Original Research Article

An Efficient Method for Computing the Inverse and Eigenvalues of Circulant Matrices with Lucas Numbers

ABSTRACT

In this article, the inverse including the determinant, and the eigenvalues of circulant matrices with entry Lucas numbers are formulated explicitly in a simple wayso that their computations can be constructed efficiently. The formulation method of the determinant and inverse is simply applying the theory of elementary row or column operations and can be unified in one theorem. Meanwhile, for the eigenvalues formulation, the recently known formulation in the case of general circulant matrices is simplified by observing the specialty of the Lucas sequence and applying cyclic group properties of unit circles in the complex plane. Then, an algorithm of those formulations is constructed efficiently.

Keywords: Circulant matrix, Eigenvalues, Determinant, Inverse, Cyclic group, Lucas sequence

1. INTRODUCTION

A circulant matrix has a good structure that the calculation of the determinants, inverse, and eigenvalues can be formulated explicitly and computed efficiently. This computation aspect is very important since so many areas of mathematical problems come from this type of matrices, such as coding theory, signal system and cybernetics [1]. The circulant matrix also plats an important role in cryptography. Yu [2] investigates the security of a cryptosystem which is based on compressed sensing. The cryptosystem encrypts a plaintext with a secret circulant matrix and transmits the ciphertext over a wireless channel. Hence, it also can be associated with computer science and engineering.

Many papers that recently observed the determinant and inverses of circulant matrices whose entries are special integer sequences. Without intending to exclude any other papers whose similar topics to the current topic, we refer to some of those in the following. Bueno [3] proposed an explicit formula for the determinant and inverse of circulant matrices with geometric sequences. Shen et al. [4] follows to formulate the determinant and inverse circulant matrices with a special entry of Fibonacci and Lucas numbers, and they also gave conditions for the invertibility. Jiang et al. [5] continued to study the problems of those circulant matrices with the more general formation of k-Fibonacci and k-Lucas numbers, and this was followed by Jiang and Li [6] with changing the matrix structure from circulant to left circulant. Then, in the same year, the explicit formula of the determinants for circulant and left circulant matrices involving Tribonacci numbers were investigated by Li et al. see in [7].

Further investigation on explicit determinant and inverse matrices continued with the matrix structure of skew circulant, but now and the entry of Tribonacci was performed by Jiang and Hong in [8]. Then, a computational approach using a symbolic algorithm for computing the determinant and inverse of general bordered tridiagonal matrices presented by Jia and Li in [9]. Radicic [10] followed the investigation on k-circulant matrices with geometric sequence, while Bozkurt and Tam [11] were interested in r-circulant matrices associated with a more general number sequence. Most recently, similar problems can be found in [12-17].

In this paper, we formulate the determinant, inverse, and eigenvalues of circulant matrices with entry of Lucas numbers. The formulation method is based on elementary row or column operations which are directed to reach a fast computation. Our main result in the eigenvalues formulation, the case of general circulant matrices is simplified by considering the specialty of the Lucas sequence and exploiting the cyclic group properties of the unit circle in the complex plane. The following is the outline of this article.

We review the circulant matrix notion of the general case in Section 2, and also discuss its previous results associated with the eigenvalues, determinant, and inverse; then reviewing the Lucas sequence and connected it to the definition of the circulant matrix. In Section 3, we present a theorem containing a simple formulation of the determinant and inverse of the defined matrix. Section 4 proposes a new theorem containing a simplified formulation for the eigenvalues of the matrix. Section 5 presents algorithms of those results and discusses the efficiency. We close this paper with a concluding remark in Section 6.

2. PRELIMINARIES

In the first subsection, we discuss the notion of a general circulant matrix and its previous results connected to the problem of formulating the determinant, inverse, and eigenvalues. In the next subsection, we review the notion of Lucas numbers, and then we discuss some of the properties associated with the next section.

2.1 General Circulant Matrix

Foranysequenceofnumbers $a_0, a_1, a_2, \dots, a_{n-2}, a_{n-1}$,the $n \times n$*circulantmatrixA* denotedas $A = \mathrm{Circ}(a_0, a_1, a_2, \dots, a_{n-2}, a_{n-1})$,isdefinedby

$$A = \begin{pmatrix} a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & \cdots & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_0 \end{pmatrix}.$$

Let λ_k be the eigenvalues of A, and v_k be the corresponding eigenvectors, for $k = 0, 1, 2, \cdots, n-1$. Then, λ_k and v_k are well-known formulated (see for examples in [18-20].

