varepsilon) + (1 - 8j\varepsilon - 3\varepsilon)x + (\varepsilon + 3j\varepsilon)x^2}{(1 - 3x + 3x^2 - x^3)}, \\ f_{\widehat{H}_n}(x) &= \frac{(3 + 3j + 3\varepsilon + 3j\varepsilon) + (-6 - 6j - 6\varepsilon - 6j\varepsilon)x + (3 + 3j + 3\varepsilon + 3j\varepsilon)x^2}{(1 - 3x + 3x^2 - x^3)}, \\ f_{\widehat{O}_n}(x) &= \frac{(2j + 6\varepsilon + 12j\varepsilon) + (2 - 16j\varepsilon - 6\varepsilon)x + (2\varepsilon + 6j\varepsilon)x^2}{(1 - 3x + 3x^2 - x^3)}, \\ f_{\widehat{P}_n}(x) &= \frac{(j + 5\varepsilon + 12j\varepsilon) + (1 + 2j - 3\varepsilon - 14j\varepsilon)x + (2 + \varepsilon + 5j\varepsilon)x^2}{(1 - 3x + 3x^2 - x^3)}, \end{aligned}$$

respectively. \square

3. Obtaining Binet Formula From Generating Function

Next, by using generating function $f_{\widehat{W}_n}(x)$, we investigate Binet formula of $\{\widehat{W}_n\}$.

THEOREM 3. (*Binet formula of dual hyperbolic generalized Guglielmo numbers*)

$$(3.1) \quad \widehat{W}_n = (\widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3) + (\widehat{a}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3n^2$$

Proof. Using gthe $\sum_{n=0}^{\infty} \widehat{W}_n x^n$ we can write

$$(3.2) \quad \sum_{n=0}^{\infty} \widehat{W}_n x^n = \frac{\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0)x^2}{(1 - 3x + 3x^2 - x^3)} = \frac{d_1}{(1-x)} + \frac{d_2}{(1-x)^2} + \frac{d_3}{(1-x)^3},$$

so that

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{W}_n x^n &= \frac{d_1}{(1-x)} + \frac{d_2}{(1-x)^2} + \frac{d_3}{(1-x)^3}, \\ &= \frac{d_1(1-x)^2 + d_2(1-x) + d_3}{(1-x)^3}, \end{aligned}$$

thus, we obtain

$$\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0)x^2 = (d_1 + d_2 + d_3) + (-2d_1 - d_2)x + d_1x^2.$$

If we equalize the coefficients of the same degree terms of x in the above equation, we get

$$(3.3) \quad \begin{aligned} \widehat{W}_0 &= d_1 + d_2 + d_3, \\ \widehat{W}_1 - 3\widehat{W}_0 &= -2d_1 - d_2, \\ \widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0 &= d_1. \end{aligned}$$

If we solve (3.3) we obtain

$$\begin{aligned} d_1 &= 3\widehat{W}_0 - 3\widehat{W}_1 + \widehat{W}_2, \\ d_2 &= 5\widehat{W}_1 - 3\widehat{W}_0 - 2\widehat{W}_2, \\ d_3 &= \widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2. \end{aligned}$$

Thus (3.2) can be given as

$$\begin{aligned}
\sum_{n=0}^{\infty} \widehat{W}_n x^n &= d_1 \sum_{n=0}^{\infty} x^n + d_2 \sum_{n=0}^{\infty} (n+1)x^n + d_3 \sum_{n=0}^{\infty} \frac{n^2 + 3n + 2}{2} x^n, \\
&= \sum_{n=0}^{\infty} (d_1 + d_2(n+1) + d_3 \frac{n^2 + 3n + 2}{2}) x^n, \\
&= \sum_{n=0}^{\infty} (\widehat{W}_0 + \frac{1}{2}(-\widehat{W}_2 + 4\widehat{W}_1 - 3\widehat{W}_0)n + \frac{1}{2}(\widehat{W}_2 - 2\widehat{W}_1 + \widehat{W}_0)n^2) x^n.
\end{aligned}$$

Hence, we get

$$\widehat{W}_n = \widehat{A}_1 + \widehat{A}_2 n + \widehat{A}_3 n^2$$

where

$$\begin{aligned}
\widehat{A}_1 &= \widehat{W}_0, \\
\widehat{A}_2 &= \frac{1}{2}(-\widehat{W}_2 + 4\widehat{W}_1 - 3\widehat{W}_0), \\
\widehat{A}_3 &= \frac{1}{2}(\widehat{W}_2 - 2\widehat{W}_1 + \widehat{W}_0).
\end{aligned}$$

Note that the following equalities given below are true,

$$\begin{aligned}
(3.4) \quad \widehat{A}_1 &= \widehat{W}_0 \\
&= W_0 + jW_1 + \varepsilon W_2 + j\varepsilon(W_0 - 3W_1 + 3W_2) \\
&= (1 + j + \varepsilon + j\varepsilon)W_0 + (j + 2\varepsilon + 3j\varepsilon)\left(\frac{1}{2}(-W_2 + 4W_1 - 3W_0)\right) \\
&\quad + (j + 4\varepsilon + 9j\varepsilon)\left(\frac{1}{2}(W_2 - 2W_1 + W_0)\right) \\
&= \widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3,
\end{aligned}$$

$$\begin{aligned}
(3.5) \quad \widehat{A}_2 &= \frac{1}{2}(-\widehat{W}_2 + 4\widehat{W}_1 - 3\widehat{W}_0) \\
&= \frac{1}{2}((-3W_0 + 4W_1 - W_2) + j\left(\frac{1}{2}(-W_0 + W_2)\right) \\
&\quad + \varepsilon(W_0 - 4W_1 + 3W_2) + j\varepsilon(3W_0 - 8W_1 + 5W_2)) \\
&= (1 + j + \varepsilon + j\varepsilon)\left(\frac{1}{2}(-W_2 + 4W_1 - 3W_0)\right) \\
&\quad + 2(j + 2\varepsilon + 3j\varepsilon)\left(\frac{1}{2}(W_2 - 2W_1 + W_0)\right) \\
&= (\widehat{a}A_2 + 2\widehat{\beta}A_3),
\end{aligned}$$

$$\begin{aligned}
(3.6) \quad \widehat{A}_3 &= \frac{1}{2}(\widehat{W}_2 - 2\widehat{W}_1 + \widehat{W}_0) \\
&= \frac{1}{2}((W_2 - 2W_1 + W_0) + j(W_2 - 2W_1 + W_0) \\
&\quad + \varepsilon(W_2 - 2W_1 + W_0) + j\varepsilon(W_2 - 2W_1 + W_0)) \\
&= \widehat{a}A_3.
\end{aligned}$$

Using (3.4), (3.5) and (3.6), we can obtain

$$\widehat{W}_n = (\widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3) + (\widehat{a}A_2 + 2\widehat{\beta}A_3)n + \widehat{a}A_3n^2. \quad \square$$

4. Some Identities

We now present certain distinctive identities for the dual hyperbolic generalized Guglielmo sequence $\{\widehat{W}_n\}$. The ensuing theorem establishes the Simpson's formula for the dual hyperbolic generalized Guglielmo numbers.

