Original Research Article

Mathematical Formulation of a co-infection Model and its Analyses for Monkeypox and HIV/AIDS Infections.

Abstract.

The dynamics of the co-infection of monkeypox and HIV/AIDS is examined from a mathematical perspective via a deterministic 13-compartment model. This consists of the investigation of the equilibrium points, the basic reproduction numbersand the equilibrium points'stability.The mathematical analysis reveals that the model is epidemiologically well-posed, and that the basic reproduction number for the monkeypox sub-model is a function of the likelihood of getting infected, the rate of effective contact, the infection coefficient of the monkeypox-infectious class, the monkeypox prevention measure, the progression rate from monkeypox-exposed class to monkeypox-infectious class, the natural death rate, the vaccination rate and the waning rate of the vaccine. It also depends on the recovery rate for the monkeypox-exposed class and the monkeypox-induced death rate. The analysis also reveals that the basic reproduction number for the HIV/AIDS sub-model is a function of the likelihood of getting infected, the rate of effective contact, the HIV/AIDS prevention measure, the infection coefficient of the monkeypox-infectious class, the HIV/AIDS-induced death rate, the natural death rate, the infection coefficient of AIDS-only class, and the progression rate of HIV-only class to AIDS-only. The stability analysis reveals that the disease-free equilibrium of the sub-models are globally asymptomatically stable, when the basic reproduction number is less than unity. Numerical simulations of the model reveals the effect of changes in certain parameter values on the population sizes. Increasing or lowering the values of certain parameters can significantly affect the sizes of some classes. The Maple 18 programming software was used to carry out all calculations and numerical simulations.

Keywords: Modeling, Stability, Monkeypox, HIV-AIDS.

1 Introduction.

Monkeypox virus was described in the 1960sas initiating a pox-like infection in monkeys. Common symptoms are skin eruptions thatgrow into papules on the soles, palms, face andtrunk. The human immunodeficiency virus (HIV) attacks the immune cells - CD4 cells. These white blood cells assist in spottinganomalies and infections in the other cells. HIV escalates the danger and the effect of other diseases. When there is no proper treatment, HIV progresses to AIDS (Acquired immunodeficiency syndrome). Monkeypox infection mortality may be higher in HIV-infected humans whose immune systems have been compromised. HIV/AIDS infection is high in Nigeria. More so, the monkeypox outbreak in Nigeria has necessitated thestudyofa possible co-infection of the two infections.

Getachew (2017) reveals that models are proposed to study the behaviours of some elements on disease transmissions and to provide plans on how to control the diseases. Ossaiugbo and Okposo (2021), and Tsetimi, Ossaiugbo and Atonuje (2022) proposed mathematical models tostudythe dynamics of Pneumonia and COVID-19 respectively. Among other researchers, Bhunu, Mushayabasa and Mac Hyman (2012), Usman and Adamu (2017), Somma, Akinwande, and Chado (2019) and Ayele, Goufo, and Mugisha (2021), have done some works on monkeypox and HIV/AIDS.

The Ayele, et. al. (2021) model considered the rate at which individuals become aware or unaware, as a function of media campaign.They carried out mathematical analyses such as calculation of the reproduction number, finding the equilibrium points, and stability pattern. Control measures such as treatment, screeningwere also introduced into the model. Usman and Adamu (2017) proposed a model for monkeypox infection and computed$R_0$for the humans and the non-humans to be $R_{0,h} = 9.1304 \times 10^{-6}$ and $R_{0,n} = 3.375 \times 10^{-3}$ respectively. Numerical simulations were done and it was shown that the individuals which are infectious will die during the intervention process. They further showed that with an increase in treatment and vaccination, R_0 decreases.

The co-infection of HIV/AIDS and Monkeypox is not a desirable condition. This work developed and mathematically analyzed a deterministic model of 13 classeswithordinary differential equations for HIV/AIDS and Monkeypox. The results shall help in the management and possible eradication of HIV/AIDS and Monkeypox co-infection.

2. Model Description and Formulation

The model divides the populationofanimalsinto four classes: susceptible (S_n), exposed (E_n), infectious (I_n) and recovered (R_n). The recruitment rateisΛ_n. The force of infection isλ_n. Exposed animals become monkeypox-infectious at rate υ_n. The exposed recover at rate ϱ_n, whilethe infectious recover at rate ρ_n. The natural death rate is μ_n while the monkeypox-induced death rateisd_n. The model shares the human population into nine compartments: susceptible (S), monkeypox-vaccinated (V_m), monkeypox-exposed (E_m), monkeypox-infectious (I_m), monkeypox-recovered (R_m), HIV-only(I_h), HIV/AIDS-infectious (A_h), HIV-only and Monkeypox co-infectious(I_{hm}), and HIV/AIDS and Monkeypox co-infectious(A_{hm}).

We assumed that monkeypox recovery does not confer permanent immunity; the waning rate of the vaccine can result in humans becoming susceptible again; animals that are exposed to the monkeypox infection recovers from the infection but at a rate different from that of the infectious animals; humans that are exposed to monkeypox also recover from the infection and the rate is different from that of the infectious humans; there is no body fluid exchange between animals and humans; different monkeypox-recovery rates exist for humans in the compartments$A_{hm}$and I_{hm}; and there is a possibility of vertical transmission for HIV infection.

The human birth rate is Λ. ε-fraction is born with HIV, hence they enter into the class I_h while$(1 - \varepsilon)$- fraction is recruited into the class S. The monkeypox vaccination rate is α_m. The waningrate of the vaccine is ω_m. Monkeypox exposure rate for humans is λ_m and the recovery rate for these exposed individuals is ϱ_m.The HIV force of infection is λ_h. The rate at which humans who are exposed to monkeypox become infectious isυ_m. The recovery rate for these infectious individuals is ρ_m. Monkeypox-recovered individuals become susceptible again at rate ζ. Humans who are exposed to monkeypox and those who are infectious of monkeypox become infected with HIV at rate λ_h, thus they join class I_{hm}.Individuals in class $R_m$also get infected with HIV at the rateλ_h, thus move to class I_h.Individuals in class $I_h$move to class A_h at the rate ρ_1. Individuals in the classes A_h and I_h become exposed to monkeypox at rates $\sigma_2\lambda_m$ and $\sigma_1\lambda_m$ respectively. $\sigma_1$andσ_2 account for increased vulnerability to monkeypox due to the HIV/AIDS infection. Individuals in class I_{hm}move to classA_{hm} at the rate ρ_2. The monkeypox-recovery rates for the classesA_{hm} andI_{hm}areτ_2 and