$$\lambda_k = \sum_{j=0}^{n-1} a_j \alpha^{jk} \ \mathrm{dan} \ v_k = \left(1, \alpha^k, \alpha^{2k}, \cdots, \alpha^{(n-2)k}, \alpha^{(n-1)k}\right), \quad (1)$$

where $\alpha = e^{\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right)$ dan $i = \sqrt{-1}$. In this matter, we have $\mathcal{H} = \{1, \alpha, \alpha^2, \cdots, \alpha^{n-1}\}$ which is a cyclic subgroup in the multiplication group $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, α is one of the generators of $\mathcal{H}$, and all the elements in $\mathcal{H}$ are *n*th roots of unity over $\mathbb{C}$, and geometrically occupy the unit circle in the complex plane and divide the circle into *n* equal parts. For simplification point of view, we rewrite Equation (1) as a matrix multiplication

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \alpha & \alpha^2 & \dots & \alpha^{(n-1)} \\ 1 & \alpha^2 & \alpha^4 & \dots & \alpha^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{(n-1)} & \alpha^{2(n-1)} & \dots & \alpha^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{pmatrix}. \quad (2)$$

Wewillrefertoallthosenotionsnextinthesectionofeigenvaluesformulation.FromEquation(1) ,thenwehave

$$\det(A) = \prod_{k=0}^{n-1} \sum_{j=0}^{n-1} a_j \alpha^{jk} \text{ and } A^{-1} = \mathrm{Circ}(b_0, b_1, \cdots, b_{n-2}, b_{n-1}), \quad (3)$$

where

$$b_j = \frac{1}{n} \sum_{k=0}^{n-1} \lambda_k \alpha^{-jk} \text{ for } j = 0, 1, \cdots, n-1.$$

Inthecomputationperspectiveofcalculatingtheeigenvalues,determinant and inverse based on Equation (1) and (3) are inefficient in particular when *n* is largeenough. Thereasonistheneedofinvolvingcomplexnumberarithmetic inthecomputationofeigenvalueseventhoughtheentryofthematrixarereal numbers.However,if the sequence $a_0, a_1, a_2, \dots, a_{n-2}, a_{n-1}$ has a good pattern, then we couldgetmoreexplicitformssuchthatthecomputationcanbedoneefficiently. In this paper, we use the Lucas sequencefor the $a_0, a_1, a_2, \dots, a_{n-2}, a_{n-1}$.

2.2 Lucas Numbers

In this subsection for the basic theory, we refer to [21]. When we define a second-order linear homogenous recurrence relation as

$$l_n + l_{n-1} + l_{n-2} = 0, \qquad n \geq 2$$

withinitialcondition$l_0 = 2,\ l_1 = 1$, then we get a sequence of Lucas numbers: 2, 1, 3, 4, 7, 11, 18, 29, .. and in the subsequent of this paper we call it simply as Lucas sequence. It is easy to prove that the solution of the relation is the explicit formula of the nth term:

$$l_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The following proposition can be proved by mathematical induction, and it will be referred to in the proof of the eigen value formulation.

Proposition 1 For any integer, $n \geq 2$ the sum of the first until the *n*th term in the Lucas sequences is formulated as$S_n = \sum_{j=1}^{n} l_j = l_{n+2} - 3$.

Furthermore

$$U_n = \sum_{j=1}^{n} (-1)^{j-1} l_j = \begin{cases} l_{n-1} - 1 & if\ n \text{ is odd} \\ -l_{n-1} - 1 & if\ n \text{ is even.} \end{cases}$$

Below is the matrix definition that will become our main object in this paper.

Definition 1 For any integer $n \geq 2$, the $n \times n$ circulant matrix wih entry of the Lucas sequence $\{l_j\}_{j=1}^{n}$ is the matrix

$$l_n = \mathrm{Circ}(1, 3, 4, 7, \cdots l_{n-1}, l_n).$$

3. A THEOREM OF INVERSE FORMULATION

In this section, we refer to [22] for the basic theory.

Theorem 1. For integer $n \geq 3$, let $A = L_n = \mathrm{Circ}(1, 3, 4, 7, \cdots, l_{n-1}, l_n)$ be the matrix defined in Definition 1 and let $x = 1 - l_{n-1} - l_n, y = l_n - 2$ then

$$\det(A) = x^{n-1} - 5y^{n-2} - \sum_{k=1}^{n-2} (l_{n-k-1} + l_{n-k+1}) y^{k-1} x^{n-k-1}.$$

If$\delta = \det(A) \neq 0$,then$A^{-1} = \frac{1}{\delta} \mathrm{Circ}(z_1, z_2, z_3, z_4, \cdots, z_{n-1}, z_n)$where

$$z_1 = \frac{\delta + 5y^{n-2}}{x}, z_2 = \frac{\delta + 5x^{n-2}}{y} \text{and } z_k = 5x^{n-k} y^{k-3} \text{for } k = 3, 4, \cdots, n.$$

