UNDER PEER REVIEW

Original Research Article

On Dual Hyperbolic Generalized Guglielmo Numbers

Abstract. In this research, the generalized dual hyperbolic Guglielmo numbers are introduced. Various special cases are explored (including dual hyperbolic triangular numbers, dual hyperbolic triangular-Lucas numbers, dual hyperbolic oblong numbers, and dual hyperbolic pentagonal numbers). Binet's formulas, generating functions and summation formulas for these numbers are presented. Moreover, Catalan's and Cassini's identities are provided, along with matrices associated with these sequences.

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Keywords. Dual hyperbolic triangular numbers, dual hyperbolic triangular-Lucas numbers, dual hyperbolic oblong numbers, dual hyperbolic pentegonal numbers

1. Introduction

A hypercomplex system, in mathematical and geometric contexts, denotes a system that extends the principles of complex numbers. These systems exhibit intriguing algebraic properties and are often explored due to their applications in physics and engineering. We give brief on the application areas of hypercomplex number systems in physics and engineering fields as follows.

- In physics

  Quantum Mechanics: Hypercomplex numbers find applications in quantum mechanics, where they are utilized to model certain quantum states and operations. The extended algebraic structure of hypercomplex systems allows for a more nuanced representation of quantum phenomena.

  Relativity: In the context of relativistic physics, hypercomplex numbers can be employed to describe spacetime transformations and events. Their mathematical properties provide a suitable framework for addressing relativistic effects in a more comprehensive manner.

- Engineering:

Robotics and Control Systems: Hypercomplex numbers can be applied in robotics and control systems to represent spatial transformations and rotations efficiently. Their use allows for more concise and elegant formulations of complex robotic movements.

Signal Processing: Hypercomplex systems find applications in signal processing, where they can be employed to analyze and manipulate multidimensional signals. The extended algebraic structure aids in handling complex transformations in signal data.

Computer Graphics: In computer graphics and computer-aided design (CAD), hypercomplex numbers play a role in representing and manipulating three-dimensional objects. Their use contributes to more sophisticated and accurate modeling.

Unlike complex numbers, hypercomplex systems offer a more intricate framework for describing transformations and symmetries in higher-dimensional spaces. As Kantor discusses in [13] , hypercomplex number systems are extensions of real numbers. The principal hypercomplex systems include complex numbers, hyperbolic numbers, and dual numbers. Complex numbers, characterized by a real part and an imaginary part, form the foundation of hypercomplex systems. Hyperbolic numbers, which extend the concept of complex numbers, have applications in various mathematical models. Dual numbers, on the other hand, introduce dual units and find utility in algebraic structures. Now, let's present some specific information about these hypercomplex number systems.

- Complex numbers are formed by extending the real number system with the imaginary unit, denoted as "i", which satisfies the equation $i^2 = -1$. Complex numbers is defined by,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\}.$$

- Hyperbolic (double, split-complex) numbers, [15], Split-complex numbers, also known as hyperbolic numbers, extend the real number system with a new element j, where $j^2 = 1$. Hyperbolic numbers is defined by,

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}.$$

- Dual numbers, [9], extend the real number system by introducing a new element ε, where $\varepsilon^2 = 0$. Dual numbers is defined by,

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

- Some non-commutative examples of hypercomplex number systems are quaternions, [10]. Quaternions extend the concept of complex numbers by introducing three imaginary units, generally denoted as i, j and k. A quaternion has the form as $a_0 + ia_1 + ja_2 + ka_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$.In addition that, the quaternion units i, j, and k are satisfy specific multiplication rules. Quaternion numbers is defined by

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

- The other number systems are octonions and sedenions, see [21], [16]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras achieved from the real numbers $\mathbb{R}$ by a doubling procedure named the Cayley-Dickson Process that extended beyond the sedenions to form what are known as the 2^n-ions. In the table below, we present papers that have been published in the literature.

Table 1. Papers that have been published in the literature raleted to 2^n-ions.

Authors and Title of the paper↓	Papers↓
Biss, D.K., Dugger, D., Isaksen, D.C., Large annihilators in Cayley-Dickson algebras	[3]
Hamilton, W.R., Elements of Quaternions	[10]
Imaeda, K., Sedenions: algebra and analysis	[12]
Moreno, G., The zero divisors of the Cayley-Dickson algebras over the real numbers	[14]
Göcen, M., Soykan, Y., Horadam 2^k-Ions	[11]
Soykan,Y., Tribonacci and Tribonacci-Lucas Sedenions	[16]
On higher order Fibonacci hyper complex numbers	[6]

A dual hyperbolic number is a type of hypercomplex number, specifically a member of the hyperbolic number system. A dual hyperbolic number is defined by

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

$\mathbb{H}_{\mathbb{D}}$, the set of all dual hyperbolic numbers, are generally denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The $\{1, j, \varepsilon, \varepsilon j\}$ is linear independent and $\mathbb{H}_{\mathbb{D}} = sp\{1, j, \varepsilon, \varepsilon j\}$ so that $\{1, j, \varepsilon, \varepsilon j\}$ is a basis of $\mathbb{H}_{\mathbb{D}}$. For more detail, see [2].

The next properties are holds for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

We claim that p and q be two dual hyperbolic numbers that $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ and then we can write the product of p and q as

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and we can write the sum dual hyperbolic numbers p and q as componentwise.

The dual hyperbolic numbers form a commutative ring, real vector space and an algebra. $\mathbb{H}_{\mathbb{D}}$ is not field since every dual hyperbolic numbers doesn't have an inverse. For more detail about dual hyperbolic numbers, see [2].

Next, we give some proporties about generalized Guglielmo numbers.

A generalized Guglielmo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3};\ W_0, W_1, W_2 \quad (n \geq 2) \tag{1.1}$$

with the initial values W_0, W_1, W_2 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be given to negative subscripts by defining

$$W_{-n} = 3W_{-(n-1)} - 3W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$. Thus, recurrence (1.1) is true for all integer n.

In the Table 2 we give the first few generalized Guglielmo numbers with positive subscript and negative subscript

Table 2. A few generalized Guglielmo numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$3W_0 - 3W_1 + W_2$
2	W_2	$6W_0 - 8W_1 + 3W_2$
3	$W_0 - 3W_1 + 3W_2$	$10W_0 - 15W_1 + 6W_2$
4	$3W_0 - 8W_1 + 6W_2$	$15W_0 - 24W_1 + 10W_2$
5	$6W_0 - 15W_1 + 10W_2$	$21W_0 - 35W_1 + 15W_2$
6	$10W_0 - 24W_1 + 15W_2$	$28W_0 - 48W_1 + 21W_2$

If we take $W_0 = 0, W_1 = 1, W_2 = 3$ then $\{T_n\}$ is the Triangular sequence, if we take $W_0 = 3, W_1 = 3, W_2 = 3$ then $\{H_n\}$ is the triangular-Lucas sequence, if we take $W_0 = 0, W_1 = 2, W_2 = 6$ then $\{O_n\}$ is the oblong sequence and if we take $W_0 = 0, W_1 = 1, W_2 = 5$ then $\{p_n\}$ is the pentegonal sequance. In other words, triangular sequence $\{T_n\}_{n\geq 0}$, triangular-Lucas sequence $\{H_n\}_{n\geq 0}$, oblong sequence $\{O_n\}_{n\geq 0}$ and pentegonal sequence $\{p_n\}_{n\geq 0}$ are given by the third-order recurrence relations

$$T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, \qquad T_0 = 0, T_1 = 1, T_2 = 3, \tag{1.2}$$

$$H_n = 3H_{n-1} - 3H_{n-2} + H_{n-3}, \qquad H_0 = 3, H_1 = 3, H_2 = 3, \tag{1.3}$$

$$O_n = 3O_{n-1} - 3O_{n-2} + O_{n-3}, \qquad O_0 = 0, O_1 = 2, O_2 = 6, \tag{1.4}$$

$$p_n = 3p_{n-1} - 3p_{n-2} + p_{n-3}, \qquad p_0 = 0, p_1 = 1, p_2 = 5. \tag{1.5}$$

In addition that the sequences given above can be extended to negative subscripts by defining,

$$\begin{aligned}T_{-n} &= 3T_{-(n-1)} - 3T_{-(n-2)} + T_{-(n-3)},\\ H_{-n} &= 3H_{-(n-1)} - 3H_{-(n-2)} + H_{-(n-3)},\\ O_{-n} &= 3O_{-(n-1)} - 3O_{-(n-2)} + O_{-(n-3)},\\ p_{-n} &= 3p_{-(n-1)} - 3p_{-(n-2)} + p_{-(n-3)},\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (1.2)-(1.5) are true for all integer n.

We can enumerate several essential properties of generalized Guglielmo numbers that are required.