$\tau_1$respectively.These individual move back to class $A_h$and $I_h$respectively. The death rate due to HIV/AIDSisd_h. The death rate due to monkeypox is d_m. The natural death rate is μ.

$$\begin{cases}
\dfrac{dS_n}{dt} = \Lambda_n - (\mu_n + \lambda_n)S_n, \\
\dfrac{dE_n}{dt} = \lambda_n S_n - (\mu_n + \varrho_n + v_n)E_n, \\
\dfrac{dI_n}{dt} = v_n E_n - (\mu_n + d_n + \rho_n)I_n, \\
\dfrac{dR_n}{dt} = \boldsymbol{\varrho_n} E_n + \rho_n I_n - \mu_n R_n, \\
\dfrac{dS}{dt} = (1-\varepsilon)\Lambda + \boldsymbol{\omega_m} V_m + \boldsymbol{\zeta} R_m - (\mu + \lambda_m + \lambda_h + \alpha_m)S, \\
\dfrac{dV_m}{dt} = \alpha_m S - (\mu + \omega_m)V_m, \\
\dfrac{dE_m}{dt} = \lambda_m S - (\mu + v_m + \boldsymbol{\varrho_m} + \lambda_h)E_m, \\
\dfrac{dI_m}{dt} = v_m E_m - (\mu + d_m + \rho_m + \lambda_h)I_m, \\
\dfrac{dR_m}{dt} = \varrho_m E_m + \rho_m I_m - (\mu + \zeta + \lambda_h)R_m, \\
\dfrac{dI_h}{dt} = \boldsymbol{\varepsilon}\Lambda + \lambda_h(S + R_m) + \boldsymbol{\tau_1} I_{hm} - (\mu + \rho_1 + \sigma_1\lambda_m)I_h, \\
\dfrac{dA_h}{dt} = \rho_1 I_h + \boldsymbol{\tau_2} A_{hm} - (\mu + d_h + \sigma_2\lambda_m)A_h, \\
\dfrac{dI_{hm}}{dt} = \sigma_1\lambda_m I_h + \lambda_h(E_m + I_m) - (\mu + d_m + \rho_2 + \tau_1)I_{hm}, \\
\dfrac{dA_{hm}}{dt} = \rho_2 I_{hm} + \sigma_2\lambda_m A_h - (\mu + d_m + d_h + \tau_2)A_{hm}.
\end{cases} \tag{1}$$

Initial conditions:

$$S_n(0) \geq 0, E_n(0) \geq 0, I_n(0) \geq 0, R_n(0) \geq 0, S(0) \geq 0, V_m(0) \geq 0, E_m(0) \geq 0, I_m(0) \geq 0, R_m(0) \geq 0, I_h(0) \geq 0, A_h(0) \geq 0, I_{hm}(0) \geq 0, A_{hm}(0) \geq 0.$$

where:

$$\lambda_n = \beta_n c_n \frac{I_n(t)}{S_n(t) + E_n(t) + I_n(t) + R_n(t)},$$

$$\lambda_m = (1-\delta_m)\left(\beta_n c_n \frac{I_n}{N_n} + \beta_m c_m \frac{\left(A_{hm}(t) + \theta_1 I_m(t) + \theta_2 I_{hm}(t)\right)}{S(t) + V_m(t) + E_m(t) + I_m(t) + R_m(t)}\right), \theta_1 < \theta_2,$$

$$\lambda_h = (1-\delta_h)\left(\beta_h c_h \frac{\left(A_{hm}(t) + \phi_1 I_h(t) + \phi_2 I_{hm}(t) + \phi_3 A_h(t)\right)}{S(t) + I_h(t) + A_h(t) + I_{hm}(t) + A_{hm}(t)}\right), \phi_1 < \phi_2 < \phi_3,$$

$$N_n(t) = S_n(t) + E_n(t) + I_n(t) + R_n(t).$$

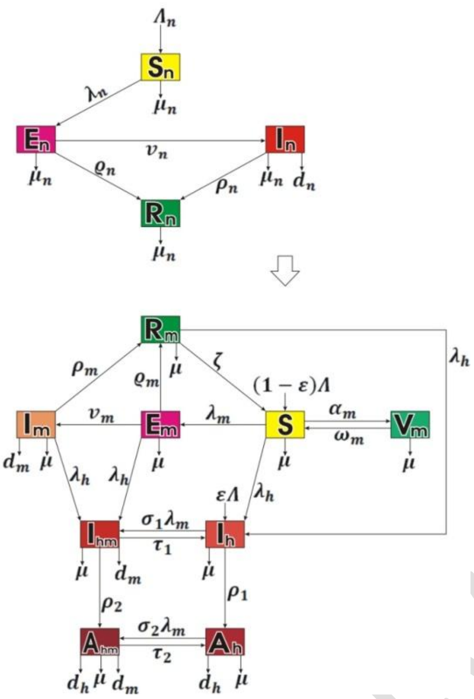

Fig. 1: Schematic diagram.

Table 1: Variables and Parameters Descriptions

Parameter	Description
Λ_n	Recruitment rate into the class S_n.
Λ	Recruitment rate into the susceptible class S.
μ_n	Natural death rate among animals.
μ	Natural death rate among humans.
ϱ_n	Recovery rate for the class E_n.
ϱ_m	Recovery rate for the class E_m.
ρ_n	Recovery rate for the class I_n.
ρ_m	Recovery rate for the class I_m.
τ_1	Monkeypox-recovery rate for the class I_{hm}.
τ_2	Monkeypox-recovery rate for the class A_{hm}.
d_n	Monkeypox-induced death rate among animals.
d_m	Monkeypox-induced death rate among humans.
d_h	HIV/AIDS-induced death rate.
ε	The fraction born infected with HIV virus.
ω_m	Monkeypox vaccine's waning rate.
ζ	Rate at which humans who recover from monkeypox infection become susceptible again.
α_m	Monkeypox vaccination rate.
v_n	Progression rate from class E_n to class I_n.
v_m	Progression rate from class E_m to class I_m.
ρ_1	Progression rate from class I_h to class A_h.
ρ_2	Progression rate from class I_{hm} to class A_{hm}.
σ_1	Parameter accounting for increased monkeypox susceptibility because of an underlying HIV infection.
σ_2	Parameter accounting for increased monkeypox susceptibility because of underlying AIDS infection.
c_n	Animals' effective rate of contact for getting monkeypox

Parameter	Description
c_m	Humans' effective rate of contact for getting monkeypox
c_h	Humans' effective rate of contact for getting HIV infection
δ_m	Monkeypox prevention measure. $0 \leq \delta_m \leq 1$.
δ_h	HIV/AIDS prevention measure. $0 \leq \delta_h \leq 1$.
β_n	Animal's probability of catching monkeypox.
β_m	Human's probability of catching monkeypox.
β_h	Probability of getting infected with HIV per sexual contact with a partner that is infected.
θ_1	The infection coefficient of the class I_m.
θ_2	The infection coefficient of the class I_{hm}.
ϕ_1	The HIV infection coefficient of the class I_h.
ϕ_2	The HIV infection coefficient of the class I_{hm}.
ϕ_3	The HIV infection coefficient of the class A_h.