Proof. We describe the following proof step by step in six steps.First, let E_1 be a series of elementary row operations on A: by substituting the *i*th row with the resulting operation of the *i*th row is subtracted by the $(i+1)$th and the $(i+2)$th rows, for $i = 2, 3, \ldots (n-2)$; the next, by substituting the $(n-1)$th row with the $(n-1)$th row is subtracted by the *n*th and the first rows; and the last, by substituting the *n*th row with the *n*th row is subtracted by 3 times the first row. Then we have $E_1(\mathrm{A}) = A \sim D_1$, that is

D_1

where

$$x = 1 - l_n - l_{n-1}, y = l_{n-2} - 2 \quad (4)$$

$$g_1 = -5, g_2 = -5, \text{and } g_j = g_{j-2} + g_{j-1} \text{for } j = 3, 4, \cdots, (n-2), \quad (5)$$

We can also formulate the g_j as

$$g_j = l_{j+2} - 3l_{j+1} = l_j - 2(l_{j-1} + l_j) = -(2l_{j-1} + l_j) = -(l_{j-1} + l_{j+1}). \quad (6)$$

Thenext,thereexiststhematrix$L_1 = E_1(I_n)$suchthat$D_1 = L_1 A$where

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ -3 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Secondly, applying K_1 as a series of elementary column operations on D_1 by substituting the jth column with the jth column is added to the result operation of the first column multiplied by $(-l_j)$, for $j = 2, 3, \cdots, n,$ then we have $K_1(D_1) = F \sim D_2$ where

$$D_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & x & -y & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x & -y & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & x & -y & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & x & -y & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & x & -y \\ 0 & g_1 & g_2 & g_3 & \cdots & g_{n-3} & g_{n-2} & 1-3l_n \end{pmatrix}$$

and then, there exists $R_1 = K_1(I_n)$ such that $D_2 = L_1 D R_1$ where

$$R_1 = \begin{pmatrix} 1 & -3 & -4 & \cdots & -l_{n-3} & -l_{n-2} & -l_{n-1} & -l_n \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Thirdly, let E_2 be a series of elementary row operations on D_2 by substituting the ith row with the ith row multiplied by $1/x$, for $i = 2,3, \dots, (n-1)$. The result is $E_2(D_2) = A \sim D_3$ and

$$D_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -q & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -q & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -q & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & -q & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -q \\ 0 & g_1 & g_2 & g_3 & \cdots & g_{n-3} & g_{n-2} & 1-3l_n \end{pmatrix}$$

where $q = \frac{y}{x}$ or

$$x = \frac{y}{q} \tag{7}$$

and then, there exists $L_2 = E_2(L_1)$ such that $D_3 = L_2 D R_1$ where

$$L_2 = \frac{1}{x}\begin{pmatrix} x & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ -3x & 0 & 0 & 0 & 0 & \cdots & 0 & x \end{pmatrix}.$$

Fourthly, letK_2beaseriesofelementarycolumnoperationsonD_3bysubstituting the(j+1)thcolumnwiththe$j$thcolumnmultipliedby$-q$andaddedto the(j+1)thcolumn,for$j$=2,3,...,(n−1).Then,$K_2(D_3) = A \sim D_4$,

$$D_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & u_1 & u_2 & u_3 & \cdots & u_{n-3} & u_{n-2} & d \end{pmatrix}$$

where

$$u_1 = g_1 = -5 \text{ and } u_j = qu_{j-1} + g_j \text{ for } j = 2, 3, \cdots, (n-2), \tag{8}$$

and it is clear that we also obtain

$$d = qu_{n-2} + 1 - 3l_n \text{ and } \det(A) = x^{n-2}d, \tag{9}$$

andinthefollowing,wewillprovetheformulaofdet(A)byformulating u_{n-2}.FromEquation(8),wehavethesequence$u_1 = g_1, u_2 = qu_1 + g_2, u_3 = q(qg_1 + g_2) + g_3 = g_1q^2 + g_2q + g_3$, $u_4 = q(g_1q^2 + g_2q + g_3) = g_1q^3 + g_2q^2 + g_3q + g_4$, $\cdots$, $u_{n-2} = \sum_{j=1}^{n-2} g_j q^{n-2-j}$, so that $\det(A) = x^{n-2}\left(\sum_{j=1}^{n-2} g_j q^{n-1-j} + 1 - 3l_n\right)$.Then, by applying Equation (7), we obtain that