- Binet formula of generalized Guglielmo sequence can be calculated using its characteristic equation given as

  $$x^3 - 3x^2 + 3x - 1 = (x-1)^3 = 0.$$

  where the roots of above equation are

  $$\alpha = \beta = \gamma = 1.$$

  Using these roots and the recurrence relation of $\{W_n\}$, we can write the Binet's formula can be written as

(1.6)
$$W_n = A_1 + A_2 n + A_3 n^2$$

  where A_1, A_2 and A_3 are given below

(1.7)
$$\begin{aligned}A_1 &= W_0,\\ A_2 &= \frac{1}{2}(-W_2 + 4W_1 - 3W_0),\\ A_3 &= \frac{1}{2}(W_2 - 2W_1 + W_0).\end{aligned}$$

  Binet's formula of triangular, triangular-Lucas, oblong and pentagonal sequences can be written as

$$\begin{aligned}T_n &= \frac{n(n+1)}{2},\\ H_n &= 3,\\ O_n &= n(n+1),\\ p_n &= \frac{1}{2}n(3n-1).\end{aligned}$$

- After then we can write the generating function of generalized Guglielmo numbers and the Cassini identity for generalized Guglielmo numbers, respectively, as

(1.8)
$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 3W_0)x + (W_2 - 3W_1 + 3W_0)x^2}{1 - 3x + 3x^2 - x^3}.$$

(1.9)
$$W_{n+1}W_{n-1} - W_n^2 = -\frac{1}{2}\left(A + Bn + Cn^2\right)$$

where A, B and C can be written as

$$\begin{aligned} A &= 2W_0^2 + 6W_1^2 - 6W_0W_1 - 2W_1W_2, \\ B &= -3W_0^2 - 8W_1^2 - W_2^2 + 10W_0W_1 - 4W_0W_2 + 6W_1W_2, \\ C &= W_0^2 + 4W_1^2 + W_2^2 - 4W_0W_1 + 2W_0W_2 - 4W_1W_2. \end{aligned}$$

For more details about generalized Guglielmo numbers, see [17]

Now, we give some information on published papers raleted to hyperbolic and dual hyperbolic numbers in litarature.

- Cockle [8] studied the hyperbolic numbers with complex coefficients.
- Cheng and Thompson [5] introduced dual numbers with complex coefficients.
- Akar, Yüce and Şahin [2] presented the dual hyperbolic numbers.

Next, we give some information related to dual hyperbolic sequences presented in literature.

- Soykan, Gümüş, Göcen [18] presented dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$ with the initial values V_0, V_1 not all being zero.

- Cihan, Azak, Güngör, Tosun, [1] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$\begin{aligned} DHF_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ DHL_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3} \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Soykan, Taşdemir and Okumuş [19] studied dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = a$, $J_1 = b$.

- Bród, Liana, Włoch [4] studied dual hyperbolic generalized balancing numbers as

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$.

Next section, we present the dual hyperbolic generalized Guglielmo numbers and give some properties of these numbers.

2. Dual Hyperbolic Generalized Guglielmo Numbers and their Generating Functions and Binet's Formulas

In this section, we define dual hyperbolic generalized Guglielmo numbers then using this defination, we present generating functions and Binet's formula of dual hyperbolic generalized Guglielmo numbers.

We now investigate dual hyperbolic generalized Guglielmo numbers over $\mathbb{H}_{\mathbb{D}}$. The nth dual hyperbolic generalized Guglielmo number is

$$\widehat{W}_n = W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}. \tag{2.1}$$

with the initial values $\widehat{W}_0, \widehat{W}_1, \widehat{W}_2$. (2.1) can be written to negative subscripts by defining,

$$\widehat{W}_{-n} = W_{-n} + jW_{-n+1} + \varepsilon W_{-n+2} + j\varepsilon W_{-n+3} \tag{2.2}$$

so identity (2.1) holds for all integers n.

Now, we define some special cases of dual hyperbolic generalized Guglielmo numbers. The nth dual hyperbolic triangular numbers, the nth dual hyperbolic triangular-Lucas numbers, the nth dual hyperbolic oblong numbers and the nth dual hyperbolic pentegonal numbers, respectively, are given as

the nth dual hyperbolic triangular numbers is given $\widehat{T}_n = T_n + jT_{n+1} + \varepsilon T_{n+2} + j\varepsilon T_{n+3}$, with the initial values

$$\begin{aligned} \widehat{T}_0 &= T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3, \\ \widehat{T}_1 &= T_1 + jT_2 + \varepsilon T_3 + j\varepsilon T_4, \\ \widehat{T}_2 &= T_2 + jT_3 + \varepsilon T_4 + j\varepsilon T_5, \end{aligned}$$

the nth dual hyperbolic triangular-Lucas numbers is given $\widehat{H}_n = H_n + jH_{n+1} + \varepsilon H_{n+2} + j\varepsilon H_{n+3}$ with the initial values

$$\begin{aligned} \widehat{H}_0 &= H_0 + jH_1 + \varepsilon H_2 + j\varepsilon H_3, \\ \widehat{H}_1 &= H_1 + jH_2 + \varepsilon H_3 + j\varepsilon H_4, \\ \widehat{H}_2 &= H_2 + jH_3 + \varepsilon H_4 + j\varepsilon H_5, \end{aligned}$$

the nth dual hyperbolic oblong numbers is given $\widehat{O}_n = O_n + jO_{n+1} + \varepsilon O_{n+2} + j\varepsilon O_{n+3}$ with the initial values

$$\begin{aligned} \widehat{O}_0 &= O_0 + jO_1 + \varepsilon O_2 + j\varepsilon O_3, \\ \widehat{O}_1 &= O_1 + jO_2 + \varepsilon O_3 + j\varepsilon O_4, \\ \widehat{O}_2 &= O_2 + jO_3 + \varepsilon O_4 + j\varepsilon O_5, \end{aligned}$$

the nth dual hyperbolic pentegonal numbers is given $\widehat{p}_n = p_n + jp_{n+1} + \varepsilon p_{n+2} + j\varepsilon p_{n+3}$ with the initial values

$$\begin{aligned}\widehat{p}_0 &= p_0 + jp_1 + \varepsilon p_2 + j\varepsilon p_3,\\ \widehat{p}_1 &= p_1 + jp_2 + \varepsilon p_3 + j\varepsilon p_4,\\ \widehat{p}_2 &= p_2 + jp_3 + \varepsilon p_4 + j\varepsilon p_5.\end{aligned}$$

Note that, for dual hyperbolic triangular numbers (by using $W_n = T_n$, $T_0 = 0$, $T_1 = 1$, $T_2 = 3$) we get

$$\begin{aligned}\widehat{T}_0 &= j + 3\varepsilon + 6j\varepsilon,\\ \widehat{T}_1 &= 1 + 3j + 6\varepsilon + 10j\varepsilon,\\ \widehat{T}_2 &= 3 + 6j + 10\varepsilon + 15j\varepsilon,\end{aligned}$$

for dual hyperbolic triangular-Lucas numbers (by using $W_n = H_n$, $H_0 = 3$, $H_1 = 3$, $H_2 = 3$) we obtain

$$\begin{aligned}\widehat{H}_0 &= 3 + 3j + 3\varepsilon + 3j\varepsilon,\\ \widehat{H}_1 &= 3 + 3j + 3\varepsilon + 3j\varepsilon,\\ \widehat{H}_2 &= 3 + 3j + 3\varepsilon + 3j\varepsilon,\end{aligned}$$

for dual hyperbolic oblong numbers (by using $W_n = O_n$, $O_0 = 0$, $O_1 = 2$, $O_2 = 6$) we get

$$\begin{aligned}\widehat{O}_0 &= 2j + 6\varepsilon + 12j\varepsilon,\\ \widehat{O}_1 &= 2 + 6j + 12\varepsilon + 20j\varepsilon,\\ \widehat{O}_2 &= 6 + 12j + 20\varepsilon + 30j\varepsilon,\end{aligned}$$

and for dual hyperbolic pentegonal numbers (by using $W_n = p_n$, $p_0 = 0$, $p_1 = 1$, $p_2 = 5$) we obtain

$$\begin{aligned}\widehat{p}_0 &= j + 5\varepsilon + 12j\varepsilon,\\ \widehat{p}_1 &= 1 + 5j + 12\varepsilon + 22j\varepsilon,\\ \widehat{p}_2 &= 5 + 12j + 22\varepsilon + 35j\varepsilon,\end{aligned}$$

So, using (2.1), we can write the following identity for non negative integers n,

$$\widehat{W}_n = 3\widehat{W}_{n-1} - 3\widehat{W}_{n-2} + \widehat{W}_{n-3}. \tag{2.3}$$

and the sequence $\{\widehat{W}_n\}_{n\geq 0}$ can be given as

$$\widehat{W}_{-n} = 3\widehat{W}_{-(n-1)} - 3\widehat{W}_{-(n-2)} + \widehat{W}_{-(n-3)},$$

for $n = 1, 2, 3, ...$ by using (2.2). As a result., recurrence (2.3) holds for all integer n.

In the Table 3, we present the first few dual hyperbolic generalized Guglielmo numbers with positive subscript and negative subscript.