THEOREM 4. (*Simpson's formula for dual hyperbolic generalized Guglielmo numbers*) For all integers n we have,

$$(4.1) \quad \begin{vmatrix} \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \end{vmatrix} = \begin{vmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{vmatrix}.$$

Proof. To proof the above theorem, we can use mathematical induction. First we assume that $n \geq 0$. For $n = 0$ identity (4.1) is true. Let (4.1) is true for $n = k$. Thus, we write the identity given below,

$$\begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k+1} & \widehat{W}_k & \widehat{W}_{k-1} \\ \widehat{W}_k & \widehat{W}_{k-1} & \widehat{W}_{k-2} \end{vmatrix} = \begin{vmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{vmatrix}.$$

For $n = k + 1$, and using above equality, we can write

$$\begin{aligned} \begin{vmatrix} \widehat{W}_{k+3} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k+1} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} &= \begin{vmatrix} 3\widehat{W}_{k+2} - 3\widehat{W}_{k+1} + \widehat{W}_k & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ 3\widehat{W}_{k+1} - 3\widehat{W}_k + \widehat{W}_{k-1} & \widehat{W}_{k+1} & \widehat{W}_k \\ 3\widehat{W}_k - 3\widehat{W}_{k-1} + \widehat{W}_{k-2} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} \\ &= 3 \begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k+1} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_k & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} - 3 \begin{vmatrix} \widehat{W}_{k+1} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_k & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k-1} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} \\ &\quad + \begin{vmatrix} \widehat{W}_k & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k-1} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k-2} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} \\ &= \begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k+1} & \widehat{W}_k & \widehat{W}_{k-1} \\ \widehat{W}_k & \widehat{W}_{k-1} & \widehat{W}_{k-2} \end{vmatrix}. \end{aligned}$$

Note that, for the case $n < 0$ the proof can be done similarly. Thus, the proof is completed. \square

From Theorem 4.1 we get following corollary.

COROLLARY 5.

$$\begin{aligned}
\text{(a): } & \begin{vmatrix} \widehat{T}_{n+2} & \widehat{T}_{n+1} & \widehat{T}_n \\ \widehat{T}_{n+1} & \widehat{T}_n & \widehat{T}_{n-1} \\ \widehat{T}_n & \widehat{T}_{n-1} & \widehat{T}_{n-2} \end{vmatrix} = -4(3\varepsilon + 1)(j + 1). \\
\text{(b): } & \begin{vmatrix} \widehat{H}_{n+2} & \widehat{H}_{n+1} & \widehat{H}_n \\ \widehat{H}_{n+1} & \widehat{H}_n & \widehat{H}_{n-1} \\ \widehat{H}_n & \widehat{H}_{n-1} & \widehat{H}_{n-2} \end{vmatrix} = 0. \\
\text{(c): } & \begin{vmatrix} \widehat{O}_{n+2} & \widehat{O}_{n+1} & \widehat{O}_n \\ \widehat{O}_{n+1} & \widehat{O}_n & \widehat{O}_{n-1} \\ \widehat{O}_n & \widehat{O}_{n-1} & \widehat{O}_{n-2} \end{vmatrix} = -32(3\varepsilon + 1)(j + 1). \\
\text{(d): } & \begin{vmatrix} \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_n \\ \widehat{P}_{n+1} & \widehat{P}_n & \widehat{P}_{n-1} \\ \widehat{P}_n & \widehat{P}_{n-1} & \widehat{P}_{n-2} \end{vmatrix} = -108(3\varepsilon + 1)(j + 1).
\end{aligned}$$

Now, we present Catalan's identity of dual hyperbolic generalized Guglielmo numbers.

THEOREM 6. (*Catalan's identity*) For all integers n and m , the following equality is valid:

$$(4.2) \quad \widehat{W}_{n+m}\widehat{W}_{n-m} - \widehat{W}_n^2 = m^2(A_3^2(2\widehat{\alpha}\widehat{\gamma} - 4\widehat{\beta}^2 + \widehat{a}^2m^2 - 2\widehat{a}^2n^2 - 4\widehat{a}n\beta) - 2\widehat{a}A_2A_3(\widehat{\beta} + \widehat{a}n) - \widehat{a}(\widehat{a}A_2^2 - 2\widehat{\alpha}A_1A_3)).$$

Proof. Using the Binet Formula of dual hyperbolic generalized Guglielmo numbers given below

$$\widehat{W}_n = (\widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3) + (\widehat{a}A_2 + 2\widehat{\beta}A_3)n + \widehat{a}A_3n^2.$$

The proof is completed. \square

As special cases of the above theorem, we give Catalan's identity of dual hyperbolic triangular, Lucas-triangular, Oblong, pentagonal numbers.

We present Catalan's identity of dual hyperbolic triangular numbers.

COROLLARY 7. (*Catalan's identity for the dual hyperbolic triangular numbers*) For all integers n and m , the following equality is valid:

$$\widehat{T}_{n+m}\widehat{T}_{n-m} - \widehat{T}_n^2 = \frac{1}{4}m^2((- \widehat{a}^2 - 2\widehat{a}\widehat{\beta} + 2\widehat{a}\widehat{\gamma} - 4\widehat{\beta}^2) - 2\widehat{a}n(\widehat{a} + 2\widehat{\beta}) + \widehat{a}^2(m^2 - 2n^2)).$$

Proof. Taking $\widehat{W}_n = \widehat{T}_n$ in Theorem 6 we get the required result. \square

We give Catalan's identity of dual hyperbolic triangular-Lucas numbers.

