Fixed Point Theorems for φ-maps on G-Cone Metric Spaces over Banach Algebra

Abstract

Aims/ objectives: In our research paper, we established some fixed point and common fixed point theorems using generalized contraction mapping in G-cone metric space over Banach algebras. This space is defined by Beg (15) through certain contractive conditions related to φ-maps. Our findings are a generalization and extension of several well-known outcomes related to fixed point theory.

Keywords: Banach algebras; common fixed point; generalized cone metric spaces; generalized Lipschitz conditions; weakly compatible maps

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1 Introduction

Fixed point theory is a significant area of study in analysis. The Banach contraction principle (9) is frequently employed in the development of existence and uniqueness theories. To determine the fixed point of contractive mappings, certain geometric assumptions are required either on the structure of mappings or spaces. Thus, identifying the fixed point of contractive mappings is a common problem in Mathematics.

In his work, Dhage (6) introduced the concept of D-metric space. However, subsequent studies have revealed that certain theorems involving this space were flawed, rendering the

majority of the results claimed by Dhage and others valid. Mustafa and Sims (7) identified these errors and proposed a new structure known as G-metric spaces that generalize metric spaces.
In their work, Huang and Zhang (2) extended the concept of metric spaces to include ordered Banach spaces instead of just the set of real numbers. This led to the definition of cone metric spaces. They also introduced the notion of completeness and described the convergence of sequences in these spaces. Additionally, they proved some fixed point theorems of contractive mappings on complete cone metric spaces, assuming a normal cone. Since then, several authors have built on Huang and Zhang's work to study fixed point theorems for both normal and non-normal cones..
In their recent work, I. Beg, M. Abbas and T. Nazir (15) introduced the concept of G-cone metric spaces, which are a generalization of both G-metric spaces and cone metric spaces. They proved several topological properties of these spaces, such as the convergence properties of sequences and completeness. Additionally, they obtained some fixed-point theorems that satisfy certain contractive conditions. Furthermore, Cristina Di Pari and Pasquale Vetro (14) have proven some theorems involving φ-maps in cone metric spaces, and W. Shatanawi (16) has also obtained some fixed-point results in G-metric spaces.
In a recent paper, Adewale and Osawaru (5) introduced the concept of a G-cone metric space, which is a more general space than a G-metric space and a cone metric space. They replaced the set of real numbers with an ordered Banach space and proved the convergence properties of sequences as well as some fixed point theorems in this space.
The purpose of this paper is to obtain fixed point and common fixed point theorems of φ-maps that satisfy contractive conditions in G-cone metric spaces over Banach algebra. Our results are generalizations of some theorems found in existing literature.

2 Preliminaries

In this paper, we assume that $\mathcal{P}$ is a cone in $\mathcal{A}$ with int $\mathcal{P} \neq \phi(\theta,$ the additive identity element of $\mathcal{A})$ and $\preceq$ is the partial ordering concerning $\mathcal{P}$ where $\mathcal{A}$ is a real Banach algebra. That is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, satisfying the following properties (3) (for all $x, y, z \in \mathcal{A}, \alpha \in \mathbb{R}$):

(i) $(xy)z = x(yz)$;

(ii) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;

(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;

(iv) there exists $e \in E$ such that $xe = ex = x$;

(v) $||e|| = 1$;

(vi) $||xy|| \leq ||x||.||y||$;

An element $x \in \mathcal{A}$ is called invertible if there exists $x^{-1} \in \mathcal{A}$ such that

$$xx^{-1} = x^{-1}x = e$$

.

Proposition 2.1. *(3) Let $x \in \mathcal{A}$ be a Banach algebra with a unit e, then the spectral radius $\rho(u)$ of $u \in \mathcal{A}$ holds*

$$\rho(u) = \lim_{n \to \infty} ||u_n||^{\frac{1}{n}} = inf||u^n||^{\frac{1}{n}} < 1$$

Further, $(e-u)$ is invertible and $(e-u)^{-1} = \sum_{i=0}^{\infty} u^i$.