Table 3. A few dual hyperbolic generalized Guglielmo numbers

n	$\widehat{W}_n$	$\widehat{W}_{-n}$
0	$\widehat{W}_0$	$\widehat{W}_0$
1	$\widehat{W}_1$	$3\widehat{W}_0-3\widehat{W}_1+\widehat{W}_2$
2	$\widehat{W}_2$	$6\widehat{W}_0-8\widehat{W}_1+3\widehat{W}_2$
3	$\widehat{W}_0-3\widehat{W}_1+3\widehat{W}_2$	$10\widehat{W}_0-15\widehat{W}_1+6\widehat{W}_2$
4	$3\widehat{W}_0-8\widehat{W}_1+6\widehat{W}_2$	$15\widehat{W}_0-24\widehat{W}_1+10\widehat{W}_2$
5	$6\widehat{W}_0-15\widehat{W}_1+10\widehat{W}_2$	$21\widehat{W}_0-35\widehat{W}_1+15\widehat{W}_2$
6	$10\widehat{W}_0-24\widehat{W}_1+15\widehat{W}_2$	$28\widehat{W}_0-48\widehat{W}_1+21\widehat{W}_2$

Note that

$$\begin{aligned}\widehat{W}_0 &= W_0+jW_1+\varepsilon W_2+j\varepsilon W_3,\\ \widehat{W}_1 &= W_1+jW_2+\varepsilon W_3+j\varepsilon W_4,\\ \widehat{W}_2 &= W_2+jW_3+\varepsilon W_4+j\varepsilon W_5.\end{aligned}$$

A few dual hyperbolic triangular numbers, dual hyperbolic triangular-Lucas numbers, dual hyperbolic oblong numbers and dual hyperbolic pentegonal numbers with positive subscript and negative subscript are given in the following Table 4, Table 5, Table 6 and Table 7.

Table 4. Dual hyperbolic triangular numbers

n	$\widehat{T}_n$	$\widehat{T}_{-n}$
0	$j+3\varepsilon+6j\varepsilon$	
1	$1+3j+6\varepsilon+10j\varepsilon$	$\varepsilon+3j\varepsilon$
2	$3+6j+10\varepsilon+15j\varepsilon$	$1+j\varepsilon$
3	$6+10j+15\varepsilon+21j\varepsilon$	$3+j$
4	$10+15j+21\varepsilon+28j\varepsilon$	$6+3j+\varepsilon$
5	$15+21j+28\varepsilon+36j\varepsilon$	$10+6j+3\varepsilon+j\varepsilon$

Table 5. Dual hyperbolic triangular-Lucas numbers

n	$\widehat{H}_n$	$\widehat{H}_{-n}$
0	$3+3j+3\varepsilon+3j\varepsilon$	
1	$3+3j+3\varepsilon+3j\varepsilon$	$3+3j+3\varepsilon+3j\varepsilon$
2	$3+3j+3\varepsilon+3j\varepsilon$	$3+3j+3\varepsilon+3j\varepsilon$
3	$3+3j+3\varepsilon+3j\varepsilon$	$3+3j+3\varepsilon+3j\varepsilon$
4	$3+3j+3\varepsilon+3j\varepsilon$	$3+3j+3\varepsilon+3j\varepsilon$
5	$3+3j+3\varepsilon+3j\varepsilon$	$3+3j+3\varepsilon+3j\varepsilon$

Table 6. Dual hyperbolic oblong numbers

n	$\widehat{O}_n$	$\widehat{O}_{-n}$
0	$2j+6\varepsilon+12j\varepsilon$	
1	$2+6j+12\varepsilon+20j\varepsilon$	$2\varepsilon+6j\varepsilon$
2	$6+12j+20\varepsilon+30j\varepsilon$	$2+2\varepsilon j$
3	$12+20j+30\varepsilon+42j\varepsilon$	$6+2j$
4	$20+30j+42\varepsilon+56j\varepsilon$	$12+6j+2\varepsilon$
5	$30+42j+56\varepsilon+72j\varepsilon$	$20+12j+6\varepsilon+2j\varepsilon$

Table 7. Dual hyperbolic pentegonal numbers

n	$\widehat{p}_n$	$\widehat{p}_{-n}$
0	$j+5\varepsilon+12j\varepsilon$	
1	$1+5j+12\varepsilon+22j\varepsilon$	$2+\varepsilon+5j\varepsilon$
2	$5+12j+22\varepsilon+35j\varepsilon$	$7+2j+j\varepsilon$
3	$12+22j+35\varepsilon+51j\varepsilon$	$15+7j+2\varepsilon$
4	$22+35j+51\varepsilon+70j\varepsilon$	$26+15j+7\varepsilon+2j\varepsilon$
5	$35+51j+70\varepsilon+92j\varepsilon$	$40+26j+15\varepsilon+7j\varepsilon$

Now, we will give some expressions that we will use in the rest of the paper and then we define Binet's formula for the dual hyperbolic generalized Guglielmo numbers.

$$\widehat{\alpha} = 1 + j + \varepsilon + j\varepsilon, \tag{2.4}$$

$$\widehat{\beta} = j + 2\varepsilon + 3j\varepsilon, \tag{2.5}$$

$$\widehat{\gamma} = j + 4\varepsilon + 9j\varepsilon. \tag{2.6}$$

Note that using above equalities we can write the following identities:

$$\begin{aligned}
\widehat{\alpha}^2 &= 2 + 2j + 4\varepsilon + 4j\varepsilon, \\
\widehat{\beta}^2 &= 1 + 6\varepsilon + 4j\varepsilon, \\
\widehat{\gamma}^2 &= 1 + 18\varepsilon + 8j\varepsilon, \\
\widehat{\alpha}\widehat{\beta} &= 1 + j + 6\varepsilon + 6j\varepsilon, \\
\widehat{\alpha}\widehat{\gamma} &= 1 + j + 14\varepsilon + +14j\varepsilon, \\
\widehat{\beta}\widehat{\gamma} &= 1 + 6j\varepsilon + 12\varepsilon,
\end{aligned}$$

THEOREM 1. *(Binet's Formula) Let n be any integer then the Binet's formula of dual hyperbolic generalized Guglielmo number is*

$$\widehat{W}_n = (\widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3) + (\widehat{\alpha}A_2 + 2\widehat{\beta}A_3)n + \widehat{a}A_3n^2. \tag{2.7}$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$ are given as (2.4)-(2.6).

Proof. Using Binet's formula of the generalized Guglielmo numbers given below

$$W_n = A_1 + A_2n + A_3n^2$$

where A_1, A_2, A_2 are given (1.7) we get

$$\begin{aligned}
\widehat{W}_n &= W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}, \\
&= A_1 + A_2n + A_3n^2 + (A_1 + A_2(n+1) + A_3(n+1)^2)j + (A_1 + A_2(n+2) + A_3(n+2)^2)\varepsilon \\
&\quad +(A_1 + A_2(n+3) + A_3(n+3)^2)j\varepsilon. \\
&= (\widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3) + (\widehat{a}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3n^2.
\end{aligned}$$

This proves (2.7). □

In particular, for any integer n, the Binet's Formula of nth dual hyperbolic triangual number, Lucas-triangular numbers, oblong numbers and pentegonal numbers, respectively, provided by

$$\begin{aligned}
\widehat{T}_n &= \frac{1}{2}((\beta+\gamma)+(\alpha+2\beta)\,n+\alpha n^2), \\
\widehat{H}_n &= 3\alpha, \\
\widehat{O}_n &= (\beta+\gamma)+(\alpha+2\beta)\,n+\alpha n^2, \\
\widehat{p}_n &= \frac{1}{2}((-\beta+3\gamma)+(6\beta-\alpha)n+3\alpha n^2).
\end{aligned}$$

In the following Theorem, we present generating function of the dual hyperbolic generalized Guglielmo numbers.

Theorem 2. *The generating function for the dual hyperbolic generalized Guglielmo numbers is*

$$f_{\widehat{W}_n}(x)=\frac{\widehat{W}_0+(\widehat{W}_1-3\widehat{W}_0)x+(\widehat{W}_2-3\widehat{W}_1+3\widehat{W}_0)x^2}{(1-3x+3x^2-x^3)}. \tag{2.8}$$

Proof. We assume that $f_{\widehat{W}_n}(x)$ is the generating function of the dual hyperbolic generalized Guglielmo numbers and then we can write

$$f_{\widehat{W}_n}(x)=\sum_{n=0}^{\infty}\widehat{W}_n x^n$$

Then, using the definition of the dual hyperbolic generalized Guglielmo numbers, and substracting $xg(x)$ and $x^2g(x)$ from $g(x)$, we get

$$\begin{aligned}
(1-3x+3x^2-x^3)f_{G\widehat{W}_n}(x) &= \sum_{n=0}^{\infty}\widehat{W}_n x^n-3x\sum_{n=0}^{\infty}\widehat{W}_n x^n+3x^2\sum_{n=0}^{\infty}\widehat{W}_n x^n-x^3\sum_{n=0}^{\infty}\widehat{W}_n x^n, \\
&= \sum_{n=0}^{\infty}\widehat{W}_n x^n-3\sum_{n=0}^{\infty}\widehat{W}_n x^{n+1}+3\sum_{n=0}^{\infty}\widehat{W}_n x^{n+2}-\sum_{n=0}^{\infty}\widehat{W}_n x^{n+3}, \\
&= \sum_{n=0}^{\infty}\widehat{W}_n x^n-3\sum_{n=1}^{\infty}\widehat{W}_{n-1} x^{n}+3\sum_{n=2}^{\infty}\widehat{W}_{n-2} x^{n}-\sum_{n=3}^{\infty}\widehat{W}_{n-3} x^{n}, \\
&= (\widehat{W}_0+\widehat{W}_1x+\widehat{W}_2x^2)-3(\widehat{W}_0x+\widehat{W}_1x^2)+3GW_0x^2 \\
&\quad +\sum_{n=3}^{\infty}(\widehat{W}_n-3\widehat{W}_{n-1}+3\widehat{W}_{n-2}-\widehat{W}_{n-3})x^n, \\
&= \widehat{W}_0+\widehat{W}_1x+\widehat{W}_2x^2-3\widehat{W}_0x-3\widehat{W}_1x^2+3\widehat{W}_0x^2, \\
&= \widehat{W}_0+(\widehat{W}_1-3\widehat{W}_0)x+(\widehat{W}_2-3\widehat{W}_1+3\widehat{W}_0)x^2.
\end{aligned}$$