3. Basic Analysis of the Model

The following mathematical analyses are carried out to ascertain the biological relevance of the model.

3.1 Non-negativity of Solutions

Theorem 1:

Suppose $\Gamma = \{(S_n, E_n, I_n, R_n, S, V_m, E_m, I_m, R_m, I_h, A_h, I_{hm}, A_{hm}) \in \mathbb{R}_+^{13} : S_n(0) > 0, E_n(0) > 0, I_n(0) > 0, R_n(0) > 0, S(0) > 0, V_m(0) > 0, E_m(0) > 0, I_m(0) > 0, R_m(0) > 0, I_h(0) > 0, A_h(0) > 0, I_{hm}(0) > 0, A_{hm}(0) > 0\}$, then $\{S_n, E_n, I_n, R_n, S, V_m, E_m, I_m, R_m, I_h, A_h, I_{hm}, A_{hm}\}$ is non-negative $\forall\ t \geq 0$.

Proof:

$$\frac{dS_n}{dt} = \Lambda_n - (\mu_n + \lambda_n)S_n.$$

$$\frac{dS_n}{dt} \geq -(\mu_n + \lambda_n)S_n,$$

$$\int \frac{1}{S_n} dS_n \geq -\int (\mu_n + \lambda_n)dt,$$

$$S_n \geq e^{-\int(\mu_n + \lambda_n)dt}.$$

$\therefore \quad S_n > 0 \quad \forall\ t \geq 0.$

Also,

$$\frac{dS}{dt} = (1 - \varepsilon)\Lambda + \omega_m V_m + \zeta R_m - (\mu + \lambda_m + \lambda_h + \alpha_m)S.$$

$$\frac{dS}{dt} \geq -(\mu + \lambda_m + \lambda_h + \alpha_m)S,$$

$$\int \frac{1}{S} dS \geq -\int (\mu + \lambda_m + \lambda_h + \alpha_m)dt,$$

$$S \geq e^{-\int(\mu + \lambda_m + \lambda_h + \alpha_m)dt}.$$

$\therefore \quad S > 0 \quad \forall\ t \geq 0.$

Similarly, $E_n(t) > 0, I_n(t) > 0, R_n(t) > 0, V_m(t) > 0, E_m(0t) > 0, I_m(t) > 0, R_m(t) > 0, I_h(t) > 0, A_h(t) > 0, I_{hm}(t) > 0, A_{hm}(t) > 0$ at all time $t \geq 0$ ■

3.2. Invariant Region and Boundedness

Theorem 2:

The sets $\Gamma_1 = \left\{(S_n, E_n, I_n, R_n) \in \mathbb{R}_+^4 : 0 \le S_n + E_n + I_n + R_n = N_n \le \frac{\Lambda_n}{\mu_n}\right\}$ and $\Gamma_2 = \left\{(S, V_m, E_m, I_m, R_m, I_h, A_h, I_{hm}, A_{hm}) \in \mathbb{R}_+^9 : 0 \le S + V_m + E_m + I_m + R_m + I_h + A_h + I_{hm} + A_{hm} = N \le \frac{\Lambda}{\mu}\right\}$are positively invariant.

Proof:

$$N_n(t) = S_n(t) + E_n(t) + I_n(t) + R_n(t), (2)$$

$$N(t) = S(t) + V_m(t) + E_m(t) + I_m(t) + R_m(t) + I_h(t) + A_h(t) + I_{hm}(t) + A_{hm}(t). \quad (3)$$

$$\frac{dN_n}{dt} = \Lambda_n - \mu_n N_n - d_n I_n \le \Lambda_n - \mu_n N_n.$$

$$\therefore \quad \frac{dN_n}{dt} + \mu_n N_n \le \Lambda_n.$$

$$\therefore \quad N_n(t) \le \frac{\Lambda_n}{\mu_n} + ke^{-\mu_n t}. \quad (4)$$

$$\frac{dN}{dt} = \Lambda - \mu N - d_m(I_m + I_{hm}) - d_h(A_h + A_{hm}) \le \Lambda - \mu N.$$

$$\therefore \quad \frac{dN}{dt} + \mu N \le \Lambda.$$

$$\therefore \quad N(t) \le \frac{\Lambda}{\mu} + ke^{-\mu t}. \quad (5)$$

The inequalities (4) and (5) are the threshold population levels for the animal population and the human respectively. They show that when the population is greater than the threshold level, the total population reduces asymptotically to the carrying capacity, and if the population is less than the threshold population level, the solution remains in the invariant-region at all time $t > 0$. Thus, the regions Γ_1 and Γ_2 are positively invariant ■

3.3. Equilibrium Points

The disease-free equilibrium (DFE) and the disease-endemic equilibrium (DEE) are obtained by equating the derivatives in system (1) to zero and solving the resulting system.

I. Monkeypox sub-model's DFE

The system is:

$$\left.\begin{aligned}
&\Lambda_n - (\mu_n + \lambda_n)S_n = 0,\\
&\lambda_n S_n - (\mu_n + \varrho_n + v_n)E_n = 0,\\
&v_n E_n - (\mu_n + d_n + \rho_n)I_n = 0,\\
&\varrho_n E_n + \rho_n I_n - \mu_n R_n = 0,\\
&\Lambda + \omega_m V_m + \zeta R_m - (\mu + \lambda_m + \alpha_m)S = 0,\\
&\alpha_m S - (\mu + \omega_m)V_m = 0,\\
&\lambda_m S - (\mu + v_m + \varrho_m)E_m = 0,\\
&v_m E_m - (\mu + d_m + \rho_m)I_m = 0,\\
&\varrho_m E_m + \rho_m I_m - (\mu + \zeta)R_m = 0,
\end{aligned}\right\} \quad (6)$$

$$\lambda_n = \beta_n c_n \frac{I_n}{S_n(t) + E_n(t) + I_n(t) + R_n(t)},$$

$$\lambda_m = (1 - \delta_m)\left(\beta_n c_n \frac{I_n}{N_n} + \beta_m c_m \frac{\theta_1 I_m}{S(t) + V_m(t) + E_m(t) + I_m(t) + R_m(t)}\right).$$