COROLLARY 8. (*Catalan's identity for the dual hyperbolic Lucas-triangular numbers*) For all integers n and m , the following equality is valid:

$$\widehat{H}_{n+m}\widehat{H}_{n-m} - \widehat{H}_n^2 = 0.$$

Proof. Taking $\widehat{W}_n = \widehat{H}_n$ in Theorem 6 we get the required result. \square

We give Catalan's identity of dual hyperbolic oblong numbers.

COROLLARY 9. (*Catalan's identity for the dual hyperbolic oblong numbers*) For all integers n and m , the following equality is valid:

$$\widehat{O}_{n+m}\widehat{O}_{n-m} - \widehat{O}_n^2 = m^2((- \widehat{a}^2 - 2\widehat{a}\widehat{\beta} + 2\widehat{a}\widehat{\gamma} - 4\widehat{\beta}^2) - 2\widehat{a}n(\widehat{a} + 2\widehat{\beta}) + \widehat{a}^2(m^2 - 2n^2)).$$

Proof. Taking $\widehat{W}_n = \widehat{O}_n$ in Theorem 6 we get the required result. \square

We give Catalan's identity of dual hyperbolic pentagonal numbers.

COROLLARY 10. (*Catalan's identity for the dual hyperbolic pentagonal numbers*) For all integers n and m , the following equality is valid:

$$\widehat{P}_{n+m}\widehat{P}_{n-m} - \widehat{P}_n^2 = \frac{1}{4}m^2((- \widehat{a}^2 + 6\widehat{a}\widehat{\beta} + 18\widehat{a}\widehat{\gamma} - 36\widehat{\beta}^2) + 6\widehat{a}n(\widehat{a} - 6\widehat{\beta}) + 9\widehat{a}^2(m^2 - 2n^2)).$$

Proof. Taking $\widehat{W}_n = \widehat{P}_n$ in Theorem 6 we get the required result. \square

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the dual hyperbolic generalized Guglielmo numbers. Hence, we present the corollary given below.

COROLLARY 11. (*Cassini's identity for the dual hyperbolic generalized Guglielmo numbers*) For all integers n , the following inequality is valid:

- (a): $\widehat{T}_{n+1}\widehat{T}_{n-1} - \widehat{T}_n^2 = -\frac{1}{2}((\widehat{a}\widehat{\beta} - \widehat{a}\widehat{\gamma} + 2\widehat{\beta}^2) + n(\widehat{a}^2 + 2\widehat{a}\widehat{\beta}) + \widehat{a}^2n^2).$
- (b): $\widehat{H}_{n+1}\widehat{H}_{n-1} - \widehat{H}_n^2 = 0.$
- (c): $\widehat{O}_{n+1}\widehat{O}_{n-1} - \widehat{O}_n^2 = -2((\widehat{a}\widehat{\beta} - \widehat{a}\widehat{\gamma} + 2\widehat{\beta}^2) + n(\widehat{a}^2 + 2\widehat{a}\widehat{\beta}) + \widehat{a}^2n^2).$
- (d): $\widehat{P}_{n+1}\widehat{P}_{n-1} - \widehat{P}_n^2 = -\frac{1}{2}((-4\widehat{a}^2 - 3\widehat{a}\widehat{\beta} - 9\widehat{a}\widehat{\gamma} + 18\widehat{\beta}^2) + 3n(6\widehat{a}\widehat{\beta} - \widehat{a}^2) + 9\widehat{a}^2n^2).$

THEOREM 12. Suppose that n and m be positive integers, T_n is triangular numbers, the following equality is valid:

$$(4.3) \quad \widehat{W}_{m+n} = T_{m-1}\widehat{W}_{n+2} + (T_{m-3} - 3T_{m-2})\widehat{W}_{n+1} + T_{m-2}\widehat{W}_n.$$

Proof. First for the proof, we assume that $m \geq 0$. The identity (12) can be proved by mathematical induction on m . Taking $m = 0$, we get

$$\widehat{W}_n = T_{-1}\widehat{W}_{n+2} + (T_{-3} - 3T_{-2})\widehat{W}_{n+1} + T_{-2}\widehat{W}_n$$

which is true by seeing that $T_{-1} = 0, T_{-2} = 1, T_{-3} = 3$. We assume that the identity (4.3) holds for $m = k$. Then for $m = k + 1$, we get

$$\begin{aligned}
\widehat{W}_{(k+1)+n} &= 3\widehat{W}_{n+k} - 3\widehat{W}_{n+k-1} + \widehat{W}_{n+k-2} \\
&= 3(T_{k-1}\widehat{W}_{n+2} + (T_{k-3} - 3T_{k-2})\widehat{W}_{n+1} + T_{k-2}\widehat{W}_n) \\
&\quad - 3(T_{k-2}\widehat{W}_{n+2} + (T_{k-4} - 3T_{k-3})\widehat{W}_{n+1} + T_{k-3}\widehat{W}_n) \\
&\quad + (T_{k-3}\widehat{W}_{n+2} + (T_{k-5} - 3T_{k-4})\widehat{W}_{n+1} + T_{k-4}\widehat{W}_n) \\
&= (3T_{k-1} - 3T_{k-2} + T_{k-3})\widehat{W}_{n+2} + ((3T_{k-3} - 3T_{k-4} + T_{k-5}) \\
&\quad - 3(3T_{k-2} - 3T_{k-3} + T_{k-4}))\widehat{W}_{n+1} + (3T_{k-2} - 3T_{k-3} + T_{k-4})\widehat{W}_n \\
&= T_k\widehat{W}_{n+2} + (T_{k-2} - 3T_{k-1})\widehat{W}_{n+1} + T_{k-1}\widehat{W}_n \\
&= T_{(k+1)-1}\widehat{W}_{n+2} + (T_{(k+1)-3} - 3T_{(k+1)-2})\widehat{W}_{n+1} + T_{(k+1)-2}\widehat{W}_n.
\end{aligned}$$

Consequently, by mathematical induction on m , this proves (12). Note that, for the other cases the proof can be done similarly. \square

5. Linear Sums

Within this section, we provide summation formulas for dual hyperbolic generalized Guglielmo numbers, encompassing both positive and negative subscripts.