Note that , using the recurrence relation $\widehat{W}_n=3\widehat{W}_{n-1}-3\widehat{W}_{n-2}+\widehat{W}_{n-3}$ and rearranging above equation, the (2.8) has been obtained. □

Now we can write the generating functions of the dual hyperbolic triangular, triangular-Lucas, oblong and pentegonal numbers as

$$\begin{aligned} f_{\widehat{T}_n}(x) &= \frac{(j+3\varepsilon+6j\varepsilon)+(1-8j\varepsilon-3\varepsilon)\,x+(\varepsilon+3j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)}, \\ f_{\widehat{H}_n}(x) &= \frac{(3+3j+3\varepsilon+3j\varepsilon)+(-6-6j-6\varepsilon-6j\varepsilon)\,x+(3+3j+3\varepsilon+3j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)}, \\ f_{\widehat{O}_n}(x) &= \frac{(2j+6\varepsilon+12j\varepsilon)+(2-16j\varepsilon-6\varepsilon)\,x+(2\varepsilon+6j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)}, \\ f_{\widehat{p}_n}(x) &= \frac{(j+5\varepsilon+12j\varepsilon)+(1+2j-3\varepsilon-14j\varepsilon)\,x+(2+\varepsilon+5j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)}, \end{aligned}$$

respectively. □

3. Obtaining Binet Formula From Generating Function

Next ,by using generating function $f_{\widehat{W}_n}(x)$, we investigate Binet formula of $\{\widehat{W}_n\}$.

THEOREM 3. *(Binet formula of dual hyperbolic generalized Guglielmo numbers)*

$$\widehat{W}_n = (\widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3) + (\widehat{\alpha}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3 n^2 \tag{3.1}$$

Proof. Using gthe $\sum_{n=0}^{\infty}\widehat{W}_n x^n$ we can write

$$\sum_{n=0}^{\infty}\widehat{W}_n x^n = \frac{\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0)x^2}{(1-3x+3x^2-x^3)} = \frac{d_1}{(1-x)} + \frac{d_2}{(1-x)^2} + \frac{d_3}{(1-x)^3}, \tag{3.2}$$

so that

$$\begin{aligned} \sum_{n=0}^{\infty}\widehat{W}_n x^n &= \frac{d_1}{(1-x)} + \frac{d_2}{(1-x)^2} + \frac{d_3}{(1-x)^3}, \\ &= \frac{d_1(1-x)^2 + d_2(1-x) + d_3}{(1-x)^3}, \end{aligned}$$

thus, we obtain

$$\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0)x^2 = (d_1+d_2+d_3) + (-2d_1-d_2)x + d_1x^2.$$

If we equalize the coefficients of the same degree terms of x in the above equation, we get

$$\begin{aligned} \widehat{W}_0 &= d_1+d_2+d_3, \\ \widehat{W}_1 - 3\widehat{W}_0 &= -2d_1-d_2, \\ \widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0 &= d_1. \end{aligned} \tag{3.3}$$

If we solve (3.3) we obtain

$$\begin{aligned} d_1 &= 3\widehat{W}_0 - 3\widehat{W}_1 + \widehat{W}_2, \\ d_2 &= 5\widehat{W}_1 - 3\widehat{W}_0 - 2\widehat{W}_2, \\ d_3 &= \widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2. \end{aligned}$$

Thus (3.2) can be given as

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{W}_n x^n &= d_1\sum_{n=0}^{\infty}x^n + d_2\sum_{n=0}^{\infty}(n+1)x^n + d_3\sum_{n=0}^{\infty}\frac{n^2+3n+2}{2}x^n,\\ &= \sum_{n=0}^{\infty}(d_1 + d_2(n+1) + d_3\frac{n^2+3n+2}{2})x^n,\\ &= \sum_{n=0}^{\infty}(\widehat{W}_0 + \frac{1}{2}(-\widehat{W}_2 + 4\widehat{W}_1 - 3\widehat{W}_0)n + \frac{1}{2}(\widehat{W}_2 - 2\widehat{W}_1 + \widehat{W}_0)n^2)x^n.\end{aligned}$$

Hence, we get

$$\widehat{W}_n = \widehat{A_1} + \widehat{A_2}n + \widehat{A_3}n^2$$

where

$$\begin{aligned}\widehat{A_1} &= \widehat{W}_0,\\ \widehat{A_2} &= \frac{1}{2}(-\widehat{W}_2 + 4\widehat{W}_1 - 3\widehat{W}_0),\\ \widehat{A_3} &= \frac{1}{2}(\widehat{W}_2 - 2\widehat{W}_1 + \widehat{W}_0).\end{aligned}$$

Note that the following equalities given below are true,

$$\begin{aligned}\widehat{A_1} &= \widehat{W}_0\\ &= W_0 + jW_1 + \varepsilon W_2 + j\varepsilon(W_0 - 3W_1 + 3W_2)\\ &= (1+j+\varepsilon+j\varepsilon)W_0 + (j+2\varepsilon+3j\varepsilon)(\frac{1}{2}(-W_2+4W_1-3W_0))\\ &\quad +(j+4\varepsilon+9j\varepsilon)(\frac{1}{2}(W_2-2W_1+W_0))\\ &= \widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\widehat{A_2} &= \frac{1}{2}(-\widehat{W}_2 + 4\widehat{W}_1 - 3\widehat{W}_0)\\ &= \frac{1}{2}((-3W_0 + 4W_1 - W_2) + j(\frac{1}{2}(-W_0 + W_2))\\ &\quad +\varepsilon(W_0 - 4W_1 + 3W_2) + j\varepsilon(3W_0 - 8W_1 + 5W_2))\\ &= (1+j+\varepsilon+j\varepsilon)(\frac{1}{2}(-W_2+4W_1-3W_0))\\ &\quad +2(j+2\varepsilon+3j\varepsilon)(\frac{1}{2}(W_2-2W_1+W_0))\\ &= (\widehat{a}A_2 + 2\widehat{\beta}A_3),\end{aligned} \tag{3.5}$$

$$\begin{aligned}\widehat{A_3} &= \frac{1}{2}(\widehat{W}_2 - 2\widehat{W}_1 + \widehat{W}_0)\\ &= \frac{1}{2}((W_2 - 2W_1 + W_0) + j(W_2 - 2W_1 + W_0)\\ &\quad +\varepsilon(W_2 - 2W_1 + W_0) + j\varepsilon(W_2 - 2W_1 + W_0)\\ &= \widehat{a}A_3.\end{aligned} \tag{3.6}$$

Using (3.4), (3.5) and (3.6), we can obtain

$$\widehat{W}_n = (\widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3) + (\widehat{\alpha}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3 n^2. \ \square$$

4. Some Identities

We now present certain distinctive identities for the dual hyperbolic generalized Guglielmo sequence $\{\widehat{W}_n\}$. The ensuing theorem establishes the Simpson's formula for the dual hyperbolic generalized Guglielmo numbers.

THEOREM 4. *(Simpson's formula for dual hyperbolic generalized Guglielmo numbers) For all integers n we have,*

$$\begin{vmatrix} \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \end{vmatrix} = \begin{vmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{vmatrix}. \tag{4.1}$$

Proof. To proof the above theorem, we can use mathematical induction. First we assume that $n \geqslant 0$. For $n = 0$ identity (4.1) is true. Let (4.1) is true for $n = k$. Thus, we write the identity given below,

$$\begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k+1} & \widehat{W}_k & \widehat{W}_{k-1} \\ \widehat{W}_k & \widehat{W}_{k-1} & \widehat{W}_{k-2} \end{vmatrix} = \begin{vmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{vmatrix}.$$

For $n = k + 1$, and using above equality, we can write

$$\begin{aligned}
\begin{vmatrix} \widehat{W}_{k+3} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k+1} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} &= \begin{vmatrix} 3\widehat{W}_{k+2} - 3\widehat{W}_{k+1} + \widehat{W}_k & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ 3\widehat{W}_{k+1} - 3\widehat{W}_k + \widehat{W}_{k-1} & \widehat{W}_{k+1} & \widehat{W}_k \\ 3\widehat{W}_k - 3\widehat{W}_{k-1} + \widehat{W}_{k-2} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} \\
&= 3\begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k+1} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_k & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} - 3\begin{vmatrix} \widehat{W}_{k+1} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_k & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k-1} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} \\
&\quad + \begin{vmatrix} \widehat{W}_k & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k-1} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k-2} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} \\
&= \begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k+1} & \widehat{W}_k & \widehat{W}_{k-1} \\ \widehat{W}_k & \widehat{W}_{k-1} & \widehat{W}_{k-2} \end{vmatrix}.
\end{aligned}$$

Note that, for the case $n < 0$ the proof can be done similarly. Thus, the proof is completed. $\square$

From Theorem 4.1 we get following corollary.

COROLLARY 5.