We set $E_n = I_n = E_m = I_m = 0$, solved the system and obtained the DFE of the monkeypox sub-model:

$$\mathbb{E}_{0_m} = \left(\frac{\Lambda_n}{\mu_n}, 0,0,0, \frac{\Lambda(\mu+\omega_m)}{\mu(\mu+\alpha_m+\omega_m)}, \frac{\Lambda\alpha_m}{\mu(\mu+\alpha_m+\omega_m)}, 0,0,0\right). \quad (7)$$

II. HIV/AIDS sub-model's DFE

The system is:

$$\left.\begin{aligned} &(1-\varepsilon)\Lambda - (\mu+\lambda_h)S = 0,\\ &\boldsymbol{\varepsilon}\Lambda + \lambda_h S - (\mu+\rho_1)I_h = 0,\\ &\rho_1 I_h - (\mu+d_h)A_h = 0. \end{aligned}\right\} (8)$$

$$\lambda_h = (1-\delta_h)\left(\beta_h c_h \frac{(\phi_1 I_h + \phi_3 A_h)}{S(t)+I_h(t)+A_h(t)}\right).$$

We solved the system and obtained the DFE:

$$\mathbb{E}_{0_h} = \left(\frac{(1-\varepsilon)\Lambda}{\mu}, 0,0\right) \quad (9)$$

III. Monkeypox sub-model's DEE

Using the maple 19 programming language, we solved the system (6) above and obtained:

$$\begin{aligned} S_n &= \frac{\Lambda_n}{\mu_n+\lambda_n}, E_n = \frac{\Lambda_n\lambda_n}{(\mu_n+\lambda_n)(\mu_n+\varrho_n+v_n)}, I_n\\ &= \frac{\lambda_n\Lambda_n v_n}{(\mu_n+\lambda_n)(\mu_n+\varrho_n+v_n)(\mu_n+d_n+\rho_n)},\\ R_n &= \frac{\lambda_n(\mu_n\varrho_n+(d_n+\rho_n)\varrho_n+v_n\rho_n)\Lambda_n}{(\mu_n+\lambda_n)(\mu_n+\varrho_n+v_n)\mu_n(\mu_n+d_n+\rho_n)},\\ S &= -((\mu+\omega_m)(\epsilon-1)(\mu+\zeta)\Lambda(\mu+d_m+\rho_m)(\mu+v_m+\varrho_m))/(\mu^5+(\lambda_m+\varrho_m\\ &\quad+\zeta+d_m+v_m+\rho_m+\alpha_m+\omega_m)\mu^4+((\lambda_m+\varrho_m+d_m+v_m+\rho_m\\ &\quad+\alpha_m+\omega_m)(\zeta)+(\lambda_m+\varrho_m+d_m+v_m+\rho_m)\omega_m+(\varrho_m+d_m\\ &\quad+v_m+\rho_m)\lambda_m+(\varrho_m+v_m+\alpha_m)d_m+(\rho_m+\alpha_m)v_m+(\varrho_m\\ &\quad+\alpha_m)\rho_m+\varrho_m\alpha_m)\mu^3+(((\lambda_m+\varrho_m+d_m+v_m+\rho_m)\omega_m+(d_m\\ &\quad+v_m+\rho_m)\lambda_m+(\varrho_m+v_m+\alpha_m)d_m+(\rho_m+\alpha_m)v_m+(\varrho_m\\ &\quad+\alpha_m)\rho_m+\varrho_m\alpha_m)(\zeta)+((\varrho_m+d_m+v_m+\rho_m)\lambda_m+(d_m\\ &\quad+\rho_m)(\varrho_m+v_m))\omega_m+(d_m+\rho_m)(\varrho_m+v_m)(\lambda_m+\alpha_m))\mu^2\\ &\quad+((((d_m+v_m+\rho_m)\lambda_m+(d_m+\rho_m)(\varrho_m+v_m))\omega_m+\lambda_m d_m v_m\\ &\quad+\alpha_m(d_m+\rho_m)(\varrho_m+v_m))(\zeta)+\lambda_m\omega_m(d_m+\rho_m)(\varrho_m+v_m))\mu\\ &\quad+(\zeta)d_m v_m\lambda_m\omega_m),\\ V_m &= -((\epsilon-1)(\mu+\zeta)\Lambda\alpha_m(\mu+d_m+\rho_m)(\mu+v_m+\varrho_m))/(\mu^5+(\lambda_m+\varrho_m+\zeta\\ &\quad+d_m+v_m+\rho_m+\alpha_m+\omega_m)\mu^4+((\lambda_m+\varrho_m+d_m+v_m+\rho_m\\ &\quad+\alpha_m+\omega_m)(\zeta)+(\lambda_m+\varrho_m+d_m+v_m+\rho_m)\omega_m+(\varrho_m+d_m\\ &\quad+v_m+\rho_m)\lambda_m+(\varrho_m+v_m+\alpha_m)d_m+(\rho_m+\alpha_m)v_m+(\varrho_m\\ &\quad+\alpha_m)\rho_m+\varrho_m\alpha_m)\mu^3+(((\lambda_m+\varrho_m+d_m+v_m+\rho_m)\omega_m+(d_m\\ &\quad+v_m+\rho_m)\lambda_m+(\varrho_m+v_m+\alpha_m)d_m+(\rho_m+\alpha_m)v_m+(\varrho_m\\ &\quad+\alpha_m)\rho_m+\varrho_m\alpha_m)(\zeta)+((\varrho_m+d_m+v_m+\rho_m)\lambda_m+(d_m\\ &\quad+\rho_m)(\varrho_m+v_m))\omega_m+(d_m+\rho_m)(\varrho_m+v_m)(\lambda_m+\alpha_m))\mu^2\\ &\quad+((((d_m+v_m+\rho_m)\lambda_m+(d_m+\rho_m)(\varrho_m+v_m))\omega_m+\lambda_m d_m v_m\\ &\quad+\alpha_m(d_m+\rho_m)(\varrho_m+v_m))(\zeta)+\lambda_m\omega_m(d_m+\rho_m)(\varrho_m+v_m))\mu\\ &\quad+(\zeta)d_m v_m\lambda_m\omega_m), \end{aligned}$$