Consider a Banach algebra $\mathcal{A}, \theta$ be the null vector, e be the identity element of $\mathcal{A}$ and a subset $\mathcal{P}$ of $\mathcal{A}$ is called a cone if it satisfies the following:

(i) $\{\theta, e\} \subset \mathcal{P}$ and $\mathcal{P}$ is closed;

(ii) $\mathcal{P}^2 = \mathcal{P}\mathcal{P} \subset \mathcal{P}$;

(iii) $\alpha\mathcal{P} + \beta\mathcal{P} \subset \mathcal{P}$, for all non-negative real numbers α and β;

(iv) $\mathcal{P} \cap (-C\mathcal{P}) = \{\theta\}$.

With respect to cone $\mathcal{P}$, a partial ordering $\preceq$ is defined as $u \preceq w$ if and only if $(w-u) \in \mathcal{P}$ and $u \prec w$ if $u \preceq w$ and $u \neq w$ whereas $u \ll w$ means $(w-u) \in int\mathcal{P}$.
If $\mathcal{A}$ is a Banach space and $\mathcal{P} \subset \mathcal{A}$, satisfies the conditions 1, 3 and 4 then $\mathcal{P}$ is called a cone of $\mathcal{A}$.

Remark 2.1. (3) If $\rho(x) < 1$, then $\|x_n\| \to 0$ as $n \to \infty$.

Definition 2.1. (4) Consider X is a non-empty set, $\mathcal{A}$ be a Banach algebra and $\mathcal{P} \subseteq \mathcal{A}$ be a cone. Suppose the mapping $d : X \times X \to \mathcal{A}$ satisfies the following for all $x, y, z \in X$:

(i) $d(x, z) = \theta$ if and only if $x = z$, and $\theta \preceq d(x, z)$;

(ii) $d(x, z) = d(z, x)$;

(iii) $d(x, z) \preceq d(x, y) + d(y, z)$ for every $x, y, z \in X$.

Here d is called a cone metric and (X, d) is called Cone metric space over a Banach algebra $\mathcal{A}$ (In Short CMSBA). Note that $d(x, z) \in \mathcal{P}$ for all $x, y \in X$.

Definition 2.2. (5) Let X be a non-empty set, $\mathcal{A}$ be a Banach algebra and $G : X^3 \to \mathcal{A}$ be a function satisfying the following properties:

(i) $G(x, y, z) = \theta$ if and only if $x = y = z$;

(ii) $\theta \prec G(x, y, z)$, for all $x, y \in X$, with $x \neq y$;

(iii) $G(x, x, y) \preceq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$;

(iv) $G(x, y, z) = G(y, z, x) = G(x, z, y) =$ ... (symmetry);

(v) $G(x, y, z) \preceq G(x, a, a) + G(a, y, z)$ for all $a, x, y, z \in X$ (rectangle inequality).

Then G is called a G-cone metric over Banach algebra $\mathcal{A}$ and the pair (X, G) denotes a G-cone metric space over Banach algebra.

Remark 2.2. (i) If $\mathcal{A}$ is a Banach space in Definition 2.3, then (X, G) becomes a G-cone metric space, and if in addition $z = y$, then it becomes a cone metric space as in Huang and Zhang (2).

(ii) If $\mathcal{A} = R$ in Definition 2.3, we obtain a G-metric space as in Mustafa and Sims (8) and if in addition, z = y in G(x, y, z), then it becomes a metric space.

Definition 2.3. (3) Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$ and $\{x_n\}$ a sequence in X. We say that

(i) $\{x_n\}$ is a convergent sequence if, for every $c \in B$ with $\theta \ll c$, there is an N such that $d(x_n, x) \ll c$ for all $n \geq N$. Ones write it by $x_n \to x(n \to \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence if, for every $c \in B$ with $\theta \ll c$, there is an N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence in X is convergent.

Lemma 2.1. *(3) Let $\mathcal{A}$ be a Banach algebra and k, a vector in $\mathcal{A}$. If $0 \leq r(k) < 1$, then we have*

$$r((e-k)^{-1}) < (1 - r(k))^{-1}.$$

Lemma 2.2. *(1) Let $\mathcal{A}$ be a Banach algebra and x, y be vectors in $\mathcal{A}$. If x and y commute, then the following holds:*

(i) $r(xy) \leq r(x)r(y)$*;*

(ii) $r(x+y) \leq r(x) + r(y)$*;*

(iii) $|r(x) - r(y)| \leq r(x-y)$.