(a): $\begin{vmatrix} \widehat{T}_{n+2} & \widehat{T}_{n+1} & \widehat{T}_{n} \\ \widehat{T}_{n+1} & \widehat{T}_{n} & \widehat{T}_{n-1} \\ \widehat{T}_{n} & \widehat{T}_{n-1} & \widehat{T}_{n-2} \end{vmatrix} = -4(3\varepsilon+1)(j+1).$

(b): $\begin{vmatrix} \widehat{H}_{n+2} & \widehat{H}_{n+1} & \widehat{H}_{n} \\ \widehat{H}_{n+1} & \widehat{H}_{n} & \widehat{H}_{n-1} \\ \widehat{H}_{n} & \widehat{H}_{n-1} & \widehat{H}_{n-2} \end{vmatrix} = 0.$

(c): $\begin{vmatrix} \widehat{O}_{n+2} & \widehat{O}_{n+1} & \widehat{O}_{n} \\ \widehat{O}_{n+1} & \widehat{O}_{n} & \widehat{O}_{n-1} \\ \widehat{O}_{n} & \widehat{O}_{n-1} & \widehat{O}_{n-2} \end{vmatrix} = -32(3\varepsilon+1)(j+1).$

(d): $\begin{vmatrix} \widehat{p}_{n+2} & \widehat{p}_{n+1} & \widehat{p}_{n} \\ \widehat{p}_{n+1} & \widehat{p}_{n} & \widehat{p}_{n-1} \\ \widehat{p}_{n} & \widehat{p}_{n-1} & \widehat{p}_{n-2} \end{vmatrix} = -108(3\varepsilon+1)(j+1).$

Now, we present Catalan's identity of dual hyperbolic generalized Guglielmo numbers.

THEOREM 6. *(Catalan's identity) For all integers n and m, the following equality is valid:*

(4.2)

$$\widehat{W}_{n+m}\widehat{W}_{n-m}-\widehat{W}_{n}^{2} = m^{2}(A_{3}^{2}\left(2\widehat{a}\widehat{\gamma}-4\widehat{\beta}^{2}+\widehat{a}^{2}m^{2}-2\widehat{a}^{2}n^{2}-4\widehat{a}n\beta\right)-2\widehat{a}A_{2}A_{3}\left(\widehat{\beta}+\widehat{a}n\right)-\widehat{a}\left(\widehat{a}A_{2}^{2}-2\widehat{\alpha}A_{1}A_{3}\right)).$$

Proof. Using the Binet Formula of dual hyperbolic generalized Guglielmo numbers given below

$$\widehat{W}_{n} = (\widehat{\alpha}A_{1}+\widehat{\beta}A_{2}+\widehat{\gamma}A_{3})+(\widehat{a}A_{2}+2\widehat{\beta}A_{3})n+\widehat{a}A_{3}n^{2}.$$

The proof is completed. □

As special cases of the above theorem, we give Catalan's identity of dual hyperbolic triangular, Lucas-triangular, Oblong, pentegonal numbers.

We present Catalan's identity of dual hyperbolic triangular numbers.

COROLLARY 7. *(Catalan's identity for the dual hyperbolic triangular numbers) For all integers n and m, the following equality is valid:*

$$\widehat{T}_{n+m}\widehat{T}_{n-m}-\widehat{T}_{n}^{2} = \frac{1}{4}m^{2}((-\widehat{a}^{2}-2\widehat{a}\widehat{\beta}+2\widehat{a}\widehat{\gamma}-4\widehat{\beta}^{2})-2\widehat{a}n\left(\widehat{a}+2\widehat{\beta}\right)+\widehat{a}^{2}\left(m^{2}-2n^{2}\right)).$$

Proof. Taking $\widehat{W}_{n}=\widehat{T}_{n}$ in Theorem 6 we get the required result. □

We give Catalan's identity of dual hyperbolic triangular-Lucas numbers.

COROLLARY 8. *(Catalan's identity for the dual hyperbolic Lucas-triangular numbers) For all integers n and m, the following equality is valid:*

$$\widehat{H}_{n+m}\widehat{H}_{n-m}-\widehat{H}_{n}^{2} = 0.$$

Proof. Taking $\widehat{W}_{n}=\widehat{H}_{n}$ in Theorem 6 we get the required result. □

We give Catalan's identity of dual hyperbolic oblong numbers.

Corollary 9. *(Catalan's identity for the dual hyperbolic oblong numbers) For all integers n and m, the following equality is valid:*

$$\widehat{O}_{n+m}\widehat{O}_{n-m}-\widehat{O}_n^2=m^2((-\widehat{a}^2-2\widehat{a}\widehat{\beta}+2\widehat{a}\widehat{\gamma}-4\widehat{\beta}^2)-2\widehat{a}n\left(\widehat{a}+2\widehat{\beta}\right)+\widehat{a}^2\left(m^2-2n^2\right)).$$

Proof. Taking $\widehat{W}_n=\widehat{O}_n$ in Theorem 6 we get the required result. □

We give Catalan's identity of dual hyperbolic pentegonal numbers.

Corollary 10. *(Catalan's identity for the dual hyperbolic pentegonal numbers) For all integers n and m, the following equality is valid:*

$$\widehat{p}_{n+m}\widehat{p}_{n-m}-\widehat{p}_n^2=\frac{1}{4}m^2((-\widehat{a}^2+6\widehat{a}\widehat{\beta}+18\widehat{a}\widehat{\gamma}-36\widehat{\beta}^2)+6\widehat{a}n\left(\widehat{a}-6\widehat{\beta}\right)+9\widehat{a}^2\left(m^2-2n^2\right)).$$

Proof. Taking $\widehat{W}_n=\widehat{p}_n$ in Theorem 6 we get the required result. □

Note that for $m=1$ in Catalan's identity, we get the Cassini's identity for the dual hyperbolic generalized Guglielmo numbers. Hence, we present the corollary given below.

Corollary 11. *(Cassini's identity for the dual hyperbolic generalized Guglielmo numbers) For all integers n, the following iequality is valid:*

(a): $\widehat{T}_{n+1}\widehat{T}_{n-1}-\widehat{T}_n^2=-\frac{1}{2}((\widehat{a}\widehat{\beta}-\widehat{a}\widehat{\gamma}+2\widehat{\beta}^2)+n(\widehat{a}^2+2\widehat{a}\widehat{\beta})+\widehat{a}^2n^2)$.
(b): $\widehat{H}_{n+1}\widehat{H}_{n-1}-\widehat{H}_n^2=0$.
(c): $\widehat{O}_{n+1}\widehat{O}_{n-1}-\widehat{O}_n^2=-2((\widehat{a}\widehat{\beta}-\widehat{a}\widehat{\gamma}+2\widehat{\beta}^2)+n(\widehat{a}^2+2\widehat{a}\widehat{\beta})+\widehat{a}^2n^2)$.
(d): $\widehat{p}_{n+1}\widehat{p}_{n-1}-\widehat{p}_n^2=-\frac{1}{2}((-4\widehat{a}^2-3\widehat{a}\widehat{\beta}-9\widehat{a}\widehat{\gamma}+18\widehat{\beta}^2)+3n(6\widehat{a}\widehat{\beta}-\widehat{a}^2)+9\widehat{a}^2n^2)$.

Theorem 12. *Suppose that n and m be positive integers, T_n is triangular numbers, the following equality is valid:*

$$\widehat{W}_{m+n}=T_{m-1}\widehat{W}_{n+2}+(T_{m-3}-3T_{m-2})\widehat{W}_{n+1}+T_{m-2}\widehat{W}_n. \tag{4.3}$$

Proof. First for the proof, we assume that $m\geq 0$ .The identity (12) can be proved by mathematical induction on m. Taking $m=0$, we get

$$\widehat{W}_n=T_{-1}\widehat{W}_{n+2}+(T_{-3}-3T_{-2})\widehat{W}_{n+1}+T_{-2}\widehat{W}_n$$

which is true by seeing that $T_{-1}=0, T_{-2}=1, T_{-3}=3$. We assume that the identity (4.3) holds for $m=k$. Then for $m=k+1$, we get

$$\begin{aligned}
\widehat{W}_{(k+1)+n} &= 3\widehat{W}_{n+k}-3\widehat{W}_{n+k-1}+\widehat{W}_{n+k-2} \\
&= 3(T_{k-1}\widehat{W}_{n+2}+(T_{k-3}-3T_{k-2})\widehat{W}_{n+1}+T_{k-2}\widehat{W}_{n}) \\
&\quad -3(T_{k-2}\widehat{W}_{n+2}+(T_{k-4}-3T_{k-3})\widehat{W}_{n+1}+T_{k-3}\widehat{W}_{n}) \\
&\quad +(T_{k-3}\widehat{W}_{n+2}+(T_{k-5}-3T_{k-4})\widehat{W}_{n+1}+T_{k-4}\widehat{W}_{n}) \\
&= (3T_{k-1}-3T_{k-2}+T_{k-3})\widehat{W}_{n+2}+((3T_{k-3}-3T_{k-4}+T_{k-5}) \\
&\quad -3(3T_{k-2}-3T_{k-3}+T_{k-4}))\widehat{W}_{n+1}+(3T_{k-2}-3T_{k-3}+T_{k-4})\widehat{W}_{n} \\
&= T_{k}\widehat{W}_{n+2}+(T_{k-2}-3T_{k-1})\widehat{W}_{n+1}+T_{k-1}\widehat{W}_{n} \\
&= T_{(k+1)-1}\widehat{W}_{n+2}+(T_{(k+1)-3}-3T_{(k+1)-2})\widehat{W}_{n+1}+T_{(k+1)-2}\widehat{W}_{n}.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves (12). Note that, for the other cases the proof can be done similarly. □

5. Linear Sums

Within this section, we provide summation formulas for dual hyperbolic generalized Guglielmo numbers, encompassing both positive and negative subscripts.