PROPOSITION 13. *For the generalized Guglielmo numbers, we have the following formulas:*

- (a): $\sum_{k=0}^n W_k = \frac{1}{12} (n+1) ((2n^2 - 2n) W_2 - 2(2n^2 - 5n) W_1 + (2n^2 - 8n + 12) W_0)$.
- (b): $\sum_{k=0}^n W_{k+1} = \frac{1}{12} (n+1) ((2n^2 + 4n) W_2 - 2(2n^2 + n - 6) W_1 + (2n^2 - 2n) W_0)$.
- (c): $\sum_{k=0}^n W_{k+2} = \frac{1}{12} (n+1) ((2n^2 + 10n + 12) W_2 - 2(2n^2 + 7n) W_1 + (2n^2 + 4n) W_0)$.
- (d): $\sum_{k=0}^n W_{k+3} = \frac{1}{12} (n+1) ((2n^2 + 16n + 36) W_2 - 2(2n^2 + 13n + 18) W_1 + (2n^2 + 10n + 12) W_0)$.

Proof. For the proof, see Soykan [17]. \square

PROPOSITION 14. *For the generalized Guglielmo numbers, we have the following formulas:*

- (a): $\sum_{k=0}^n W_{2k} = \frac{1}{12} (n+1) ((8n^2 - 2n) W_2 - 2(8n^2 - 8n) W_1 + (8n^2 - 14n + 12) W_0)$.
- (b): $\sum_{k=0}^n W_{2k+1} = \frac{1}{12} (n+1) (W_2 (8n^2 + 10n) - 2W_1 (8n^2 + 4n - 6) + W_0 (8n^2 - 2n))$.
- (c): $\sum_{k=0}^n W_{2k+2} = \frac{1}{12} (n+1) ((8n^2 + 22n + 12) W_2 - 2(8n^2 + 16n) W_1 + (8n^2 + 10n) W_0)$.
- (d): $\sum_{k=0}^n W_{2k+3} = \frac{1}{12} (n+1) ((8n^2 + 34n + 36) W_2 - 2(8n^2 + 28n + 18) W_1$
 $+ (8n^2 + 22n + 12) W_0)$.
- (e): $\sum_{k=0}^n W_{2k+4} = \frac{1}{12} (n+1) ((8n^2 + 46n + 72) W_2 - 2(8n^2 + 40n + 48) W_1 + (8n^2 + 34n + 36) W_0)$.

Proof. For the proof, see Soykan [17]. \square

PROPOSITION 15. *For the generalized Guglielmo numbers, we have the following formulas:*

- (a): $\sum_{k=0}^n W_{-k} = \frac{1}{12} (n+1) ((2n^2 + 4n) W_2 - 2(2n^2 + 7n) W_1 + (2n^2 + 10n + 12) W_0)$.
(b): $\sum_{k=0}^n W_{-k+1} = \frac{1}{12} (n+1) ((2n^2 - 2n) W_2 - 2(2n^2 + n - 6) W_1 + (2n^2 + 4n) W_0)$.
(c): $\sum_{k=0}^n W_{-k+2} = \frac{1}{12} (n+1) ((2n^2 - 8n + 12) W_2 - 2(2n^2 - 5n) W_1 + (2n^2 - 2n) W_0)$.
(d): $\sum_{k=0}^n W_{-k+3} = \frac{1}{12} (n+1) ((2n^2 - 14n + 36) W_2 - 2(2n^2 - 11n + 18) W_1 + (2n^2 - 8n + 12) W_0)$.

Proof. For the proof, see Soykan [17]. \square

PROPOSITION 16. *For the generalized Guglielmo numbers, we have the following formulas:*

- (a): $\sum_{k=0}^n W_{-2k} = \frac{1}{12} (n+1) ((8n^2 + 10n) W_2 - 2(8n^2 + 16n) W_1 + (8n^2 + 22n + 12) W_0)$.
(b): $\sum_{k=0}^n W_{-2k+1} = \frac{1}{12} (n+1) ((8n^2 - 2n) W_2 - 2(8n^2 + 4n - 6) W_1 + (8n^2 + 10n) W_0)$.
(c): $\sum_{k=0}^n W_{-2k+2} = \frac{1}{12} (n+1) ((8n^2 - 14n + 12) W_2 - 2(8n^2 - 8n) W_1 + (8n^2 - 2n) W_0)$.
(d): $\sum_{k=0}^n W_{-2k+3} = \frac{1}{12} (n+1) ((8n^2 - 26n + 36) W_2 - 2(8n^2 - 20n + 18) W_1 + (8n^2 - 14n + 12) W_0)$.
(e): $\sum_{k=0}^n W_{2k+4} = \frac{1}{12} (n+1) ((8n^2 + 46n + 72) W_2 - 2(8n^2 + 40n + 48) W_1 + (8n^2 + 34n + 36) W_0)$.

Proof. For the proof, see Soykan [17]. \square

Next, we present the formulas which give the summation of the dual hyperbolic generalized Guglielmo numbers.

THEOREM 17. *For $n \geq 0$ then the following sum formulas holds for dual hyperbolic generalized Guglielmo numbers.*

- (a): $\sum_{k=0}^n \widehat{W}_k = \frac{1}{6} (n+1) ((-n+6\varepsilon+18j\varepsilon+5n\varepsilon+jn^2+n^2\varepsilon+2jn+n^2+8jn\varepsilon+jn^2\varepsilon)W_2 + (6j+5n-18j\varepsilon-7n\varepsilon-2jn^2-2n^2\varepsilon-jn-2n^2-13jn\varepsilon-2jn^2\varepsilon)W_1 + (-4n+6j\varepsilon+2n\varepsilon+jn^2+n^2\varepsilon-jn+n^2+5jn\varepsilon+jn^2\varepsilon+6)W_0)$.
(b): $\sum_{k=0}^n \widehat{W}_{2k} = \frac{1}{6} (n+1) ((-n+6\varepsilon+18j\varepsilon+11n\varepsilon+4jn^2+4n^2\varepsilon+5jn+4n^2+17jn\varepsilon+4jn^2\varepsilon)W_2 + (6j+8n-18j\varepsilon-16n\varepsilon-8jn^2-8n^2\varepsilon-4jn-8n^2-28jn\varepsilon-8jn^2\varepsilon)W_1 + (-7n+6j\varepsilon+5n\varepsilon+4jn^2+4n^2\varepsilon-jn+4n^2+11jn\varepsilon+4jn^2\varepsilon+6)W_0)$.
(c): $\sum_{k=0}^n \widehat{W}_{2k+1} = \frac{1}{6} (n+1) ((6j+5n+18\varepsilon+36j\varepsilon+17n\varepsilon+4jn^2+4n^2\varepsilon+11jn+4n^2+23jn\varepsilon+4jn^2\varepsilon)W_2 + (6-18\varepsilon-48j\varepsilon-28n\varepsilon-8jn^2-8n^2\varepsilon-16jn-8n^2-40jn\varepsilon-8jn^2\varepsilon-4n)W_1 + (-n+6\varepsilon+18j\varepsilon+11n\varepsilon+4jn^2+4n^2\varepsilon+5jn+4n^2+17jn\varepsilon+4jn^2\varepsilon)W_0)$.