$$\begin{aligned} \det(A) &= x^{n-2}\left(\sum_{j=1}^{n-2} g_j \left(\frac{y}{x}\right)^{n-1-j} + 1 - 3l_n\right) \\ &= \sum_{j=1}^{n-2} g_j \left(\frac{y}{x}\right)^{n-1-j} x^{j-1} + (1 - 3l_n)x^{n-2} \\ &= \sum_{j=1}^{n-1} g_j \left(\frac{y}{x}\right)^{n-1-j} x^{j-1} - g_{n-1}x^{n-2} + (1 - 3l_n)x^{n-2}, \end{aligned}$$

and we apply Equation (5) and (6) to get

$$\begin{aligned} \det(A) &= (1 + l_{n-2} + l_n - 3l_n)x^{n-2} + \sum_{j=1}^{n-1} g_j y^{n-1-j} x^{j-1} \\ &= x^{n-1} + \sum_{j=1}^{n-1} g_j y^{n-1-j} = x^{n-1} - \sum_{j=1}^{n-1} \left(l_{j-1} + l_{j+1}\right) y^{n-1-j} x^{j-1}. \end{aligned}$$

Finally, by transforming the counter variable $t = n - 1 - j$for $j$from $(n-1)$down to 1, here we reach our formula of $\det(A)$as

$$\begin{aligned}\det(A) &= x^{n-1} - \sum_{t=0}^{n-2}(l_{n-2-t} + l_{n-t})y^t x^{n-2-t} \\ &= x^{n-1} - 5y^{n-2} - \sum_{k=1}^{n-2}(l_{n-k-1} + l_{n-k+1})y^{k-1}x^{n-k-1}.\end{aligned}$$

From all the above steps, we also get $R = K_2(R_1)$ such that $D_4 = L_2AR$ with

$$R = \begin{pmatrix} 1 & v_2 & v_3 & \cdots & v_{n-2} & v_{n-1} & v_n \\ 0 & 1 & q & q^2 & \cdots & q^{n-3} & q^{n-2} \\ 0 & 0 & 1 & q & q^2 & \cdots & q^{n-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & 1 & q & q^2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & q \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

where $v_2 = -3$, and for $j = 3,4,\ldots,n, v_j = qv_{j-1} - l_j$. Fifthly, let E_3 be a series of elementary row operations on D_4 by substituting the nth row with ith row multiplied by $-u_i$ and added to the nth row, for $i = 2,3,\ldots,(n-2)$. The result is $E_3(D_4) = A \sim D$ which is a diagonal matrix on the form $D = \begin{pmatrix} I_n & O \\ O & d \end{pmatrix}$, and then there exists the matrix $L = E_3(L_2)$ such that $D = LAR$ with

$$L = \frac{1}{x}\begin{pmatrix} x & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ s_1 & s_2 & s_3 & s_4 & \cdots & s_{n-2} & s_{n-1} & s_n \end{pmatrix},$$

where

$$(s_1 \quad s_2 \quad \cdots \quad s_{n-1} \quad s_n) = (0 \quad -u_1 \quad -u_2 \quad \cdots \quad u_{n-2} \quad 1)(xL_2). \tag{10}$$

Now, we refer Equations (5)-(10) to obtain the formulations of s_j for $j = 1, 2, \cdots, n$ as follows:

$$\begin{aligned} s_1 &= u_{n-2} - 3x = \frac{qu_{n-2} - 3y}{q} = \frac{qu_{n-2} + 1 - 3l_n + 5}{q} = \frac{d+5}{q} \\ s_2 &= -u_1 = 5, g_j = -(2l_{j-1} + l_j) \text{ so that} \\ s_3 &= u_1 - u_2 = (-5) - (q(-5) - 5) = 5q \\ s_4 &= u_1 + u_2 - u_3 = (-5) + (-5q - 5) - u_3 \\ &= (-5q - 10) - (q(-5q - 5) - 5(2)) = 5q^2 \\ s_5 &= u_2 + u_3 - u_4 = (-5q - 5) + (-5q^2 - 5q - 10) - u_4 \\ &= (-5q^2 - 10q - 15) - (q(-5q^2 - 5q - 10) - 5(3)) = 5q^3 \end{aligned} \tag{11}$$

And so on, follows the pattern to get inductively that

$$s_j = u_{j-3} + u_{j-2} - u_{j-1} = 5q^{j-2}, \qquad \text{for } j = 2, 3, 4, 5, \cdots, n-1. \tag{12}$$