PROPOSITION 13. *For the generalized Guglielmo numbers, we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} W_k = \frac{1}{12}(n+1)\left(\left(2n^2-2n\right)W_2-2\left(2n^2-5n\right)W_1+\left(2n^2-8n+12\right)W_0\right).$
- **(b):** $\sum_{k=0}^{n} W_{k+1} = \frac{1}{12}(n+1)\left(\left(2n^2+4n\right)W_2-2\left(2n^2+n-6\right)W_1+\left(2n^2-2n\right)W_0\right).$
- **(c):** $\sum_{k=0}^{n} W_{k+2} = \frac{1}{12}(n+1)\left(\left(2n^2+10n+12\right)W_2-2\left(2n^2+7n\right)W_1+\left(2n^2+4n\right)W_0\right).$
- **(d):** $\sum_{k=0}^{n} W_{k+3} = \frac{1}{12}(n+1)\left(\left(2n^2+16n+36\right)W_2-2\left(2n^2+13n+18\right)W_1+\left(2n^2+10n+12\right)W_0\right).$

Proof. For the proof, see Soykan [17]. □

PROPOSITION 14. *For the generalized Guglielmo numbers, we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} W_{2k} = \frac{1}{12}(n+1)(\left(8n^2-2n\right)W_2-2\left(8n^2-8n\right)W_1+\left(8n^2-14n+12\right)W_0).$
- **(b):** $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{12}(n+1)(W_2\left(8n^2+10n\right)-2W_1\left(8n^2+4n-6\right)+W_0\left(8n^2-2n\right)).$
- **(c):** $\sum_{k=0}^{n} W_{2k+2} = \frac{1}{12}(n+1)(\left(8n^2+22n+12\right)W_2-2\left(8n^2+16n\right)W_1+\left(8n^2+10n\right)W_0).$
- **(d):** $\sum_{k=0}^{n} W_{2k+3} = \frac{1}{12}(n+1)(\left(8n^2+34n+36\right)W_2-2\left(8n^2+28n+18\right)W_1$
  $+\left(8n^2+22n+12\right)W_0).$
- **(e):** $\sum_{k=0}^{n} W_{2k+4} = \frac{1}{12}(n+1)(\left(8n^2+46n+72\right)W_2-2\left(8n^2+40n+48\right)W_1+\left(8n^2+34n+36\right)W_0).$

Proof. For the proof, see Soykan [17]. □

PROPOSITION 15. *For the generalized Guglielmo numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_{-k} = \frac{1}{12}(n+1)\left(\left(2n^2+4n\right)W_2 - 2\left(2n^2+7n\right)W_1 + \left(2n^2+10n+12\right)W_0\right)$.

(b): $\sum_{k=0}^{n} W_{-k+1} = \frac{1}{12}(n+1)\left(\left(2n^2-2n\right)W_2 - 2\left(2n^2+n-6\right)W_1 + \left(2n^2+4n\right)W_0\right)$.

(c): $\sum_{k=0}^{n} W_{-k+2} = \frac{1}{12}(n+1)\left(\left(2n^2-8n+12\right)W_2 - 2\left(2n^2-5n\right)W_1 + \left(2n^2-2n\right)W_0\right)$.

(d): $\sum_{k=0}^{n} W_{-k+3} = \frac{1}{12}(n+1)\left(\left(2n^2-14n+36\right)W_2 - 2\left(2n^2-11n+18\right)W_1 + \left(2n^2-8n+12\right)W_0\right)$.

Proof. For the proof, see Soykan [17]. □

PROPOSITION 16. *For the generalized Guglielmo numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_{-2k} = \frac{1}{12}(n+1)\left(\left(8n^2+10n\right)W_2 - 2\left(8n^2+16n\right)W_1 + \left(8n^2+22n+12\right)W_0\right)$.

(b): $\sum_{k=0}^{n} W_{-2k+1} = \frac{1}{12}(n+1)\left(\left(8n^2-2n\right)W_2 - 2\left(8n^2+4n-6\right)W_1 + \left(8n^2+10n\right)W_0\right)$.

(c): $\sum_{k=0}^{n} W_{-2k+2} = \frac{1}{12}(n+1)\left(\left(8n^2-14n+12\right)W_2 - 2\left(8n^2-8n\right)W_1 + \left(8n^2-2n\right)W_0\right)$.

(d): $\sum_{k=0}^{n} W_{-2k+3} = \frac{1}{12}(n+1)\left(\left(8n^2-26n+36\right)W_2 - 2\left(8n^2-20n+18\right)W_1 + \left(8n^2-14n+12\right)W_0\right)$.

(e): $\sum_{k=0}^{n} W_{2k+4} = \frac{1}{12}(n+1)\left(\left(8n^2+46n+72\right)W_2 - 2\left(8n^2+40n+48\right)W_1 + \left(8n^2+34n+36\right)W_0\right)$.

Proof. For the proof, see Soykan [17]. □

Next, we present the formulas which give the summation of the dual hyperbolic generalized Guglielmo numbers.

THEOREM 17. *For $n \geq 0$ then the following sum formulas holds for dual hyperbolic generalized Guglielmo numbers.*

(a): $\sum_{k=0}^{n} \widehat{W}_k = \frac{1}{6}(n+1)((-n+6\varepsilon+18j\varepsilon+5n\varepsilon+jn^2+n^2\varepsilon+2jn+n^2+8jn\varepsilon+jn^2\varepsilon)W_2+(6j+5n-18j\varepsilon-7n\varepsilon-2jn^2-2n^2\varepsilon-jn-2n^2-13jn\varepsilon-2jn^2\varepsilon)W_1+(-4n+6j\varepsilon+2n\varepsilon+jn^2+n^2\varepsilon-jn+n^2+5jn\varepsilon+jn^2\varepsilon+6)W_0)$.

(b): $\sum_{k=0}^{n} \widehat{W}_{2k} = \frac{1}{6}(n+1)((-n+6\varepsilon+18j\varepsilon+11n\varepsilon+4jn^2+4n^2\varepsilon+5jn+4n^2+17jn\varepsilon+4jn^2\varepsilon)W_2+(6j+8n-18j\varepsilon-16n\varepsilon-8jn^2-8n^2\varepsilon-4jn-8n^2-28jn\varepsilon-8jn^2\varepsilon)W_1+(-7n+6j\varepsilon+5n\varepsilon+4jn^2+4n^2\varepsilon-jn+4n^2+11jn\varepsilon+4jn^2\varepsilon+6)W_0)$.

(c): $\sum_{k=0}^{n} \widehat{W}_{2k+1} = \frac{1}{6}(n+1)((6j+5n+18\varepsilon+36j\varepsilon+17n\varepsilon+4jn^2+4n^2\varepsilon+11jn+4n^2+23jn\varepsilon+4jn^2\varepsilon)W_2+(6-18\varepsilon-48j\varepsilon-28n\varepsilon-8jn^2-8n^2\varepsilon-16jn-8n^2-40jn\varepsilon-8jn^2\varepsilon-4n)W_1+(-n+6\varepsilon+18j\varepsilon+11n\varepsilon+4jn^2+4n^2\varepsilon+5jn+4n^2+17jn\varepsilon+4jn^2\varepsilon)W_0)$.

Proof.

(a): Note that using (2.1), we get

$$\sum_{k=0}^{n} \widehat{W}_k = \sum_{k=0}^{n} W_k + j\sum_{k=0}^{n} W_{k+1} + \varepsilon\sum_{k=0}^{n} W_{k+2} + j\varepsilon\sum_{k=0}^{n} W_{k+3}$$

and using Proposition 13 the proof is easily attainable.

(b): Note that using (2.1), we get

$$\sum_{k=0}^{n} \widehat{W}_{2k} = \sum_{k=0}^{n} W_{2k} + j\sum_{k=0}^{n} W_{2k+1} + \varepsilon\sum_{k=0}^{n} W_{2k+2} + j\varepsilon\sum_{k=0}^{n} W_{2k+3}$$

and using Proposition 14 the proof is easily attainable.

(c): Note that using (2.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{2k+1}=\sum_{k=0}^{n}W_{2k+1}+j\sum_{k=0}^{n}W_{2k+2}+\varepsilon\sum_{k=0}^{n}W_{2k+3}+j\varepsilon\sum_{k=0}^{n}W_{2k+4}$$

and using Proposition 14 the proof is easily attainable. □

As a particular case of the theorem 17,(a), we present following corollary.

Corollary 18.

(a): $\sum_{k=0}^{n}\widehat{T}_k=\frac{1}{6}(n+1)((6j+18\varepsilon+36j\varepsilon)+(5j+8\varepsilon+11j\varepsilon+2)n+(j+\varepsilon+j\varepsilon+1)n^2)$.

(b): $\sum_{k=0}^{n}\widehat{H}_k=(3j+3\varepsilon+3j\varepsilon+3)(n+1)$.

(c): $\sum_{k=0}^{n}\widehat{O}_k=\frac{1}{6}(n+1)((12j+36\varepsilon+72j\varepsilon)+(10j+16\varepsilon+22j\varepsilon+4)n+(2j+2\varepsilon+2j\varepsilon+2)n^2)$.