Proof.

- (a): Note that using (2.1), we get

$$\sum_{k=0}^n \widehat{W}_k = \sum_{k=0}^n W_k + j \sum_{k=0}^n W_{k+1} + \varepsilon \sum_{k=0}^n W_{k+2} + j\varepsilon \sum_{k=0}^n W_{k+3}$$

and using Proposition 13 the proof is easily attainable.

- (b): Note that using (2.1), we get

$$\sum_{k=0}^n \widehat{W}_{2k} = \sum_{k=0}^n W_{2k} + j \sum_{k=0}^n W_{2k+1} + \varepsilon \sum_{k=0}^n W_{2k+2} + j\varepsilon \sum_{k=0}^n W_{2k+3}$$

and using Proposition 14 the proof is easily attainable.

(c): Note that using (2.1), we get

$$\sum_{k=0}^n \widehat{W}_{2k+1} = \sum_{k=0}^n W_{2k+1} + j \sum_{k=0}^n W_{2k+2} + \varepsilon \sum_{k=0}^n W_{2k+3} + j\varepsilon \sum_{k=0}^n W_{2k+4}$$

and using Proposition 14 the proof is easily attainable. \square

As a particular case of the theorem 17,(a), we present following corollary.

COROLLARY 18.

- (a): $\sum_{k=0}^n \widehat{T}_k = \frac{1}{6} (n+1) ((6j+18\varepsilon+36j\varepsilon) + (5j+8\varepsilon+11j\varepsilon+2)n + (j+\varepsilon+j\varepsilon+1)n^2)$.
- (b): $\sum_{k=0}^n \widehat{H}_k = (3j+3\varepsilon+3j\varepsilon+3)(n+1)$.
- (c): $\sum_{k=0}^n \widehat{O}_k = \frac{1}{6} (n+1) ((12j+36\varepsilon+72j\varepsilon) + (10j+16\varepsilon+22j\varepsilon+4)n + (2j+2\varepsilon+2j\varepsilon+2)n^2)$.
- (d): $\sum_{k=0}^n \widehat{p}_k = \frac{1}{6} (n+1) ((6j+30\varepsilon+72j\varepsilon) + (18\varepsilon+9j+27j\varepsilon)n + (3j+3\varepsilon+3+3j\varepsilon)n^2)$.

As a particular case of the theorem 17, (b), we present following corollary.

COROLLARY 19.

- (a): $\sum_{k=0}^n \widehat{T}_{2k} = \frac{1}{6} (n+1) ((6j+18\varepsilon+36j\varepsilon) + (5+17\varepsilon+11j+23j\varepsilon)n + (4+4j+4\varepsilon+4j\varepsilon)n^2)$.
- (b): $\sum_{k=0}^n \widehat{H}_{2k} = (3j+3\varepsilon+3j\varepsilon+3)(n+1)$.
- (c): $\sum_{k=0}^n \widehat{O}_{2k} = \frac{1}{6} (n+1) ((12j+36\varepsilon+72j\varepsilon) + (10+22j+34\varepsilon+46j\varepsilon)n + (8+8j+8\varepsilon+8j\varepsilon)n^2)$.
- (d): $\sum_{k=0}^n \widehat{p}_{2k} = \frac{1}{6} (n+1) ((6j+30\varepsilon+72j\varepsilon) + (3+21j+39\varepsilon+57j\varepsilon)n + (12+12j+12\varepsilon+12j\varepsilon)n^2)$.

As a particular case of the theorem 17, (c), we present following corollary.

COROLLARY 20.

- (a): $\sum_{k=0}^n \widehat{T}_{2k+1} = \frac{1}{6} (n+1) ((6+18j+36\varepsilon+60j\varepsilon) + (11+17j+23\varepsilon+29j\varepsilon)n + (4+4j+4\varepsilon+4j\varepsilon)n^2)$.
- (b): $\sum_{k=0}^n \widehat{H}_{2k+1} = (3j+3\varepsilon+3j\varepsilon+3)(n+1)$.
- (c): $\sum_{k=0}^n \widehat{O}_{2k+1} = \frac{1}{6} (n+1) ((12+36j+72\varepsilon+120j\varepsilon) + (22+46\varepsilon+34j+58j\varepsilon)n + (8+8j+8\varepsilon+8j\varepsilon)n^2)$.
- (d): $\sum_{k=0}^n \widehat{p}_{2k+1} = \frac{1}{6} (n+1) ((6+30j+72\varepsilon+132j\varepsilon) + (21+39j+57\varepsilon+75j\varepsilon)n + (12+12j+12\varepsilon+12j\varepsilon)n^2)$.

Now, we present the formulas to give us the summation formulas of the generalized Guglielmo numbers with negative subscripts.

THEOREM 21. *For $n \geq 0$ then the following sum formulas holds for dual hyperbolic generalized Guglielmo numbers.*

- (a): $\sum_{k=0}^n \widehat{W}_{-k} = \frac{1}{6} (n+1) ((2n+6\varepsilon+18j\varepsilon-4n\varepsilon+jn^2+n^2\varepsilon-jn+n^2-7jn\varepsilon+jn^2\varepsilon)W_2 + (6j-7n-18j\varepsilon+5n\varepsilon-2jn^2-2n^2\varepsilon-jn-2n^2+11jn\varepsilon-2jn^2\varepsilon)W_1 + (5n+6j\varepsilon-n\varepsilon+jn^2+n^2\varepsilon+2jn+n^2-4jn\varepsilon+jn^2\varepsilon+6)W_0)$.