For the formulation of s_n, notice that $s_n = u_{n-3} + u_{n-2} + x$ or

$$\begin{aligned} s_n &= q(u_{n-4} + u_{n-3}) - l_{n-2} - l_n + 1 - l_n - l_{n-1} \\ &= q(u_{n-4} + u_{n-3} - u_{n-2}) + qu_{n-2} + 1 - 3l_n \end{aligned}$$

and we focus on using Equation (9) to obtain that

$$s_n = 5q^{n-2} + d. \tag{13}$$

Sixthly, since $D = LAR$ or $A^{-1} = (RD^{-1})L$, then

$$A^{-1} = \begin{pmatrix} 1 & v_2 & v_3 & \cdots & v_{n-1} & \frac{v_n}{d} \\ 0 & 1 & q & q^2 & \cdots & \frac{q^{n-2}}{d} \\ 0 & 0 & 1 & q & \ddots & \frac{q^{n-3}}{d} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{-q}{d} \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{d} \end{pmatrix} L,$$

and since A^{-1} is also circulant (see Equation (3)), then the last row ofA^{-1} is $\frac{1}{xd}(s_1 \quad s_2 \quad \cdots \quad s_{n-1} \quad s_n)$, and we may denote $A^{-1} = \frac{1}{xd}\mathrm{Circ}(s_n \quad s_1 \quad s_2 \quad \cdots \quad s_{n-2} \quad s_{n-1})$ and again since $d = \frac{\delta}{x^{n-2}}$, then

$$A^{-1} = \frac{x^{n-2}}{\delta}\mathrm{Circ}(s_n \quad s_1 \quad s_2 \quad \cdots \quad s_{n-2} \quad s_{n-1}).$$

For simplification, now we write

$$A^{-1} = \frac{1}{\delta}\mathrm{Circ}(z_1 \quad z_2 \quad z_3 \quad \cdots \quad z_{n-1} \quad z_n),$$

where z_j can be formulated using the formula of s_j and substituting back that $q = \frac{y}{x}$ as follows. We refer to Equations (7) and (11)-(14) to complete this proof that$z_1 = x^{n-3}s_n$ so

$$z_1 = x^{n-3}(5q^{n-2} + d) = x^{n-3}\left(5\left(\frac{y}{x}\right)^{n-2} + \frac{\delta}{x^{n-2}}\right) = \frac{\delta + 5y^{n-2}}{x}$$

$$z_2 = x^{n-3}s_1 = x^{n-3}\left(\frac{d+5}{q}\right) = x^{n-3}\left(\frac{x^{\frac{\delta}{n-2}}}{\frac{y}{x}}\right) = \frac{\delta + 5x^{n-2}}{y}.$$

and for $k = 3, 4, 5, \cdots, n$, we have

$$z_k = x^{n-3}s_{k-1} = x^{n-3}(5q^{k-3}) = 5x^{n-3}\left(\frac{y}{x}\right)^{k-3} = 5x^{n-k}y^{k-3}.$$

4 A THEOREM OF EIGENVALUES FORMULATION

Recallthecyclicgroup $\mathcal{H} = \{1, \alpha, \alpha^2, \cdots, \alpha^{n-1}\}$ from Section 2.All n elements of $\mathcal{H}$ geometrically occupy the unit circle in the complex plane and divide the circleinto*n*equalparts,thenitisveryclearfromthedefinitionof*S*that for $l = 1, 2, \cdots, \left\lfloor\frac{n}{2}\right\rfloor$, we have

$$\alpha^l + \alpha^{n-l} = \alpha^l + \alpha^{-l} = 2\cos(l\theta) \text{ and } \alpha^l - \alpha^{n-l} = \alpha^l - \alpha^{-l} = 2i\sin(l\theta) \qquad (15)$$

where $\theta = \frac{2\pi}{n}$.These equations will be used as an important part in the proof of the following theorem.

Theorem 2. For integer $n \geq 3$, let $A = \mathrm{Circ}(l_1, l_2, \cdots, l_{n-1}, l_n)$ be the matrix defined in Definition 1 and for $j = 0, 1, 2, \cdots, n-1$, let λ_j be eigenvalues of A. If $\theta = \frac{2\pi}{n}$ and $m = \left\lfloor\frac{n-1}{2}\right\rfloor$,then $\lambda_0 = 2l_n + l_{n-1} - 3$, and for $k = 1, 2, \cdots, m$, we have $\lambda_k = R_k + C_k i$ and $\lambda_{n-k} = \overline{\lambda_k}$ where

$$R_k = 1 + \sum_{s=1}^{m}(l_{s+1} + l_{n-s+1})\cos(sk\theta) \text{ and}$$

$$C_k = \sum_{s=1}^{m} (l_{s+1} + l_{n-s+1}) \sin(sk\theta).$$

Forthecaseofniseven,wealsoinclude$\lambda_{m+1} = -(l_{n-1} + 1)andR_k$becomes

$$R_k = 1 + (-1)^k l_{\frac{n}{2}+1} + \sum_{s=1}^{m} (l_{s+1} + l_{n-s+1}) \cos(sk\theta).$$