$$
\begin{aligned}
E_m = {} & -((\mu + d_m + \rho_m)\Lambda\lambda_m(\epsilon - 1)(\mu + \omega_m)(\mu + \zeta))/(\mu^5 + (\lambda_m + \varrho_m + \zeta + d_m \\
& + v_m + \rho_m + \alpha_m + \omega_m)\mu^4 + ((\lambda_m + \varrho_m + d_m + v_m + \rho_m + \alpha_m \\
& + \omega_m)(\zeta) + (\lambda_m + \varrho_m + d_m + v_m + \rho_m)\omega_m + (\varrho_m + d_m + v_m \\
& + \rho_m)\lambda_m + (\varrho_m + v_m + \alpha_m)d_m + (\rho_m + \alpha_m)v_m + (\varrho_m + \alpha_m)\rho_m \\
& + \varrho_m\alpha_m)\mu^3 + (((\lambda_m + \varrho_m + d_m + v_m + \rho_m)\omega_m + (d_m + v_m \\
& + \rho_m)\lambda_m + (\varrho_m + v_m + \alpha_m)d_m + (\rho_m + \alpha_m)v_m + (\varrho_m + \alpha_m)\rho_m \\
& + \varrho_m\alpha_m)(\zeta) + ((\varrho_m + d_m + v_m + \rho_m)\lambda_m + (d_m + \rho_m)(\varrho_m \\
& + v_m))\omega_m + (d_m + \rho_m)(\varrho_m + v_m)(\lambda_m + \alpha_m))\mu^2 + ((((d_m + v_m \\
& + \rho_m)\lambda_m + (d_m + \rho_m)(\varrho_m + v_m))\omega_m + \lambda_m d_m v_m + \alpha_m(d_m \\
& + \rho_m)(\varrho_m + v_m))(\zeta) + \lambda_m\omega_m(d_m + \rho_m)(\varrho_m + v_m))\mu \\
& + (\zeta)d_m v_m \lambda_m \omega_m), \\
I_m = {} & -(\Lambda v_m \lambda_m(\epsilon - 1)(\mu + \omega_m)(\mu + \zeta))/(\mu^5 + (\lambda_m + \varrho_m + \zeta + d_m + v_m + \rho_m \\
& + \alpha_m + \omega_m)\mu^4 + ((\lambda_m + \varrho_m + d_m + v_m + \rho_m + \alpha_m + \omega_m)(\zeta) \\
& + (\lambda_m + \varrho_m + d_m + v_m + \rho_m)\omega_m + (\varrho_m + d_m + v_m + \rho_m)\lambda_m \\
& + (d_m + \rho_m + \alpha_m)v_m + (\varrho_m + \alpha_m)d_m + (\varrho_m + \alpha_m)\rho_m + \varrho_m\alpha_m)\mu^3 \\
& + (((\lambda_m + \varrho_m + d_m + v_m + \rho_m)\omega_m + (d_m + v_m + \rho_m)\lambda_m + (d_m \\
& + \rho_m + \alpha_m)v_m + (\varrho_m + \alpha_m)d_m + (\varrho_m + \alpha_m)\rho_m + \varrho_m\alpha_m)(\zeta) \\
& + ((\varrho_m + d_m + v_m + \rho_m)\lambda_m + (d_m + \rho_m)(\varrho_m + v_m))\omega_m + (d_m \\
& + \rho_m)(\varrho_m + v_m)(\lambda_m + \alpha_m))\mu^2 + ((((d_m + v_m + \rho_m)\lambda_m + (d_m \\
& + \rho_m)(\varrho_m + v_m))\omega_m + \lambda_m d_m v_m + \alpha_m(d_m + \rho_m)(\varrho_m + v_m))(\zeta) \\
& + \lambda_m\omega_m(d_m + \rho_m)(\varrho_m + v_m))\mu + (\zeta)d_m v_m \lambda_m \omega_m), \\
R_m = {} & -((\epsilon - 1)(\mu\varrho_m + d_m\varrho_m + \rho_m(\varrho_m + v_m))(\mu + \omega_m)\lambda_m\Lambda)/(\mu^5 + (\lambda_m + \varrho_m \\
& + \zeta + d_m + v_m + \rho_m + \alpha_m + \omega_m)\mu^4 + ((\lambda_m + \varrho_m + d_m + v_m + \rho_m \\
& + \alpha_m + \omega_m)(\zeta) + (\lambda_m + \varrho_m + d_m + v_m + \rho_m)\omega_m + (\varrho_m + d_m \\
& + v_m + \rho_m)\lambda_m + (\varrho_m + v_m + \alpha_m)d_m + (\rho_m + \alpha_m)v_m + (\varrho_m \\
& + \alpha_m)\rho_m + \varrho_m\alpha_m)\mu^3 + (((\lambda_m + \varrho_m + d_m + v_m + \rho_m)\omega_m + (d_m \\
& + v_m + \rho_m)\lambda_m + (\varrho_m + v_m + \alpha_m)d_m + (\rho_m + \alpha_m)v_m + (\varrho_m \\
& + \alpha_m)\rho_m + \varrho_m\alpha_m)(\zeta) + ((\varrho_m + d_m + v_m + \rho_m)\lambda_m + (d_m \\
& + \rho_m)(\varrho_m + v_m))\omega_m + (d_m + \rho_m)(\varrho_m + v_m)(\lambda_m + \alpha_m))\mu^2 \\
& + ((((d_m + v_m + \rho_m)\lambda_m + (d_m + \rho_m)(\varrho_m + v_m))\omega_m + \lambda_m d_m v_m \\
& + \alpha_m(d_m + \rho_m)(\varrho_m + v_m))(\zeta) + \lambda_m\omega_m(d_m + \rho_m)(\varrho_m + v_m))\mu \\
& + (\zeta)d_m v_m \lambda_m \omega_m).
\end{aligned}
$$

IV HIV/AIDS sub-model's DEE

Using the maple 18 programing language, we solved the system (8) above and obtained

$$S = \frac{(1-\varepsilon)\Lambda}{\mu + \lambda_h}, \qquad I_h = \frac{(\varepsilon\mu + \lambda_h)\Lambda}{(\mu + \lambda_h)(\mu + \rho_1)}, \qquad A_h = \frac{\Lambda\rho_1(\epsilon\mu + \lambda_h)}{(\mu + \lambda_h)(\mu + \rho_1)(\mu + d_h)}.$$

3.4. Basic Reproduction Numbers

The basic reproduction number is the average number of infectious cases generated by a single infectious individual in an entirely susceptible population.

We shall employ the approach used by Ossaiugbo and Okposo (2021). That is the next generation matrix approach. We use the notations R_{0_m}, R_{0_h} and $R_{0_{hm}}$ to denote the basic reproduction numbers of the monkeypox sub-model, the HIV/AIDS sub-model, and the co-infection model respectively.

(i) Calculating R_{0_m}.