Lemma 2.3. *(1) If $\mathcal{A}$ is real Banach algebra with a solid cone $\mathcal{P}$ and $\{x_n\}$ is a sequence in $\mathcal{A}$. Suppose $||x_n|| \to 0(n \to \infty)$ for any $\theta \ll c$. Then $x_n \ll c$ for all $n > N^1, N^1 \in N$.*

Lemma 2.4. *(5) If E is a real Banach space with a solid cone $\mathcal{P}$ and if $||x_n|| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$.*

Lemma 2.5. *(11) Let $\mathcal{A}$ be a Banach algebra and $k \in A$. If $\rho(k) < 1$, then*

$$\lim_{n\to\infty} ||k^n|| = 0$$

.

Definition 2.4. (5) Let (X, G) be a G-cone metric space over Banach algebra. G is said to be symmetric if

$$G(x, y, y) = G(x, x, y)$$

for all $x, y, z \in X$.

Lemma 2.6. *(11) If X is a symmetric G-cone metric space, then*

$$d_G(x, y) = 2G(x, y, y).$$

Example 2.7. *(8) Let $\mathcal{A}$ be the Banach space of all continuous real-valued functions $C(K)$ on a compact Hausdorff topological space K, with multiplication defined pointwise. Then $\mathcal{A}$ is a Banach algebra, and the constant function $f(t) = 1$ is the unit of $\mathcal{A}$.*
Let $P = \{f \in \mathcal{A} : f(t) \geq 0$ for all $t \in K\}$. Then $\mathcal{P} \subset \mathcal{A}$ is a normal cone with a normal constant M = 1. Let $X = C(K)$ with the metric $d : X \times X \to \mathcal{A}$ defined by

$$d(f, g) = |f(t) - g(t)|$$

where $t \in K$. Then (X, d) is a cone metric space over a Banach algebra $\mathcal{A}$.

Example 2.8. *(4) Let (X,d) be a cone metric space. Define $G: X^3 \to B$, by*

$$G(x,y,z) = d(x,y) + d(y,z) + d(x,z).$$

Definition 2.5. (5) A G-cone metric space over Banach algebra $\mathcal{A}$ is said to be G-bounded if for any $x, y, z \in X$, there exists $K \succ \theta$ such that

$$||G(x,y,z)|| \preceq K$$

Definition 2.6. (5) Let (X, G) be a G-cone metric space over Banach algebra and $\{x_n\}$ a sequence in X , $c \gg \theta$ with $c \in \mathcal{A}$. Then

(i) $\{x_n\}$ converges to $x \in X$ if and only if $G(x_m, x_n, x) \ll c$ for all $n, m > N^1, N^1 \in N$.

(ii) $\{x_n\}$ is Cauchy sequence if and only if $G(x_n, x_m, x_p) \ll c$ for all $n, m > p > N^1, N^1 \in N$.

(iii) (X, G) is complete G-cone metric space over Banach algebra if every Cauchy sequence converges.

Definition 2.7. (12) Let f and g be self-maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Lemma 2.9. *(13) Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Definition 2.8. (13) The mappings $f, g : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$fgx = gfx$$

whenever $gx = fx$.

Definition 2.9. (14) Let $\mathcal{P}$ be a cone defined as above. A nondecreasing function $\varphi : \mathcal{P} \to \mathcal{P}$ is called a φ-map if the following conditions hold,

(i) $\varphi(\theta) = \theta$ and $\theta < \varphi(z) < z$ for $z \in \mathcal{P}\backslash\{\theta\}$,

(ii) $z \in int\mathcal{P}$ implies $z - \varphi(z) \in int\mathcal{P}$,

(iii) $\lim_{n\to\infty} \varphi^n(z) = \theta$ for every $z \in \mathcal{P}\backslash\{\theta\}$.