Proof. In this proof, we consider the fact that $\mathcal{H}$is a cyclic group. Based on Equation (2) in Section 2, in the context of matrix A here we have

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{-2} & \alpha^{-1} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{-4} & \alpha^{-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \alpha^{-2} & \alpha^{-4} & \cdots & \alpha^4 & \alpha^2 \\ 1 & \alpha^{-1} & \alpha^{-2} & \cdots & \alpha^2 & \alpha \end{pmatrix} \begin{pmatrix} l_1 \\ l_1 \\ l_3 \\ \vdots \\ l_{n-1} \\ l_n \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-2} \\ \lambda_{n-1} \end{pmatrix}$$

and so by Proposition 1 it is very simple that

$$\lambda_0 = \sum_{t=1}^{n} l_t = l_{n+2} - 3 = l_n + l_{n+1} - 3 = l_{n-1} + 2l_n - 3$$

and for the specific case of n is even, we also have

$$\lambda_{m+1} = \lambda_{\frac{n}{2}} = \sum_{t=1}^{n} l_t \alpha^{\frac{n}{2}t} = \sum_{t=1}^{n} (-1)^t l_t = -(l_{n-1} + 1)$$

Next,for $k = 1, 2, \cdots, m = \left\lfloor \frac{n-1}{2} \right\rfloor$,considerthat

$$\begin{aligned} \lambda_k + \lambda_{n-k} &= \sum_{t=0}^{n-1} l_{t+1}\left(\alpha^{tk} + \alpha^{t(n-k)}\right) = 2l_1 + \sum_{t=1}^{n-1} l_{t+1}(\alpha^{tk} + \alpha^{-tk}) \\ &= 2 + \sum_{t=1}^{m} l_{t+1}(\alpha^{tk} + \alpha^{-tk}) + \sum_{t=n-m}^{n-1} l_{t+1}(\alpha^{tk} + \alpha^{-tk}) \end{aligned}$$

butforthespecificcaseof*n*iseven,

$$\lambda_k + \lambda_{n-k} = 2 + \sum_{t=1}^{m} l_{t+1}(\alpha^{tk} + \alpha^{-tk}) + \sum_{t=n-m}^{n-1} l_{t+1}(\alpha^{tk} + \alpha^{-tk}) + 2(-1)^k l_{\frac{n}{2}+1}$$

Transforming the counter variable:$s = t$when $t = 1, \ldots, m$and $s = n — t$when $t = n — m, \ldots, n — 1$,we have

$$\begin{aligned} \lambda_k + \lambda_{n-k} &= 2 + \sum_{s=1}^{m} l_{s+1}(\alpha^{sk} + \alpha^{-sk}) + \sum_{s=1}^{m} l_{n-s+1}\left(\alpha^{(n-s)k} + \alpha^{-(n-s)s}\right) \\ &= 2 + \sum_{s=1}^{m} (l_{s+1} + l_{n-s+1}) (\alpha^{sk} + \alpha^{-sk}) \end{aligned}$$

andforthecaseof*n*iseven,

$$\lambda_k + \lambda_{n-k} = 2 + 2(-1)^k l_{\frac{n}{2}+1} + \sum_{s=1}^{m} (l_{s+1} + l_{n-s+1})(\alpha^{sk} + \alpha^{-sk})$$

Then, applying Equation (15),

$$\lambda_k + \lambda_{n-k} = 2\left(1 + \sum_{s=1}^{m} (l_{s+1} + l_{n-s+1}) \cos(sk\theta)\right) \quad (16)$$

and when *n* is even

$$\lambda_k + \lambda_{n-k} = 2\left(1 + (-1)^k l_{\frac{n}{2}+1} + \sum_{s=1}^{m} (l_{s+1} + l_{n-s+1}) \cos(sk\theta)\right) \quad (17)$$

Analogously, consider that

$$\begin{aligned} \lambda_k - \lambda_{n-k} &= \sum_{t=0}^{n-1} l_{t+1}\left(\alpha^{tk} - \alpha^{t(n-k)}\right) = \sum_{t=1}^{n-1} l_{t+1}(\alpha^{tk} - \alpha^{-tk}) \\ &= \sum_{s=1}^{m} (l_{s+1} - l_{n-s+1})(\alpha^{sk} - \alpha^{-sk}) \end{aligned}$$

Then, applying Equation (15),

$$\lambda_k - \lambda_{n-k} = 2i \sum_{s=1}^{m} (l_{s+1} - l_{n-s+1}) \sin(sk\theta) \quad (18)$$