Let $X(t)$ represent the infected classes. Then

$$X' = \mathcal{F}(t,X) - \mathcal{V}(t,X)$$

where$\mathcal{F}$stands for the new infection terms and $\mathcal{V}$stands for the remaining terms. Now,

$$X = \begin{pmatrix} E_m \\ I_m \end{pmatrix}, \qquad \mathcal{F} = \begin{pmatrix} \lambda_m S \\ 0 \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} (\mu + v_m + \boldsymbol{\varrho_m})E_m \\ -v_m E_m + (\mu + d_m + \rho_m)I_m \end{pmatrix}.$$

The next generation matrices F and V which are the Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$calculated at the DFE.

$$F = \begin{bmatrix} 0 & -\dfrac{(-1+\delta_m)\beta_m c_m \theta_1 (\mu + \omega_m)}{\mu + \alpha_m + \omega_m} \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} \mu + v_m + \varrho_m & 0 \\ -v_m & \mu + d_m + \rho_m \end{bmatrix}.$$

$$V^{-1} = \begin{bmatrix} \dfrac{1}{\mu + v_m + \varrho_m} & 0 \\ \dfrac{v_m}{(\mu + v_m + \varrho_m)(\mu + d_m + \rho_m)} & \dfrac{1}{\mu + d_m + \rho_m} \end{bmatrix}.$$

$$FV^{-1} = \begin{bmatrix} \dfrac{(1-\delta_m)\beta_m c_m \theta_1 (\mu + \omega_m) v_m}{(\mu + \alpha_m + \omega_m)(\mu + v_m + \varrho_m)(\mu + d_m + \rho_m)} & \dfrac{(1-\delta_m)\beta_m c_m \theta_1 (\mu + \omega_m)}{(\mu + \alpha_m + \omega_m)(\mu + d_m + \rho_m)} \\ 0 & 0 \end{bmatrix}.$$

R_{0_m} is the spectral radius of FV^{-1}.

$$\therefore \; R_{0_m} = (\beta_m c_m \theta_1 (1-\delta_m) v_m)\left(\frac{\mu + \omega_m}{\mu + \alpha_m + \omega_m}\right)\left(\frac{1}{(\mu + v_m + \varrho_m)(\mu + d_m + \rho_m)}\right). \quad (3.13)$$

Thussusceptible humans become exposed at rate $\beta_m c_m \theta_1 (1-\delta_m)$when an infectious human is introduced into a completely susceptible population. Susceptible individuals are vaccinated but due to the vaccine waning effect, vaccinated individuals can become susceptible again. The time taken by these humansas susceptible and vaccinated is $\frac{\mu+\omega_m}{\mu+\alpha_m+\omega_m}$. Exposed humans become infectious at rate v_m. The time spent by these humansas exposed and infectious is $\frac{1}{(\mu+v_m+\varrho_m)(\mu+d_m+\rho_m)}$.

(ii) *Calculating R_{0_h}.*

$$X = \begin{pmatrix} I_h \\ A_h \end{pmatrix}, \qquad \mathcal{F} = \begin{pmatrix} \varepsilon\Lambda + \lambda_h S \\ 0 \end{pmatrix}, \mathcal{V} = \begin{pmatrix} (\mu + \rho_1) I_h \\ -\rho_1 I_h + (\mu + d_h) A_h \end{pmatrix}.$$

$$F = \begin{bmatrix} -(-1+\delta_h)\beta_h c_h \phi_1 & -(-1+\delta_h)\beta_h c_h \phi_3 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} \mu + \rho_1 & 0 \\ -\rho_1 & d_h + \mu \end{bmatrix}.$$

$$V^{-1} = \begin{bmatrix} (\mu + \rho_1)^{-1} & 0 \\ \dfrac{\rho_1}{(\mu + \rho_1)(d_h + \mu)} & (d_h + \mu)^{-1} \end{bmatrix}.$$

$$FV^{-1} = \begin{bmatrix} -\dfrac{(-1+\delta_h)\beta_h c_h \phi_1}{\mu + \rho_1} - \dfrac{(-1+\delta_h)\beta_h c_h \phi_3 \rho_1}{(\mu + \rho_1)(d_h + \mu)} & -\dfrac{(-1+\delta_h)\beta_h c_h \phi_3}{d_h + \mu} \\ 0 & 0 \end{bmatrix}.$$

$$\therefore \qquad R_{0_h} = \beta_h c_h (1-\delta_h)(\phi_1 (d_h + \mu) + \phi_3 \rho_1)\frac{1}{(\mu + \rho_1)(d_h + \mu)}. \qquad (10)$$

(iii) *Calculating $R_{0_{hm}}$*

$$X = \begin{pmatrix} E_m \\ I_m \\ I_h \\ A_h \\ I_{hm} \\ A_{hm} \end{pmatrix}, \qquad \mathcal{F} = \begin{pmatrix} \lambda_m S \\ 0 \\ \varepsilon \Lambda + \lambda_h (S + R_m) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathcal{V} = \begin{pmatrix} (\mu + v_m + \varrho_m + \lambda_h) E_m \\ -v_m E_m + (\mu + d_m + \rho_m + \lambda_h) I_m \\ -\tau_1 I_{hm} + (\mu + \rho_1 + \sigma_1 \lambda_m) I_h \\ -\rho_1 I_h - \tau_2 A_{hm} + (\mu + d_h + \sigma_2 \lambda_m) A_h \\ -\sigma_1 \lambda_m I_h - \lambda_h (E_m + I_m) + (\mu + d_m + \rho_2 + \tau_1) I_{hm} \\ -\rho_2 I_{hm} - \sigma_2 \lambda_m A_h + (\mu + d_m + d_h + \tau_2) A_{hm} \end{pmatrix}.$$

$$F = \begin{bmatrix} 0 & c_1 & 0 & 0 & c_2 & c_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-\delta_h)\beta_h c_h \phi_1 & (1-\delta_h)\beta_h c_h \phi_3 & (1-\delta_h)\beta_h c_h \phi_2 & (1-\delta_h)\beta_h c_h \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} \mu + v_m + \varrho_m & 0 & 0 & 0 & 0 & 0 \\ -v_m & \mu + d_m + \rho_m & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu + \rho_1 & 0 & -\tau_1 & 0 \\ 0 & 0 & -\rho_1 & d_h + \mu & 0 & -\tau_2 \\ 0 & 0 & 0 & 0 & \mu + d_m + \rho_2 + \tau_1 & 0 \\ 0 & 0 & 0 & 0 & -\rho_2 & \mu + d_m + d_h + \tau_2 \end{bmatrix}.$$