$$(b): \sum_{k=0}^n \widehat{W}_{-2k} = \frac{1}{6} (n+1) ((5n+6\varepsilon+18j\varepsilon-7n\varepsilon+4jn^2+4n^2\varepsilon-jn+4n^2-13jn\varepsilon+4jn^2\varepsilon)W_2 + (6j-16n-18j\varepsilon+8n\varepsilon-8jn^2-8n^2\varepsilon-4jn-8n^2+20jn\varepsilon-8jn^2\varepsilon)W_1 + (11n+6j\varepsilon-n\varepsilon+4jn^2+4n^2\varepsilon+5jn+4n^2-7jn\varepsilon+4jn^2\varepsilon+6)W_0).$$

$$(c): \sum_{k=0}^n \widehat{W}_{-2k+1} = \frac{1}{6} (n+1) ((6j-n+18\varepsilon+36j\varepsilon-13n\varepsilon+4jn^2+4n^2\varepsilon-7jn+4n^2-19jn\varepsilon+4jn^2\varepsilon)W_2 + (20n\varepsilon-18\varepsilon-48j\varepsilon-4n-8jn^2-8n^2\varepsilon+8jn-8n^2+32jn\varepsilon-8jn^2\varepsilon+6)W_1 + (5n+6\varepsilon+18j\varepsilon-7n\varepsilon+4jn^2+4n^2\varepsilon-jn+4n^2-13jn\varepsilon+4jn^2\varepsilon)W_0).$$

Proof.

(a): Note that using (2.1), we get

$$\sum_{k=0}^n \widehat{W}_{-k} = \sum_{k=0}^n W_{-k} + j \sum_{k=0}^n W_{-k+1} + \varepsilon \sum_{k=0}^n W_{-k+2} + j\varepsilon \sum_{k=0}^n W_{-k+3}$$

and using Proposition 15 the proof is easily attainable.

(b): Note that using (2.1), we get

$$\sum_{k=0}^n \widehat{W}_{-2k} = \sum_{k=0}^n W_{-2k} + j \sum_{k=0}^n W_{-2k+1} + \varepsilon \sum_{k=0}^n W_{-2k+2} + j\varepsilon \sum_{k=0}^n W_{-2k+3}$$

and using Proposition 16 the proof is easily attainable.

(c): Note that using (2.1), we get using Proposition (16), we get

$$\sum_{k=0}^n \widehat{W}_{-2k+1} = \sum_{k=0}^n W_{-2k+1} + j \sum_{k=0}^n W_{-2k+2} + \varepsilon \sum_{k=0}^n W_{-2k+3} + j\varepsilon \sum_{k=0}^n W_{-2k+4}$$

and using Proposition 16 the proof is easily attainable. \square

Next, we present different sum formulas of the dual hyperbolic generalized Guglielmo numbers.

As a particular case of the theorem 21, (a), we obtain the following corollary.

COROLLARY 22.

$$(a): \sum_{k=0}^n \widehat{T}_{-k} = \frac{1}{6} (n+1) ((6j+18\varepsilon+36j\varepsilon) + (-1-4j-7\varepsilon-10j\varepsilon)n + (1+j+\varepsilon+j\varepsilon)n^2).$$

$$(b): \sum_{k=0}^n \widehat{H}_{-k} = (3j+3\varepsilon+3j\varepsilon+3)(n+1).$$

$$(c): \sum_{k=0}^n \widehat{O}_{-k} = \frac{1}{6} (n+1) ((12j+36\varepsilon+72j\varepsilon) + (-2-8j-14\varepsilon-20j\varepsilon)n + (2+2j+2\varepsilon+2j\varepsilon)n^2).$$

$$(d): \sum_{k=0}^n \widehat{p}_{-k} = \frac{1}{2} (n+1) ((2j+10\varepsilon+24j\varepsilon) + (1-2j-5\varepsilon-8j\varepsilon)n + (1+j+\varepsilon+j\varepsilon)n^2).$$

As a particular case of the theorem 21, (b), we obtain the following corollary.

COROLLARY 23.

$$(a): \sum_{k=0}^n \widehat{T}_{-2k} = \frac{1}{6} (n+1) ((6j+18\varepsilon+36j\varepsilon) + (-1-7j-13\varepsilon-19j\varepsilon)n + (4+4j+4\varepsilon+4j\varepsilon)n^2).$$

$$(b): \sum_{k=0}^n \widehat{H}_{-2k} = (3j+3\varepsilon+3j\varepsilon+3)(n+1).$$

$$(c): \sum_{k=0}^n \widehat{O}_{-2k} = \frac{1}{3} (n+1) ((6j+18\varepsilon+36j\varepsilon) + (-1-7j-13\varepsilon-19j\varepsilon)n + (4+4j+4\varepsilon+4j\varepsilon)n^2).$$

$$(d): \sum_{k=0}^n \widehat{p}_{-2k} = \frac{1}{6} (n+1) ((6j+30\varepsilon+72j\varepsilon) + (9-9j-27\varepsilon-45j\varepsilon)n + (12+12j+12\varepsilon+12j\varepsilon)n^2).$$

As a particular case of the theorem 21, (c), we obtain the following corollary.

COROLLARY 24.

- (a): $\sum_{k=0}^n \widehat{T}_{-2k+1} = \frac{1}{6} (n+1) ((6+18j+36\varepsilon+60j\varepsilon) + (-7-13j-19\varepsilon-25j\varepsilon)n + (4+4j+4\varepsilon+4j\varepsilon)n^2)$.
(b): $\sum_{k=0}^n \widehat{H}_{-2k+1} = (3j+3\varepsilon+3j\varepsilon+3)(n+1)$.
(c): $\sum_{k=0}^n \widehat{O}_{-2k+1} = \frac{1}{3} (n+1) ((6+18j+36\varepsilon+60j\varepsilon) + (-7-13j-19\varepsilon-25j\varepsilon)n + (4+4j+4\varepsilon+4j\varepsilon)n^2)$.
(d): $\sum_{k=0}^n \widehat{P}_{-2k+1} = \frac{1}{6} (n+1) ((6+30j+72\varepsilon+132j\varepsilon) + (-9-27j-45\varepsilon-63j\varepsilon)n + (12+12j+12\varepsilon+12j\varepsilon)n^2)$.

6. Matrices related with Dual Hyperbolic Generalized Guglielmo Numbers

In this section, using dual hyperbolic Guglielmo numbers, we give some matrices related to dual hyperbolic Guglielmo numbers.