3 Main Result

Theorem 3.1. *Let (X,d) be a complete symmetric G-cone metric space over a Banach algebra $\mathcal{A}$ and $\mathcal{P}$ be a nonnormal cone in $\mathcal{A}$. Suppose that the mapping $f, g : X \to X$ satisfies the following contractive condition*

$$G(fx, fy, fz) \preceq \varphi G(gx, gy, gz) \tag{3.1}$$

for all $x, y, z \in X$. Suppose f and g are weakly compatible with $f(X) \subset g(X)$. If $f(X)$ or $g(X)$ is a complete subspace of X, then mappings f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Choose $x_1 \in X$ such that $fx_0 = gx_1$. This is true since $f(X) \subset g(X)$. Continuing this process, having chosen $x_n \in X$, we choose $x_{n+1} \in X$ such that $fx_n = gx_{n+1}$ for all $n \in N$. If $fx_n = fx_{n-1}$ for some $n \in N$, then $fx_m = fx_n$ for all $m \in N$ with $m > n$ and sofx_n is a Cauchy sequence. We assume that $fx_n \neq fx_{n-1}$ for all $n \in N$. By (3.1), we have

$$\begin{aligned} G(fx_{n+1}, fx_{n+1}, fx_n) &\preceq \varphi G(gx_{n+1}, gx_{n+1}, gx_n) \\ &= \varphi G(fx_n, fx_n, fx_{n-1}) \\ &\preceq \varphi^2 G(gx_n, gx_n, gx_{n-1}) \\ &= \varphi^2 G(fx_{n-1}, fx_{n-1}, fx_{n-2}) \\ &\preceq ....\varphi^n G(fx_1, fx_1, fx_0) \end{aligned}$$

Given $\theta \ll c$ and we choose a positive real number δ such that $c - \varphi(c) + N(\theta + \delta) \subset int\mathcal{P}$, where $N(\theta + \delta) = \{x \in \mathcal{A} : \|x\| < \delta\}$. Also, choose a natural number N such that $\varphi^n G(fx_1, fx_1, fx_0) \ll c - \varphi(c)$ for all $m \geq N$. Consequently

$$G(fx_{m+1}, fx_{m+1}, fx_m) \ll c - \varphi(c)$$

for all $m \geq N$. Fix $m \geq N$and we have

$$G(fx_m, fx_{n+1}, fx_{n+1}) \ll c \tag{3.2}$$

for all $n \geq m$. We write (3.2) holds when n = m. We suppose that (3.2) holds for some $n \geq m$. Then we have by using the definition(2.4),

$$\begin{aligned} G(fx_m, fx_{n+2}, fx_{n+2}) &\preceq G(fx_m, fx_{m+1}, fx_{m+1}) + G(fx_{m+1}, fx_{n+2}, fx_{n+2}) \\ &\ll c - \varphi(c) + \varphi G(gx_{m+1}, gx_{n+2}, gx_{n+2}) \\ &\ll c - \varphi(c) + \varphi G(fx_m, fx_{n+1}, fx_{n+1}) \\ &\ll c - \varphi(c) + \varphi(c) \\ &= c \end{aligned}$$

Therefore, (3.2) holds when $m = n+1$. By induction, we deduce (3.2) holds for all $m, n \geq N$. Hence fx_n is a Cauchy sequence. Suppose $f(X)$ is a complete subspace of X, then there exists $w \in f(X) \subset g(X)$ such that $fx_n \to w$ and also $gx_n \to w$. Let $v \in X$ be such that $gv = w$. We prove that $gv = fv$.
Fix $\theta \ll c$ and we choose a natural number N such that $G(w, fx_n, fx_n) \ll \frac{c}{2}$ and $G(gx_n, gv, gv) \ll \frac{c}{2}$. Then by using the definition(2.4),

$$\begin{aligned} G(w, fv, fv) &\preceq G(w, fx_n, fx_n) + G(fx_n, fv, fv) \\ &\preceq G(w, fx_n, fx_n) + \varphi G(gx_n, gv, gv) \end{aligned}$$

Using the property of φ we get

$$\begin{aligned} F(G(w, fv, fv)) &< G(w, fx_n, fx_n) + G(gx_n, gv, gv) \\ &\ll \frac{c}{2} + \frac{c}{2} \\ &= c \end{aligned}$$