$$c_1 = -\frac{(-1+\delta_m)\beta_m c_m \theta_1 (\mu + \omega_m)}{\mu + \alpha_m + \omega_m}, \quad c_2 = -\frac{(-1+\delta_m)\beta_m c_m \theta_2 (\mu + \omega_m)}{\mu + \alpha_m + \omega_m},$$
$$c_3 = -\frac{(-1+\delta_m)\beta_m c_m (\mu + \omega_m)}{\mu + \alpha_m + \omega_m}.$$

$$V^{-1} = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ k_2 & k_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\mu + \rho_1)^{-1} & 0 & k_5 & 0 \\ 0 & 0 & k_4 & (d_h + \mu)^{-1} & k_6 & k_9 \\ 0 & 0 & 0 & 0 & k_7 & 0 \\ 0 & 0 & 0 & 0 & k_8 & k_{10} \end{bmatrix}$$

$$k_1 = (\mu + v_m + \varrho_m)^{-1}, k_2 = \frac{v_m}{(\mu + v_m + \varrho_m)(\mu + d_m + \rho_m)}, k_3 = (\mu + d_m + \rho_m)^{-1},$$
$$k_4 = \frac{\rho_1}{(\mu + \rho_1)(d_h + \mu)}, k_5 = \frac{\tau_1}{(\mu + d_m + \rho_2 + \tau_1)(\mu + \rho_1)},$$
$$k_6 = \frac{d_h \rho_1 \tau_1 + d_m \rho_1 \tau_1 + \mu \rho_1 \tau_1 + \mu \rho_2 \tau_2 + \rho_1 \rho_2 \tau_2 + \rho_1 \tau_1 \tau_2}{(\mu + d_m + \rho_2 + \tau_1)(\mu + \rho_1)(\mu + d_m + d_h + \tau_2)(d_h + \mu)},$$
$$k_7 = (\mu + d_m + \rho_2 + \tau_1)^{-1}, \quad k_8 = \frac{\rho_2}{(\mu + d_m + \rho_2 + \tau_1)(\mu + d_m + d_h + \tau_2)},$$
$$k_9 = \frac{\tau_2}{(\mu + d_m + d_h + \tau_2)(d_h + \mu)}, k_{10} = (\mu + d_m + d_h + \tau_2)^{-1}.$$

$$FV^{-1} = \begin{bmatrix} k_{11} & k_{12} & 0 & 0 & k_{14} & -\frac{(-1+\delta_m)\beta_m c_m(\mu+\omega_m)}{(\mu+\alpha_m+\omega_m)(\mu+d_m+d_h+\tau_2)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{13} & -\frac{(-1+\delta_h)\beta_h c_h \phi_3}{d_h+\mu} & k_{15} & -\frac{(-1+\delta_h)\beta_h c_h(\phi_3\tau_2+d_h+\mu)}{(\mu+d_m+d_h+\tau_2)(d_h+\mu)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

where

$$k_{11} = -\frac{(-1+\delta_m)\beta_m c_m \theta_1(\mu+\omega_m)v_m}{(\mu+\alpha_m+\omega_m)(\mu+v_m+\varrho_m)(\mu+d_m+\rho_m)};$$

$$k_{12} = -\frac{(-1+\delta_m)\beta_m c_m \theta_1(\mu+\omega_m)}{(\mu+\alpha_m+\omega_m)(\mu+d_m+\rho_m)};$$

$$k_{13} = -\frac{(-1+\delta_h)\beta_h c_h(d_h\phi_1+\mu\phi_1+\phi_3\rho_1)}{(\mu+\rho_1)(d_h+\mu)};$$

$$k_{14} = -\frac{(-1+\delta_m)\beta_m c_m(\mu+\omega_m)(d_h\theta_2+d_m\theta_2+\mu\theta_2+\tau_2\theta_2+\rho_2)}{(\mu+\alpha_m+\omega_m)(\mu+d_m+\rho_2+\tau_1)(\mu+d_m+d_h+\tau_2)};$$

$$\begin{aligned} k_{15} = -((-1+\delta_h)\beta_h c_h(&{d_h}^2\mu\phi_2 + {d_h}^2\phi_1\tau_1 + {d_h}^2\phi_2\rho_1 + d_h d_m\mu\phi_2 + d_h d_m\phi_1\tau_1 \\ &+ d_h d_m\phi_2\rho_1 + 2d_h\mu^2\phi_2 + 2d_h\mu\phi_1\tau_1 + 2d_h\mu\phi_2\rho_1 + d_h\mu\phi_2\tau_2 \\ &+ d_h\phi_1\tau_1\tau_2 + d_h\phi_2\rho_1\tau_2 + d_h\phi_3\rho_1\tau_1 + d_m\mu^2\phi_2 + d_m\mu\phi_1\tau_1 \\ &+ d_m\mu\phi_2\rho_1 + d_m\phi_3\rho_1\tau_1 + \mu^3\phi_2 + \mu^2\phi_1\tau_1 + \mu^2\phi_2\rho_1 + \mu^2\phi_2\tau_2 \\ &+ \mu\phi_1\tau_1\tau_2 + \mu\phi_2\rho_1\tau_2 + \mu\phi_3\rho_1\tau_1 + \mu\phi_3\rho_2\tau_2 + \phi_3\rho_1\rho_2\tau_2 + \phi_3\rho_1\tau_1\tau_2 \\ &+ d_h\mu\rho_2 + d_h\rho_1\rho_2 + \mu^2\rho_2 + \mu\rho_1\rho_2))/((\mu+d_m+\rho_2+\tau_1)(\mu+\rho_1)(\mu \\ &+ d_m + d_h + \tau_2)(d_h+\mu)). \end{aligned}$$

$R_{0_{hm}}$ isobtained as the spectral radius of FV^{-1}.

$$\therefore \quad R_{0_{hm}} = max\left(\frac{(1-\delta_m)\beta_m c_m \theta_1(\mu+\omega_m)v_m}{(\mu+\alpha_m+\omega_m)(\mu+v_m+\varrho_m)(\mu+d_m+\rho_m)}, \frac{(1-\delta_h)\beta_h c_h((d_h+\mu)\phi_1+\phi_3\rho_1)}{(\mu+\rho_1)(d_h+\mu)}\right).$$

$$i.e. \quad R_{0_{hm}} = max\left(R_{0_m}, R_{0_h}\right).$$

3.4.5. Stability Analysis

Lemma 1 (Maia, 2015)

Assume that J is a square matrix with constant entries and $\text{Det}\, J \neq 0$. Assume that J has been obtained as a linearization around the equilibrium (x^*, y^*). Then the equilibrium (x^*, y^*) is locally asymptotically stable if and only if

$$\text{Trace}\, J < 0 \quad \text{and} \quad \text{Det}\, J > 0.$$

The equilibrium (x^*, y^*) is unstable if and only if

$$\text{Trace}\, J > 0 \quad \text{and} \quad \text{Det}\, J < 0.$$

I. Local stability of DEE

Theorem 3:

$\mathbb{E}_{0_m}$ is locally asymptotically stable if $R_{0_m} < 1$, otherwise, it is unstable.