Finally, by adding and subtracting of Equations: (16) with (18), and when *n* is even of Equations: (17) with (18), we have $\lambda_k = R_k + iC_k$ and $\lambda_{n-k} = R_k - iC_k$ where

$$R_k = 1 + \sum_{s=1}^{m} (l_{s+1} + l_{n-s+1}) \cos(sk\theta) \text{ and } C_k = \sum_{s=1}^{m} (l_{s+1} - l_{n-s+1}) \sin(sk\theta)$$

and for the case of *n* is even, R_k becomes

$$R_k = 1 + (-1)^k l_{\frac{n}{2}+1} + \sum_{s=1}^{m} (l_{s+1} + l_{n-s+1}) \cos(sk\theta).$$

5 NUMERICAL EXAMPLES

In this section, we present a simple illustration to figure out how to apply the formulations to compute the determinant dan inverse based on Theorem 1 and the eigenvalues based on Theorem 2. Then, by considering that illustration, we construct the algorithms.

Example 1. For $n = 5$, we have *A*=Circ(1,3,4,7,11). Then,

$$x = 1 - 11 - 7 = -17, \qquad y = 9,$$

and determinant is

$$\begin{aligned}\delta = (-17)^4 - 5(9^3) - (4+11)(9^0)(-17)^3 + (3+7)(9^1)(-17)^2 \\ + (1+4)(9^2)(-17)^1 = 134446.\end{aligned}$$

For the invers, $A^{-1} = \frac{1}{\delta}\mathrm{Circ}(z_1, z_2, z_3, z_5)$ where

$$z_1 = \frac{\delta + 5(9^3)}{-17} = -8123 \qquad z_3 = 5(-17)^2(9^0) = 1445 \qquad z_5 = 5(-17)^0(9^2) = 405$$

$$z_2 = \frac{\delta + 5 \times (-17)^3}{9} = 12209 \qquad z_4 = 5(-17)^1(9^1) = -765,$$

For the eigenvalues, we have $\lambda_0 = 2(11) + 7 - 3 = 26$, moreover we obtain $\lambda_1 = R_1 + C_1 i, \lambda_4 = \overline{\lambda_1}$ and $\lambda_2 = R_2 + C_2 i, \lambda_3 = \overline{\lambda_2}$ where

$$R_1 = 1 + (3+11)\cos\frac{2\pi}{5} + (4+7)\cos\frac{4\pi}{5} \approx -3{,}57$$
$$R_2 = 1 + 14\cos\frac{4\pi}{5} + 11\cos\frac{8\pi}{5} \approx -6{,}93$$
$$C_1 = (3-11)\sin\frac{2\pi}{5} + (4-7)\sin\frac{4\pi}{5} \approx -9{,}37 \quad \text{and}$$
$$C_2 = -8\sin\frac{4\pi}{5} - 3\sin\frac{8\pi}{5} \approx -1{,}85$$

Example 2. For *n*=6, we have *A*=Circ(1,3,4,7,11,18). Then, we have
$$\lambda_0 = 2(18) + 11 - 3 = 44, \lambda_3 = -(11+1) = -12.$$
Moreover, we obtain $\lambda_1 = R_1 + C_1 i, \lambda_5 = \overline{\lambda_1}$ and $\lambda_2 = R_2 + C_2 i, \lambda_4 = \overline{\lambda_2}$ where
$$R_1 = 1 + 7(-1) + (3+18)\cos\frac{2\pi}{6} + (4+11)\cos\frac{4\pi}{6} = -3$$
$$R_2 = 1 + 7(-1)^2 + 11\cos\frac{4\pi}{6} + 15\cos\frac{8\pi}{6} = -10$$
$$C_1 = (3-18)\sin\frac{2\pi}{6} + (4-11)\sin\frac{4\pi}{6} \approx -19{,}05 \text{ and}$$
$$C_2 = -15\sin\frac{4\pi}{6} - 7\sin\frac{8\pi}{6} \approx -6{,}93.$$

From above illustrations, it is easy to see that in the iterative process of computing the determinant, some data can be stored for the next process of computing the inverse. So, the computation process can be done in one function and in a parallel way to get very fast and efficient performance.

Algorithm 1.
INPUT: *L*=Circ$(l_1, l_2, \cdots, l_{n-1}, l_n)$ with the entries of Lucas sequences.
OUTPUT: $\delta = \det(L)$ and $L^{-1} = \frac{1}{\delta}\mathrm{Circ}(z_1, z_2, \cdots, z_{n-1}, z_n)$.