(d): $\sum_{k=0}^{n}\widehat{p}_k=\frac{1}{6}(n+1)((6j+30\varepsilon+72j\varepsilon)+(18\varepsilon+9j+27j\varepsilon)n+(3j+3\varepsilon+3+3j\varepsilon)n^2)$.

As a particular case of the theorem 17, (b), we present following corollary.

Corollary 19.

(a): $\sum_{k=0}^{n}\widehat{T}_{2k}=\frac{1}{6}(n+1)((6j+18\varepsilon+36j\varepsilon)+(5+17\varepsilon+11j+23j\varepsilon)n+(4+4j+4\varepsilon+4j\varepsilon)n^2)$.

(b): $\sum_{k=0}^{n}\widehat{H}_{2k}=(3j+3\varepsilon+3j\varepsilon+3)(n+1)$.

(c): $\sum_{k=0}^{n}\widehat{O}_{2k}=\frac{1}{6}(n+1)((12j+36\varepsilon+72j\varepsilon)+(10+22j+34\varepsilon+46j\varepsilon)n+(8+8j+8\varepsilon+8j\varepsilon)n^2)$.

(d): $\sum_{k=0}^{n}\widehat{p}_{2k}=\frac{1}{6}(n+1)((6j+30\varepsilon+72j\varepsilon)+(3+21j+39\varepsilon+57j\varepsilon)n+(12+12j+12\varepsilon+12j\varepsilon)n^2)$.

As a particular case of the theorem 17, (c), we present following corollary.

Corollary 20.

(a): $\sum_{k=0}^{n}\widehat{T}_{2k+1}=\frac{1}{6}(n+1)((6+18j+36\varepsilon+60j\varepsilon)+(11+17j+23\varepsilon+29j\varepsilon)n+(4+4j+4\varepsilon+4j\varepsilon)n^2)$.

(b): $\sum_{k=0}^{n}\widehat{H}_{2k+1}=(3j+3\varepsilon+3j\varepsilon+3)(n+1)$.

(c): $\sum_{k=0}^{n}\widehat{O}_{2k+1}=\frac{1}{6}(n+1)((12+36j+72\varepsilon+120j\varepsilon)+(22+46\varepsilon+34j+58j\varepsilon)n+(8+8j+8\varepsilon+8j\varepsilon)n^2)$.

(d): $\sum_{k=0}^{n}\widehat{p}_{2k+1}=\frac{1}{6}(n+1)((6+30j+72\varepsilon+132j\varepsilon)+(21+39j+57\varepsilon+75j\varepsilon)n+(12+12j+12\varepsilon+12j\varepsilon)n^2)$.

Now, we present the formulas to give us the summation formulas of the generalized Guglielmo numbers with negative subscripts.

Theorem 21. *For $n\geq 0$ then the following sum formulas holds for dual hyperbolic generalized Guglielmo numbers.*

(a): $\sum_{k=0}^{n}\widehat{W}_{-k}=\frac{1}{6}(n+1)((2n+6\varepsilon+18j\varepsilon-4n\varepsilon+jn^2+n^2\varepsilon-jn+n^2-7jn\varepsilon+jn^2\varepsilon)W_2+(6j-7n-18j\varepsilon+5n\varepsilon-2jn^2-2n^2\varepsilon-jn-2n^2+11jn\varepsilon-2jn^2\varepsilon)W_1+(5n+6j\varepsilon-n\varepsilon+jn^2+n^2\varepsilon+2jn+n^2-4jn\varepsilon+jn^2\varepsilon+6)W_0)$.

(b): $\sum_{k=0}^{n}\widehat{W}_{-2k} = \frac{1}{6}(n+1)((5n+6\varepsilon+18j\varepsilon-7n\varepsilon+4jn^2+4n^2\varepsilon-jn+4n^2-13jn\varepsilon+4jn^2\varepsilon)$ $W_2+(6j-16n-18j\varepsilon+8n\varepsilon-8jn^2-8n^2\varepsilon-4jn-8n^2+20jn\varepsilon-8jn^2\varepsilon)W_1+(11n+6j\varepsilon-n\varepsilon+4jn^2+4n^2\varepsilon+5jn+4n^2-7jn\varepsilon+4jn^2\varepsilon+6)W_0)$.

(c): $\sum_{k=0}^{n}\widehat{W}_{-2k+1} = \frac{1}{6}(n+1)((6j-n+18\varepsilon+36j\varepsilon-13n\varepsilon+4jn^2+4n^2\varepsilon-7jn+4n^2-19jn\varepsilon+4jn^2\varepsilon)$ $W_2+(20n\varepsilon-18\varepsilon-48j\varepsilon-4n-8jn^2-8n^2\varepsilon+8jn-8n^2+32jn\varepsilon-8jn^2\varepsilon+6)W_1+(5n+6\varepsilon+18j\varepsilon-7n\varepsilon+4jn^2+4n^2\varepsilon-jn+4n^2-13jn\varepsilon+4jn^2\varepsilon)W_0)$.

Proof.

(a): Note that using (2.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{-k}=\sum_{k=0}^{n}W_{-k}+j\sum_{k=0}^{n}W_{-k+1}+\varepsilon\sum_{k=0}^{n}W_{-k+2}+j\varepsilon\sum_{k=0}^{n}W_{-k+3}$$

and using Proposition 15 the proof is easily attainable.

(b): Note that using (2.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{-2k}=\sum_{k=0}^{n}W_{-2k}+j\sum_{k=0}^{n}W_{-2k+1}+\varepsilon\sum_{k=0}^{n}W_{-2k+2}+j\varepsilon\sum_{k=0}^{n}W_{-2k+3}$$

and using Proposition 16 the proof is easily attainable.

(c): Note that using (2.1), we get using Proposition (16), we get

$$\sum_{k=0}^{n}\widehat{W}_{-2k+1}=\sum_{k=0}^{n}W_{-2k+1}+j\sum_{k=0}^{n}W_{-2k+2}+\varepsilon\sum_{k=0}^{n}W_{-2k+3}+j\varepsilon\sum_{k=0}^{n}W_{-2k+4}$$

and using Proposition 16 the proof is easily attainable. □

Next, we present different sum formulas of the dual hyperbolic generalized Guglielmo numbers.

As a particular case of the theorem 21, (a), we obtain the following corollary.

Corollary 22.

(a): $\sum_{k=0}^{n}\widehat{T}_{-k} = \frac{1}{6}(n+1)((6j+18\varepsilon+36j\varepsilon)+(-1-4j-7\varepsilon-10j\varepsilon)n+(1+j+\varepsilon+j\varepsilon)n^2)$.

(b): $\sum_{k=0}^{n}\widehat{H}_{-k} = (3j+3\varepsilon+3j\varepsilon+3)(n+1)$.

(c): $\sum_{k=0}^{n}\widehat{O}_{-k} = \frac{1}{6}(n+1)((12j+36\varepsilon+72j\varepsilon)+(-2-8j-14\varepsilon-20j\varepsilon)n+(2+2j+2\varepsilon+2j\varepsilon)n^2)$.

(d): $\sum_{k=0}^{n}\widehat{p}_{-k} = \frac{1}{2}(n+1)((2j+10\varepsilon+24j\varepsilon)+(1-2j-5\varepsilon-8j\varepsilon)n+(1+j+\varepsilon+j\varepsilon)n^2)$.

As a particular case of the theorem 21, (b), we obtain the following corollary.

Corollary 23.

(a): $\sum_{k=0}^{n}\widehat{T}_{-2k} = \frac{1}{6}(n+1)((6j+18\varepsilon+36j\varepsilon)+(-1-7j-13\varepsilon-19j\varepsilon)n+(4+4j+4\varepsilon+4j\varepsilon)n^2)$.

(b): $\sum_{k=0}^{n}\widehat{H}_{-2k} = (3j+3\varepsilon+3j\varepsilon+3)(n+1)$.

(c): $\sum_{k=0}^{n}\widehat{O}_{-2k} = \frac{1}{3}(n+1)((6j+18\varepsilon+36j\varepsilon)+(-1-7j-13\varepsilon-19j\varepsilon)n+(4+4j+4\varepsilon+4j\varepsilon)n^2)$.

(d): $\sum_{k=0}^{n}\widehat{p}_{-2k} = \frac{1}{6}(n+1)((6j+30\varepsilon+72j\varepsilon)+(9-9j-27\varepsilon-45j\varepsilon)n+(12+12j+12\varepsilon+12j\varepsilon)n^2)$.

As a particular case of the theorem 21, (c), we obtain the following corollary.

Corollary 24.

(a): $\sum_{k=0}^{n}\widehat{T}_{-2k+1}=\frac{1}{6}(n+1)((6+18j+36\varepsilon+60j\varepsilon)+(-7-13j-19\varepsilon-25j\varepsilon)n+(4+4j+4\varepsilon+4j\varepsilon)n^2)$.

(b): $\sum_{k=0}^{n}\widehat{H}_{-2k+1}=(3j+3\varepsilon+3j\varepsilon+3)(n+1)$.

(c): $\sum_{k=0}^{n}\widehat{O}_{-2k+1}=\frac{1}{3}(n+1)((6+18j+36\varepsilon+60j\varepsilon)+(-7-13j-19\varepsilon-25j\varepsilon)n+(4+4j+4\varepsilon+4j\varepsilon)n^2)$.