We consider the triangular sequence $\{T_n\}$ defined by the third-order recurrence relation as follows

$$T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}$$

with the initial conditions

$$T_0 = 0, T_1 = 1, T_2 = 3.$$

We present the square matrix A of order 3 as

$$A = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

under the condition that $\det A = 1$. Then, we give the following Lemma.

LEMMA 25. *For any integers n the following identity can be written*

$$(6.1) \quad \begin{pmatrix} \widehat{W}_{n+2} \\ \widehat{W}_{n+1} \\ \widehat{W}_n \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}.$$

Proof. First, we prove the assertion for the case $n \geq 0$. Lemma 25 can be given by mathematical induction on n . If $n = 0$ we get

$$\begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}$$

which is true. We assume that (6.1) is true for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_k \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned}
\begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} &= \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} \\
&= \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_k \end{pmatrix} \\
&= \begin{pmatrix} 3\widehat{W}_{k+2} - 3\widehat{W}_{k+1} + \widehat{W}_k \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix}.
\end{aligned}$$

For the other case $n < 0$ the proof is easily attainable. Consequently, using mathematical induction on n , the proof is completed.

Note that, see [20],

$$A^n = \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}.$$

THEOREM 26. *If we define the matrices $N_{\widehat{W}}$ and $E_{\widehat{W}}$ as follow*

$$\begin{aligned}
N_{\widehat{W}} &= \begin{pmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{pmatrix}, \\
E_{\widehat{W}} &= \begin{pmatrix} \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \end{pmatrix}.
\end{aligned}$$

then the following identity is true:

$$A^n N_{\widehat{W}} = E_{\widehat{W}}.$$

Proof. For the proof, we can use the following identities

$$\begin{aligned}
A^n N_{\widehat{W}} &= \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix} \begin{pmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{pmatrix}, \\
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
a_{11} &= \widehat{W}_2 T_{n+1} + \widehat{W}_1 (T_{n-1} - 3T_n) + \widehat{W}_0 T_n, \\
a_{12} &= \widehat{W}_1 T_{n+1} + \widehat{W}_0 (T_{n-1} - 3T_n) + \widehat{W}_{-1} T_n, \\
a_{13} &= \widehat{W}_0 T_{n+1} + \widehat{W}_{-1} (T_{n-1} - 3T_n) + \widehat{W}_{-2} T_n, \\
a_{21} &= \widehat{W}_2 T_n + \widehat{W}_1 (T_{n-2} - 3T_{n-1}) + \widehat{W}_0 T_{n-1}, \\
a_{22} &= \widehat{W}_1 T_n + \widehat{W}_0 (T_{n-2} - 3T_{n-1}) + \widehat{W}_{-1} T_{n-1}, \\
a_{23} &= \widehat{W}_0 T_n + \widehat{W}_{-1} (T_{n-2} - 3T_{n-1}) + \widehat{W}_{-2} T_{n-1}, \\
a_{31} &= \widehat{W}_2 T_{n-1} + \widehat{W}_1 (T_{n-3} - 3T_{n-2}) + \widehat{W}_0 T_{n-2}, \\
a_{32} &= \widehat{W}_1 T_{n-1} + \widehat{W}_0 (T_{n-3} - 3T_{n-2}) + \widehat{W}_{-1} T_{n-2}, \\
a_{33} &= \widehat{W}_0 T_{n-1} + \widehat{W}_{-1} (T_{n-3} - 3T_{n-2}) + \widehat{W}_{-2} T_{n-2}.
\end{aligned}$$

Using the Theorem 12, the proof is done. \square

From Theorem 26, we can write the following corollary.

COROLLARY 27.

(a): We suppose that the matrices $N_{\widehat{T}}$ and $E_{\widehat{T}}$ are defined as following

$$\begin{aligned}
N_T &= \begin{pmatrix} \widehat{T}_2 & \widehat{T}_1 & \widehat{T}_0 \\ \widehat{T}_1 & \widehat{T}_0 & \widehat{T}_{-1} \\ \widehat{T}_0 & \widehat{T}_{-1} & \widehat{T}_{-2} \end{pmatrix}, \\
E_{\widehat{T}} &= \begin{pmatrix} \widehat{T}_{n+2} & \widehat{T}_{n+1} & \widehat{T}_n \\ \widehat{T}_{n+1} & \widehat{T}_n & \widehat{T}_{n-1} \\ \widehat{T}_n & \widehat{T}_{n-1} & \widehat{T}_{n-2} \end{pmatrix},
\end{aligned}$$

so that the following identity is true for A^n , $N_{\widehat{T}}$, $E_{\widehat{T}}$,

$$A^n N_{\widehat{T}} = E_{\widehat{T}},$$

(b): We suppose that the matrices $N_{\widehat{O}}$ and $E_{\widehat{O}}$ are defined as following

$$N_{\widehat{O}} = \begin{pmatrix} \widehat{O}_2 & \widehat{O}_1 & \widehat{O}_0 \\ \widehat{O}_1 & \widehat{O}_0 & \widehat{O}_{-1} \\ \widehat{O}_0 & \widehat{O}_{-1} & \widehat{O}_{-2} \end{pmatrix},$$

$$E_{\widehat{O}} = \begin{pmatrix} \widehat{O}_{n+2} & \widehat{O}_{n+1} & \widehat{O}_n \\ \widehat{O}_{n+1} & \widehat{O}_n & \widehat{O}_{n-1} \\ \widehat{O}_n & \widehat{O}_{n-1} & \widehat{O}_{n-2} \end{pmatrix},$$

so that the following identity is true for A^n , $N_{\widehat{O}}$, $E_{\widehat{O}}$,

$$A^n N_{\widehat{O}} = E_{\widehat{O}}.$$

(c): We suppose that the matrices $N_{\widehat{H}}$ and $E_{\widehat{H}}$ are defined as following

$$N_{\widehat{H}} = \begin{pmatrix} \widehat{H}_2 & \widehat{H}_1 & \widehat{H}_0 \\ \widehat{H}_1 & \widehat{H}_0 & \widehat{H}_{-1} \\ \widehat{H}_0 & \widehat{H}_{-1} & \widehat{H}_{-2} \end{pmatrix},$$

$$E_{\widehat{H}} = \begin{pmatrix} \widehat{H}_{n+2} & \widehat{H}_{n+1} & \widehat{H}_n \\ \widehat{H}_{n+1} & \widehat{H}_n & \widehat{H}_{n-1} \\ \widehat{H}_n & \widehat{H}_{n-1} & \widehat{H}_{n-2} \end{pmatrix},$$