1. $x \leftarrow 1 - l_n - l_{n-1}; y \leftarrow l_{n-1} - 2;$

2. $r \leftarrow x^{n-2}; \delta \leftarrow xr;\ z_2 \leftarrow 5r; s \leftarrow 1;$

3. **For** $k = 1$ **to** $n-2$ **do**

$$t \leftarrow (l_{n-k-1} + l_{n-k+1})rs;\ \delta \leftarrow \delta - t;$$

$$r \leftarrow \frac{r}{x}; z_{k+2} \leftarrow 5rs; s \leftarrow sy;$$

End do;

4. $u \leftarrow 5s;\ \delta \leftarrow \delta - u;\ z_1 \leftarrow \frac{\delta + s}{x};\ z_2 \leftarrow \frac{\delta + z_2}{y};$

5. **Return**$(\delta, L^{-1});$

For the eigenvalues, it is also very easy to see that only $\left\lfloor\frac{n-1}{2}\right\rfloor$ eigenvalues are computed iteratively and all without any complex number arithmetic used. So, it must be much faster than applying the general formula as mentioned in Section 2.

Algorithm 2.
INPUT: *A*=Circ$(l_1, l_2, \cdots, l_{n-1}, l_n)$ with the entries of Lucas sequences.

OUTPUT: $\lambda_0, \lambda_1, \cdots, \lambda_{n-2}, \lambda_{n-1}$; the eigenvalues of A.

1. **If**$(n \, mod \, 2) = 0$**then**$\lambda_{\frac{n}{2}} \leftarrow ((-1)^{n-1} f_{n-1} + 1)$**endif***;*

2. $\lambda_0 \leftarrow (1 - f_n - f_{n-1}); m \leftarrow \left\lfloor \frac{n-1}{2} \right\rfloor; \; \theta \leftarrow \frac{2\pi}{n};$

3. **for** $k = 1$**to***m***do**

$$R \leftarrow 1; C \leftarrow 0; S \leftarrow 0; A \leftarrow k\theta;$$

$$\mathbf{for}\; s = 1 \;\mathbf{to}\; m \;\mathbf{do}$$

$$S \leftarrow S + A; x \leftarrow (f_{s+1} + f_{n-s+1}) \cos S; \; R \leftarrow R + x;$$

$$y \leftarrow (f_{s+1} - f_{n-s+1}) \sin S; C \leftarrow C + y;$$

End do*;*

If$(n \, mod \, 2) = 0$**then**$R \leftarrow \left(R + (-1)^k a_{\frac{n}{2}+1}\right)$**endif***;*

$$\lambda_k \leftarrow R + Ci; \; \lambda_{n-k} \leftarrow R - Ci;$$

End do*;*

4. **return**$(\lambda_0, \lambda_1, \cdots, \lambda_{n-2}, \lambda_{n-1})$.

We have already implemented Algorithm 1 and Algorithm 2 by running in MAPLE codes. All the experiments were done in the same computation environment and computer specifications. We set the values of *n* are large enough, then we compared the performanceAlgorithm 1 and Algorithm 2 to the related algorithm built in the MAPLE Library via LinearAlgebra package. Below we describe some facts as the results of our observations from the experimental aspect.

1. Based on our experiments, the implementation of Algorithm 1 is able to compute the determinant and the inverse at the same time (in parallel) on the circulant matrices with Lucas number for $n = 1000$ on average took only 2.90 seconds by 10 trials. Meanwhile, experiments on the MAPLE Library, we set for only relatively much smaller the value of $n = 100$, calculating the inverse on average took 57.67 seconds by 10 trials.
2. Algorithm 2 is able to compute the eigenvalues on the circulant matrices with Lucas number for $n = 500$ on avagare took only 3.74 seconds by 10 trials. Meanwhile, experiments on the MAPLE Library, we set only for relatively smaller the value of $n = 150$, calculating the eigenvalues on average took only 17.35 seconds by 10 trials.

6 CONCLUSIONS

The formulation for the determinant and inverse of the matrices involving Lucas sequence can be presented in one theorem and in a simple way, so an efficient algorithm can be constructed for its perspective computation. The method of deriving the formulas is simply using elementary row or column operations. For the eigenvalues, the previous formulation from the case of general circulant matrices can be simplified by considering the specialty of the Lucas numbers and using cyclic group properties, so the computation can be done efficiently without involving any complex number arithmetic, i.e. all complex number eigenvalues are constructed. From the implementation facts, algorithms derived from those formulations show much faster than the related algorithm built in the MAPLE Library via LinearAlgebra package.

The methods in this article should be applicable for any variant of circulant matrices (such as skew or more general r-circulant) with any specific formation of numbers (such as Fibonacci, Lucas, Pell, etc.). These would become the nearly future works.

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