Thus $G(w, fv, fv) \ll \frac{c}{i}$ for all $i \geq 1$. Since $\frac{c}{i} - G(w, fv, fv) \in \mathcal{P}$, for all i, as $i \to \infty$ we get $-G(w, fv, fv) \in \mathcal{P}$. But $G(w, fv, fv) \in \mathcal{P}$. Therefore $G(w, fv, fv) = \theta$ which implies that $gv = fv = w$, that is w, is a coincidence point of f and g. We need to use the hypothesis of weak compatibility of the mappings to show that w is a common fixed point of f and g. As $fv = gv$, by weak compatibility of f and g, it follows that

$$fw = fgv = gfv = gw.$$

We show that $fw = gw = w$. If $gw \neq w$, by condition (3.1), we get

$$\begin{aligned} G(fw, fw, fv) &\preceq \varphi G(gw, gw, gv) \\ &< G(gw, gw, gv) \\ &= G(fw, fw, fv) \end{aligned}$$

which gives us that $fw = w = gw$. Then w is a common fixed point for the mappings f and g.
Finally, suppose that w^* is another common fixed point of f and g. For the proof we use (3.1).

$$\begin{aligned} G(w, w, w^*) &= G(fw, fw, fw^*) \\ &\preceq \varphi G(gw, gw, gw^*) \\ &= G(gw, gw, gw^*) \\ &< G(w, w, w^*) \end{aligned}$$

which is a contradiction, so uniqueness is obtained. □

Theorem 3.2. *Let X be a complete symmetric G-cone metric space over a Banach algebra $\mathcal{A}$ and $\mathcal{P}$ be a nonnormal cone in $\mathcal{A}$. Suppose that the mapping $f : X \to X$ satisfies the following contractive condition*

$$G(fx, fy, fz) \preceq \varphi N(x, y, z) \tag{3.3}$$

where

$$N(x, y, z) \in \{G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(fx, y, z)\} \tag{3.4}$$

for all $x, y, z \in X$, then f has a unique fixed point in x.

Proof. Choose $x_0 \in X$. Let $x_n = fx_{n+1}$ for all $n \in N$. Suppose that $x_n \neq x_{n-1}$, for each $n \in N$. Thus we have

$$G(x_n, x_{n+1}, x_{n+1}) = G(fx_{n-1}, fx_n, fx_n) \preceq \varphi N(x_{n-1}, x_n, x_n)$$

where

$$\begin{aligned} N(x_{n-1}, x_n, x_n) \in \{&G(x_{n-1}, x_n, x_n), G(x_{n-1}, fx_{n-1}, fx_{n-1}), \\ &G(x_n, fx_n, fx_n), G(fx_{n-1}, x_n, x_n)\} \\ = \{&G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), \\ &G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n)\} \\ = \{&G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \theta\} \end{aligned}$$

If $N(x_{n-1}, x_n, x_n) = G(x_n, x_{n+1}, x_{n+1})$,then

$$G(x_n, x_{n+1}, x_{n+1}) \preceq \varphi G(x_n, x_{n+1}, x_{n+1})$$

By the property of φ maps, we have

$$G(x_n, x_{n+1}, x_{n+1}) < G(x_n, x_{n+1}, x_{n+1})$$

which is impossible. If $N(x_{n-1}, x_n, x_n) = \theta$,then

$$G(x_n, x_{n+1}, x_{n+1}) \preceq \varphi(\theta) < \theta$$

which is a contradiction. And at last, if

$$N(x_{n-1}, x_n, x_n) = G(x_{n-1}, x_n, x_n),$$

Then

$$G(x_n, x_{n+1}, x_{n+1}) \preceq \varphi G(x_{n-1}, x_n, x_n)$$

Using the same method as in Theorem 3.1, we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ is convergent to $u \in X$. Now we show that $u = fu$.For $n \in N$, we have by using definition 2.4

$$\begin{aligned} G(u, u, fu) &\preceq G(u, u, x_n) + G(x_n, x_n, fu) \\ &= G(u, u, x_n) + G(fx_{n-1}, fx_{n-1}, fu) \\ &\preceq G(u, u, x_n) + \varphi N(x_{n-1}, x_{n-1}, u) \end{aligned}$$