so that the following identity is true for A^n , $N_{\widehat{H}}$, $E_{\widehat{H}}$,

$$A^n N_{\widehat{H}} = E_{\widehat{H}}.$$

(d): We suppose that the matrices $N_{\widehat{p}}$ and $E_{\widehat{p}}$ are defined as following

$$N_{\widehat{p}} = \begin{pmatrix} \widehat{p}_2 & \widehat{p}_1 & \widehat{p}_0 \\ \widehat{p}_1 & \widehat{p}_0 & \widehat{p}_{-1} \\ \widehat{p}_0 & \widehat{p}_{-1} & \widehat{p}_{-2} \end{pmatrix},$$

$$E_{\widehat{p}} = \begin{pmatrix} \widehat{p}_{n+2} & \widehat{p}_{n+1} & \widehat{p}_n \\ \widehat{p}_{n+1} & \widehat{p}_n & \widehat{p}_{n-1} \\ \widehat{p}_n & \widehat{p}_{n-1} & \widehat{p}_{n-2} \end{pmatrix}.$$

so that the following identity is true for A^n , $N_{\widehat{p}}$, $E_{\widehat{p}}$,

$$A^n N_{\widehat{p}} = E_{\widehat{p}}.$$

7. Conclusion

In the literature, there have been numerous studies on sequences of numbers, which have found wide applications in various research fields, including physics, engineering, architecture, nature, and art. In this study, we introduce the concept of dual hyperbolic generalized Guglielmo sequences and focus on four specific cases: dual hyperbolic triangular numbers, dual hyperbolic Lucas-triangular numbers, dual hyperbolic oblong numbers, and dual hyperbolic pentagonal numbers.

- In section 1, we give brief on the application areas of hypercomplex number systems in physics and engineering fields as follows. Also, we give some properties about generalized Guglielmo numbers.
- In section 2, we define dual hyperbolic generalized Guglielmo numbers then using this definition, we present generating functions and Binet's formula of dual hyperbolic generalized Guglielmo numbers.
- In section 3, we present certain distinctive identities for the dual hyperbolic generalized Guglielmo sequence $\{\widehat{W}_n\}$ that named Simpson's formula, Catalan's identity and Cassani's.
- In section 4, we present summation formulas for dual hyperbolic generalized Guglielmo numbers, which encompass both positive and negative subscripts.
- In section 5, we give some matrices related to dual hyperbolic Guglielmo numbers.

References

- [1] A. Cihan, A. Z. Azak, M. A. Güngör, M. Tosun, A Study on Dual Hyperbolic Fibonacci and Lucas Numbers, *An. Şt. Univ. Ovidius Constanta*, 27(1), 35–48, 2019.
- [2] Akar, M., Yüce, S., Şahin, Ş., On the Dual Hyperbolic Numbers and the Complex Hyperbolic Numbers, *Journal of Computer Science & Computational Mathematics*, 8(1), 1-6, 2018.
- [3] Biss, D.K., Dugger, D., Isaksen, D.C., Large annihilators in Cayley-Dickson algebras, *Communication in Algebra*, 36 (2), 632-664, 2008.
- [4] Bród, D., Liana, A., Włoch, I., Two Generalizations of Dual-Hyperbolic Balancing Numbers, *Symmetry*, 12(11), 1866, 2020
- [5] Cheng, H. H., Thompson, S., Dual Polynomials and Complex Dual Numbers for Analysis of Spatial Mechanisms, *Proc. of ASME 24th Biennial Mechanisms Conference*, Irvine, CA, August, 19-22, 1996.
- [6] Kızılateş, C., Kone, T., On higher order Fibonacci hyper complex numbers, *Chaos Solitons & Fractals*, 148(1), 6 pages (2021)
- [7] Cihan, A., Azak, A. Z., Güngör, M. A., Tosun, M., A Study on Dual Hyperbolic Fibonacci and Lucas Numbers, *An. Şt. Univ. Ovidius Constanta*, 27(1), 35–48, (2019).
- [8] Cockle, J., On a New Imaginary in Algebra, *Philosophical magazine*, London-Dublin-Edinburgh, 3(34), 37-47, 1849.
- [9] Fjelstad, P., Gal, S.G., n-dimensional Hyperbolic Complex Numbers, *Advances in Applied Clifford Algebras*, 8(1), 47-68, 1998, .
- [10] Hamilton, W.R., *Elements of Quaternions*, Chelsea Publishing Company, New York , 1969.
- [11] Göcen, M., Soykan, Y., Horadam 2^k -Ions, *Konuralp Journal of Mathematics*, 7(2), 492-501, 2019.
- [12] Imaeda, K., Imaeda, M., Sedenions: algebra and analysis, *Applied Mathematics and Computation*, 115, 77-88, 2000.
- [13] Kantor, I., Solodovnikov, A., *Hypercomplex Numbers*, Springer-Verlag, New York, 1989.
- [14] Moreno, G., The zero divisors of the Cayley-Dickson algebras over the real numbers, *Bol. Soc. Mat. Mexicana* 3(4), 13-28, 1998.

- [15] Sobczyk, G., The Hyperbolic Number Plane, The College Mathematics Journal, 26(4), 268-280, 1995.
- [16] Soykan, Y., Tribonacci and Tribonacci-Lucas Sedenions. Mathematics 7(1), 74, 2019.
- [17] Soykan, Y., Generalized Guglielmo Numbers: An Investigation of Properties of Triangular, Oblong and Pentagonal Numbers via Their Third Order Linear Recurrence Relations, Earthline Journal of Mathematical Sciences, 9(1), 1-39, 2022.
- [18] Soykan, Y., Gümüş, M., Göcen, M., A study on dual hyperbolic generalized Pell numbers, Malaya Journal Of Matematik, 09(03), 99-116, 2021.
- [19] Soykan, Y., Taşdemir, E., Okumuş, İ., On dual hyperbolic numbers with generalized Jacobsthal numbers components, Indian J Pure Appl Math, 54, 824-840, 2023.
- [20] Soykan Y., A Study On Generalized (r,s,t) -Numbers, MathLAB Journal, 7, 101-129, 2020.
- [21] J. Baez, The octonions, Bull. Amer. Math. Soc. 39(2), 145-205, 2002.