(d): $\sum_{k=0}^{n}\widehat{p}_{-2k+1}=\frac{1}{6}(n+1)((6+30j+72\varepsilon+132j\varepsilon)+(-9-27j-45\varepsilon-63j\varepsilon)n+(12+12j+12\varepsilon+12j\varepsilon)n^2)$.

6. Matrices related with Dual Hyperbolic Generalized Guglielmo Numbers

In this sectiion, using dual hyperbolic Guglielmo numbers, we give some matrices related to dual hyperbolic Guglielmo numbers.

We consider the triangular sequence $\{T_n\}$ defined by the third-order recurrence relation as follows

$$T_n=3T_{n-1}-3T_{n-2}+T_{n-3}$$

with the initial conditions

$$T_0=0,\ T_1=1,\ T_2=3.$$

We present the square matrix A of order 3 as

$$A=\begin{pmatrix}3&-3&1\\1&0&0\\0&1&0\end{pmatrix}$$

under the condition that $\det A=1$. Then, we give the following Lemma.

Lemma 25. *For any integers n the following identity can be written*

$$\begin{pmatrix}\widehat{W}_{n+2}\\\widehat{W}_{n+1}\\\widehat{W}_{n}\end{pmatrix}=\begin{pmatrix}3&-3&1\\1&0&0\\0&1&0\end{pmatrix}^{n}\begin{pmatrix}\widehat{W}_{2}\\\widehat{W}_{1}\\\widehat{W}_{0}\end{pmatrix}. \tag{6.1}$$

Proof. First, we prove the assertion for the case $n\geq 0$. Lemma 25 can be given by mathematical induction on n. If $n=0$ we get

$$\begin{pmatrix}\widehat{W}_{2}\\\widehat{W}_{1}\\\widehat{W}_{0}\end{pmatrix}=\begin{pmatrix}3&-3&1\\1&0&0\\0&1&0\end{pmatrix}^{0}\begin{pmatrix}\widehat{W}_{2}\\\widehat{W}_{1}\\\widehat{W}_{0}\end{pmatrix}$$

which is true. We assume that (6.1) is true for $n=k$. Thus the following identity is true.

$$\begin{pmatrix}\widehat{W}_{k+2}\\\widehat{W}_{k+1}\\\widehat{W}_{k}\end{pmatrix}=\begin{pmatrix}3&-3&1\\1&0&0\\0&1&0\end{pmatrix}^{k}\begin{pmatrix}\widehat{W}_{2}\\\widehat{W}_{1}\\\widehat{W}_{0}\end{pmatrix}.$$

For $n = k+1$, we get

$$\begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_k \end{pmatrix}$$

$$= \begin{pmatrix} 3\widehat{W}_{k+2} - 3\widehat{W}_{k+1} + \widehat{W}_k \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix}.$$

For the other case $n < 0$ the proof is easily attainable. Consequently, using mathematical induction on n, the proof is completed.

Note that, see [20],

$$A^n = \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}.$$

THEOREM 26. *If we define the matrices $N_{\widehat{W}}$ and $E_{\widehat{W}}$ as follow*

$$N_{\widehat{W}} = \begin{pmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{pmatrix},$$

$$E_{\widehat{W}} = \begin{pmatrix} \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \end{pmatrix}.$$

then the following identity is true:

$$A^n N_{\widehat{W}} = E_{\widehat{W}}.$$

Proof. For the proof, we can use the following identities

$$\begin{aligned} A^n N_{\widehat{W}} &= \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix} \begin{pmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{pmatrix}, \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a_{11} &= G\widehat{W}_2 T_{n+1} + \widehat{W}_1 (T_{n-1} - 3T_n) + \widehat{W}_0 T_n, \\ a_{12} &= \widehat{W}_1 T_{n+1} + \widehat{W}_0 (T_{n-1} - 3T_n) + \widehat{W}_{-1} T_n, \\ a_{13} &= \widehat{W}_0 T_{n+1} + \widehat{W}_{-1} (T_{n-1} - 3T_n) + \widehat{W}_{-2} T_n, \\ a_{21} &= \widehat{W}_2 T_n + \widehat{W}_1 (T_{n-2} - 3T_{n-1}) + \widehat{W}_0 T_{n-1}, \\ a_{22} &= \widehat{W}_1 T_n + \widehat{W}_0 (T_{n-2} - 3T_{n-1}) + \widehat{W}_{-1} T_{n-1}, \\ a_{23} &= \widehat{W}_0 T_n + \widehat{W}_{-1} (T_{n-2} - 3T_{n-1}) + \widehat{W}_{-2} T_{n-1}, \\ a_{31} &= \widehat{W}_2 T_{n-1} + \widehat{W}_1 (T_{n-3} - 3T_{n-2}) + \widehat{W}_0 T_{n-2}, \\ a_{32} &= \widehat{W}_1 T_{n-1} + \widehat{W}_0 (T_{n-3} - 3T_{n-2}) + \widehat{W}_{-1} T_{n-2}, \\ a_{33} &= \widehat{W}_0 T_{n-1} + \widehat{W}_{-1} (T_{n-3} - 3T_{n-2}) + \widehat{W}_{-2} T_{n-2}. \end{aligned}$$

Using the Theorem 12, the proof is done. □

From Theorem 26, we can write the following corollary.

Corollary 27.

(a): *We suppose that the matrices* $N_{\widehat{T}}$ *and* $E_{\widehat{T}}$ *are defined as following*

$$N_T = \begin{pmatrix} \widehat{T}_2 & \widehat{T}_1 & \widehat{T}_0 \\ \widehat{T}_1 & \widehat{T}_0 & \widehat{T}_{-1} \\ \widehat{T}_0 & \widehat{T}_{-1} & \widehat{T}_{-2} \end{pmatrix},$$

$$E_{\widehat{T}} = \begin{pmatrix} \widehat{T}_{n+2} & \widehat{T}_{n+1} & \widehat{T}_n \\ \widehat{T}_{n+1} & \widehat{T}_n & \widehat{T}_{n-1} \\ \widehat{T}_n & \widehat{T}_{n-1} & \widehat{T}_{n-2} \end{pmatrix},$$

so that the following identity is true for A^n, $N_{\widehat{T}}$, $E_{\widehat{T}}$,

$$A^n N_{\widehat{T}} = E_{\widehat{T}},$$

(b): *We suppose that the matrices* $N_{\widehat{O}}$ *and* $E_{\widehat{O}}$ *are defined as following*

$$N_{\widehat{O}} = \begin{pmatrix} \widehat{O}_2 & \widehat{O}_1 & \widehat{O}_0 \\ \widehat{O}_1 & \widehat{O}_0 & \widehat{O}_{-1} \\ \widehat{O}_0 & \widehat{O}_{-1} & \widehat{O}_{-2} \end{pmatrix},$$

$$E_{\widehat{O}} = \begin{pmatrix} \widehat{O}_{n+2} & \widehat{O}_{n+1} & \widehat{O}_n \\ \widehat{O}_{n+1} & \widehat{O}_n & \widehat{O}_{n-1} \\ \widehat{O}_n & \widehat{O}_{n-1} & \widehat{O}_{n-2} \end{pmatrix},$$

so that the following identity is true for A^n, $N_{\widehat{O}}$, $E_{\widehat{O}}$,

$$A^n N_{\widehat{O}} = E_{\widehat{O}}.$$

(c): *We suppose that the matrices* $N_{\widehat{H}}$ *and* $E_{\widehat{H}}$ *are defined as following*

$$N_{\widehat{H}} = \begin{pmatrix} \widehat{H}_2 & \widehat{H}_1 & \widehat{H}_0 \\ \widehat{H}_1 & \widehat{H}_0 & \widehat{H}_{-1} \\ \widehat{H}_0 & \widehat{H}_{-1} & \widehat{H}_{-2} \end{pmatrix},$$

$$E_{\widehat{H}} = \begin{pmatrix} \widehat{H}_{n+2} & \widehat{H}_{n+1} & \widehat{H}_n \\ \widehat{H}_{n+1} & \widehat{H}_n & \widehat{H}_{n-1} \\ \widehat{H}_n & \widehat{H}_{n-1} & \widehat{H}_{n-2} \end{pmatrix},$$

so that the following identity is true for A^n, $N_{\widehat{H}}$, $E_{\widehat{H}}$,

$$A^n N_{\widehat{H}} = E_{\widehat{H}}.$$

(d): *We suppose that the matrices* $N_{\widehat{p}}$ *and* $E_{\widehat{p}}$ *are defined as following*

$$N_{\widehat{p}} = \begin{pmatrix} \widehat{p}_2 & \widehat{p}_1 & \widehat{p}_0 \\ \widehat{p}_1 & \widehat{p}_0 & \widehat{p}_{-1} \\ \widehat{p}_0 & \widehat{p}_{-1} & \widehat{p}_{-2} \end{pmatrix},$$

$$E_{\widehat{p}} = \begin{pmatrix} \widehat{p}_{n+2} & \widehat{p}_{n+1} & \widehat{p}_n \\ \widehat{p}_{n+1} & \widehat{p}_n & \widehat{p}_{n-1} \\ \widehat{p}_n & \widehat{p}_{n-1} & \widehat{p}_{n-2} \end{pmatrix}.$$

so that he following identity is true for A^n, $N_{\widehat{p}}$, $E_{\widehat{p}}$,

$$A^n N_{\widehat{p}} = E_{\widehat{p}}.$$

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