And

$$\begin{aligned} N(x_{n-1}, x_{n-1}, u) \in \{&G(x_{n-1}, x_{n-1}, u), G(x_{n-1}, fx_{n-1}, fx_{n-1}), \\ &G(x_{n-1}, fx_{n-1}, fx_{n-1}), G(fx_{n-1}, x_{n-1}, u)\} \\ = \{&G(x_{n-1}, x_{n-1}, u), G(x_{n-1}, x_n, x_n), G(x_n, x_{n-1}, u). \end{aligned}$$

Choose a natural number N_1 such that $G(u, u, x_n) \ll \frac{c}{2}$, for all $n \geq N_1$. We investigate these situations as follows;
Case I: If $N(x_{n-1}, x_{n-1}, u) = G(x_{n-1}, x_{n-1}, u)$, then

$$\begin{aligned} G(u, u, fu) &\preceq G(u, u, x_n) + \varphi G(x_{n-1}, x_{n-1}, u) \\ &< G(u, u, x_n) + G(x_{n-1}, x_{n-1}, u) \\ &\ll \frac{c}{2} + \frac{c}{2} \\ &= c \end{aligned}$$

Case II: If $N(x_{n-1}, x_{n-1}, u) = G(x_{n-1}, x_n, x_n)$, then

$$\begin{aligned} G(u, u, fu) &\preceq G(u, u, x_n) + \varphi G(x_{n-1}, x_n, x_n) \\ &< G(u, u, x_n) + G(x_{n-1}, x_n, x_n) \\ &\ll \frac{c}{2} + \frac{c}{2} \\ &= c \end{aligned}$$

Case III: If $N(x_{n-1}, x_{n-1}, u) = G(x_n, x_{n-1}, u)$, then

$$\begin{aligned} G(u, u, fu) &\preceq G(u, u, x_n) + \varphi G(x_n, x_{n-1}, u) \\ &< G(u, u, x_n) + G(x_n, x_{n-1}, u) \\ &\preceq G(u, u, x_n) + G(x_n, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_{n-1}, u) \\ &\ll c \end{aligned}$$

Whenever $n \in N$. Thus in all cases $G(u, u, fu) \ll \frac{c}{i}$, for all $i \geq 1$. So $\frac{c}{i} - G(u, u, fu) \in \mathcal{P}$,for all $i \geq 1$. Since $\frac{c}{i} \to 0$ as $i \to \infty$ and $\mathcal{P}$ is closed, hence $-G(u, u, fu) \in \mathcal{P}$ and $G(u, u, fu) = \theta$ therefore $u = fu$. □

Theorem 3.3. *Let X be a complete G-cone metric space over a Banach algebra $\mathcal{A}$ and $\mathcal{P}$ be a nonnormal cone in $\mathcal{A}$. Suppose that the f is a self-map of X satisfying for all $x, y, z \in X$*

$$G(fx, fy, fz) \preceq \mu N(x, y, z) \tag{3.5}$$

where

$$\begin{aligned} N(x, y, z) \in \{&G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), \\ &\frac{[G(x, fy, fy) + G(z, fx, fx)]}{2}, \frac{G(x, Ty, Ty) + G(y, Tx, Tx)}{2}\} \end{aligned}$$

and μ is a constant satisfying $0 \leq \mu < 1$. Then f has a unique fixed point.

Proof. Applying the similar method as in Theorem 3.2 with taking $\varphi(x) = \mu(x)$, where $\mu \in [0, 1)$. □

Theorem 3.4. *Let X be a complete G-cone metric space over a Banach algebra $\mathcal{A}$ and $\mathcal{P}$ be a nonnormal cone in $\mathcal{A}$. Suppose that the f is a self-map of X satisfying for all $x, y, z \in X$*

$$G(fx, fy, fz) \preceq \mu N(x, y, z) \tag{3.6}$$

where

$$\begin{aligned} N(x, y, z) \in \{&G(x, y, z), G(x, fx, fx), G(y, fy, fy), \\ &G(x, x, fy), G(y, y, fx), G(z, z, fz)\} \end{aligned}$$

or

$$\begin{aligned} N'(x, y, z) \in \{&G(x, y, z), G(x, x, fx), G(y, y, fy), \\ &G(x, x, fy), G(y, y, fx), G(z, z, fz)\} \end{aligned}$$

and μ is a constant satisfying $0 \leq \mu < 1$. Then f has a unique fixed point.

Proof. Assume that f satisfies (3.6). Using (3.6) with $z = y$ we have

$$G(fx, fy, fy) \preceq \mu N(x, y, y) \tag{3.7}$$

we have

$$\begin{aligned} N(x, y, y) \in \{&G(x, y, y), G(x, fx, fx), G(y, fy, fy), \\ &G(x, x, fy), G(y, y, fx)\} \end{aligned}$$

and

$$N'(x,y,y) \in \{G(x,y,y), G(x,x,fx), G(y,y,fy), \\ G(x,x,fy), G(y,y,fx)\}$$

By Lemma 2.13 we know that $d_G(x,y) = 2G(x,y,y)$ makes X to cone metric space

$$N(x,y) \in \{d_G(x,y), d_G(x,fx), d_G(y,fy), \\ d_G(x,fy), d_G(y,fx)\}$$

Let $x_0 \in X$ and $x_n = fx_{n-1}$. Suppose that $x_n \neq x_{n+1}$, then

$$d_G(x_n, x_{n+1}) = d_G(fx_{n-1}, fx_n) \preceq \mu N''(x_{n-1}, x_n)$$

where

$$N''(x_{n-1}, x_n) \in \{d_G(x_{n-1}, x_n), d_G(x_n, x_{n+1}), d_G(x_{n-1}, x_{n+1}), \theta\}$$

We investigate some possibilities:
Case I: If $N''(x_{n-1}, x_n) = d_G(x_{n-1}, x_{n+1})$, then

$$\begin{aligned} d_G(x_n, x_{n+1}) &\preceq \mu d_G(x_{n-1}, x_{n+1}) \\ &\preceq \mu\{d_G(x_{n-1}, x_n) + \mu d_G(x_n, x_{n+1})\} \\ &\preceq \frac{\mu}{e-\mu} d_G(x_{n-1}, x_n) \end{aligned}$$

Case II: If $N''(x_{n-1}, x_n) = d_G(x_{n-1}, x_{n+1})$, then

$$d_G(x_n, x_{n+1}) \preceq \mu d_G(x_n, x_{n+1})$$

we have $d_G(x_n, x_{n+1})(1-\mu) \preceq \theta$, since $\mu \in [0,1)$ this a contradiction.
Case III: If $N''(x_{n-1}, x_n) = \theta$, then

$$d_G(x_n, x_{n+1}) \preceq \mu\theta$$

which contradict with the assumption of $x_n \neq x_{n+1}$
Case IV: If $N''(x_{n-1}, x_n) = d_G(x_{n-1}, x_n)$, then

$$\begin{aligned} d_G(x_n, x_{n+1}) &\preceq \mu d_G(x_{n-1}, x_n) \\ &\preceq \mu^2 d_G(x_{n-2}, x_{n-1} \\ &\preceq .... \preceq \mu^n d_G(x_0, x_1) \end{aligned}$$

By the technique used in (17) Theorem 2.3, we can obtain the desired result of f having a unique fixed point. □

Example 3.5. *Let $\mathcal{A} = R^3$ and $\mathcal{P} = \{u \in R : u \geq 0\}$ be a cone. Let $X = [1, \infty)$ with the following G-cone metric space*

$$G(u,v,w) = d(u,v) + d(v,w) + d(u,w)$$

and the usual metric $d(u,v) = |u-v|$. Define the two maps $f, g : X \to X$ by

$$fu = u,$$

$$gu = 4u - 3,$$

for all $u \in X$. *Let's define the function* $\varphi : \mathcal{P} \to \mathcal{P}$ *by* $\varphi q = \frac{1}{3}q$, *for all* $q \in \mathcal{P}$. *Then we have*

$$\begin{aligned} G(fu, fv, fw) &= d(fu, fv) + d(fv, fw) + d(fu, fw) \\ &= |fu - fv| + |fv - fw| + |fu - fv| \\ &= |u - v| + |v - w| + |u - w| \\ &\preceq \frac{4}{3}(|u - v| + |v - w| + |u - w|) \\ &= \frac{1}{3}(|4u - 4v| + |4v - 4w| + |4u - 4w|) \\ &= \frac{1}{3}(|4u - 3 - 4v + 3| + |4v - 3 - 4w + 3| + |4u - 3 - 4w + 3|) \\ &= \frac{1}{3}(|gu - gv| + |gv - gw| + |gu - gw|) \\ &= \frac{1}{3}G(gu, gv, gw) \\ &\preceq \varphi G(gu, gv, gw) \end{aligned}$$

$$f1 = g1 = 1$$

Hence we have the conditions of Theorem 3.1 and see that $u = 1$ is a unique common fixed point for f and g.

4 CONCLUTIONS

We have proved fixed point and common fixed point theorems for φ-mappings in G-cone metric spaces with Banach algebra $\mathcal{A}$, which generalize several existing results in the literature.

References

[1] Ahmed A. and Salunke J.N., Fixed Point Theorem of Expanding Mapping without Continuity in Cone Metric Space over Banach Algebra, In International Conference on Recent Trends in Engineering and Science, vol. 20,(2017), 19-22.

[2] Huang L.and Zhang X., Cone metric spaces and fixed point theorems of contractive mappings, J. Math.Anal.Appl., 332(2), (2007), 1468-1476.

[3] Hao L. and Shaoyuan X. Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings, Fixed Point Theory and Applications,(2013),1-10.

[4] Xu S. and Radenovic S., Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebra without assumption of normality, Fixed Point Theory Appl., (2014), 1-12.

[5] Adewale O.K. and Osawaru E.K., G-cone metric Spaces over Banach Algebras and Some Fixed Point Results, International Journal of Mathematical Analysis and Optimization: Theory and Applications, Vol. No.2, (2019), 546 - 557.

[6] Dhage B., Pathan A.M., and Rhoades A.M., A general existence principle for fixed point theorems in D-metric spaces, International Journal of Mathematics and Mathematical Sciences, 23, (2000), 441-448.

[7] Mustafa and Sims, B., Existence of Fixed Point Results in G-Metric Spaces, International Journal of Mathematics and Mathematical Sciences, (2009), 10.

[8] Mustafa Z.and Sims B., Fixed point theorems for contractive mappings in complete G-metric spaces Applications, Fixed Point Theory and Applications, (2009), 1-10.

[9] Banach S., Sur les operations dans ensembles abstraits et leur application aux equations integrales, Fundamenta Mathematicae, 3, (1922), 133-181.

[10] Mishra A.K. and Padmavati, On Some Fixed Point Theorem in Ordered G-Cone Metric Spaces Over Banach Algebra, International Journal of Mathematics And its Applications, 10(4), (2022), 29-38.

[11] Mishra A.K., Padmawati, Rathour L. and Mishra V.N., Some Fixed Point Theorem using Generalized Cone Metric Spaces with Banach Algebra, High Technology Letters, 29(1), (2023), 153-162.

[12] Singh N. and Reena Jain, *Common Fixed Point Theorems in Generalized Cone Metric*, European Journal of Business and Management, Vol 3, No.4,(2017), 184-190

[13] M.A. Öztürk and M. Basarir, *On some common fixed point theorems with φ-maps on G-cone metric spaces*, Bull. Math. Anal. Appl, 3(1), (2011),121-133.

[14] C. Di Bari, P. Vetro, *φ-Pairs and Common Fixed Points in Cone Metric Spaces,* Rendiconti del Circolo Matematico di Palermo 57 (2008) 279–285.

[15] I. Beg, M. Abbas, T. Nazır, *Generalized Cone Metric Spaces*, The Journal of Nonlinear Science And Applications, 1 (2010), 21–31.

[16] W. Shatanawi, *Fixed Point Theory for Contractive Mappings Satisfying φ-Maps in G-Metric Spaces,* Fixed Point Theory and Appications, 2010 (2010) 1–9.

[17] Rezapour S. and Hamlbarani R. Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", Journal of Mathematical Analysis and Applications. 2008;345(2):719-724.