Proof:

The Jacobian matrix $J(\mathbb{E}_{0_m})$ of the monkeypox sub-model, evaluated at the disease-free equilibrium is

$$J\left(\mathbb{E}_{0_m}\right)=\begin{bmatrix}-\mu_n & 0 & -\beta_n c_n & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & p_1 & \beta_n c_n & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & v_n & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \varrho_n & \rho_n & -\mu_n & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & p_2 & 0 & -\mu-\alpha_m & \omega_m & 0 & p_5 & \zeta\\ 0 & 0 & 0 & 0 & \alpha_m & -\mu-\omega_m & 0 & 0 & 0\\ 0 & 0 & p_3 & 0 & 0 & 0 & p_4 & p_6 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & v_m & p_7 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \varrho_m & \rho_m & -\mu-\zeta\end{bmatrix},$$

$$p_1=-\mu_n-\varrho_n-v_n,\qquad p_2=-\frac{(-1+\delta_m)\beta_n c_n\mu_n\Lambda(-1+\epsilon)(\mu+\omega_m)}{\Lambda_n\mu(\mu+\alpha_m+\omega_m)},$$
$$p_3=\frac{(-1+\delta_m)\beta_n c_n\mu_n\Lambda(-1+\epsilon)(\mu+\omega_m)}{\Lambda_n\mu(\mu+\alpha_m+\omega_m)},p_4=-\mu-v_m-\varrho_m,$$
$$p_5=\frac{(-1+\delta_m)\beta_m c_m\theta_1(\mu+\omega_m)}{\mu+\alpha_m+\omega_m},\qquad p_6=-\frac{(-1+\delta_m)\beta_m c_m\theta_1(\mu+\omega_m)}{\mu+\alpha_m+\omega_m},$$
$$p_7=-\mu-d_m-\rho_m.$$

$\therefore\ Trace\, J\left(\mathbb{E}_{0_m}\right)=-3\mu_n-\varrho_n-v_n-5\mu-\alpha_m-\omega_m-v_m-\varrho_m-d_m-\rho_m-\zeta<0,$

$$\begin{aligned}Det\, J\left(\mathbb{E}_{0_m}\right)=\ &{\mu_n}^2 c_n\beta_n v_n(\mu+\zeta)\Big(\mu^3+(d_m+v_m+\alpha_m+\omega_m+\rho_m+\varrho_m)\mu^2\\&+\big((\omega_m+\alpha_m+\rho_m+d_m-(1-\delta_m)\beta_m c_m\theta_1)v_m+(\varrho_m+d_m+\rho_m)\omega_m\\&+(\alpha_m+d_m+\rho_m)\varrho_m+\alpha_m(d_m+\rho_m)\big)\mu\\&+\big((\rho_m+d_m-(1-\delta_m)\beta_m c_m\theta_1)\omega_m+\alpha_m(d_m+\rho_m)\big)v_m\\&+\varrho_m(d_m+\rho_m)(\alpha_m+\omega_m)\Big)>0.\end{aligned}$$

Observe that: $\omega_m+\alpha_m+\rho_m+d_m>(1-\delta_m)\beta_m c_m\theta_1$ and $\rho_m+d_m>(1-\delta_m)\beta_m c_m\theta_1$. Hence, by Lemma 1, $\mathbb{E}_{0_m}$ is locally asymptotically stable. Now substituting

$$R_{0_m}=(\beta_m c_m\theta_1(1-\delta_m)v_m)\left(\frac{\mu+\omega_m}{\mu+\alpha_m+\omega_m}\right)\left(\frac{1}{(\mu+v_m+\varrho_m)(\mu+d_m+\rho_m)}\right)$$

into the expression for $Det\, J\left(\mathbb{E}_{0_m}\right)$, we obtained

$$R_{0_m}<1.$$

It follows that the disease-free equilibrium $\mathbb{E}_{0_m}$ is locally asymptotically stable. ■

Similarly, we established the following result:

Theorem 4
$\mathbb{E}_{0_h}$ is locally asymptotically stable if $R_{0_h}<1$, otherwise, it is unstable.

Proof:

$$J\left(\mathbb{E}_{0_h}\right)=\begin{bmatrix}-\mu & -(1-\delta_h)\beta_h c_h\phi_1 & -(1-\delta_h)\beta_h c_h\phi_3\\ 0 & (1-\delta_h)\beta_h c_h\phi_1 & (1-\delta_h)\beta_h c_h\phi_3\\ 0 & \rho_1 & -\mu-d_h\end{bmatrix}.$$

$$Trace\, J\left(\mathbb{E}_{0_h}\right)=(1-\delta_h)\beta_h c_h\phi_1-(\rho_1+d_h+3\mu)<0,$$
$$Det\, J\left(\mathbb{E}_{0_h}\right)=\mu\big(c_h(1-\delta_h)(d_h\phi_1+\mu\phi_1+\phi_3\rho_1)\beta_h+(\mu+\rho_1)(\mu+d_h)\big)>0.$$

Observe that,

$$(1-\delta_h)\beta_h c_h\phi_1<(\rho_1+d_h+3\mu).$$

Hence, by Lemma 1, $\mathbb{E}_{0_h}$ is locally asymptotically stable. Substituting

$$R_{0_h} = \beta_h c_h (1-\delta_h)(\phi_1(d_h+\mu)+\phi_3\rho_1)\frac{1}{(\mu+\rho_1)(d_h+\mu)}$$

into the expression for $Det\, J(\mathbb{E}_{0_h})$, we obtained

$$R_{0_h} < 1.$$

It follows that $\mathbb{E}_{0_m}$ is locally asymptotically stable. ■

II. DFE's Global Stability

Lemma 2: If a function $V(x)$ is globally positively definite and radially bounded, and its time derivative globally negative,$V^{'}(x) < 0$ for all $x \neq x^*$,then x^* is globally stable.

Theorem 5

$\mathbb{E}_{0_m}$is globally asymptotically stable if$R_{0_m} < 1$.

Proof:

$$V = k_1\left(S_n - S_n{}^* - S_n{}^*\ln\frac{S_n}{S_n{}^*}\right) + \frac{1}{(\mu_n+\varrho_n+\upsilon_n)}E_n + \frac{1}{\upsilon_n}I_n + k_2\left(S - S^* - S^*\ln\frac{S}{S^*}\right) + k_3\left(V_m - V_m{}^* - V_m{}^*\ln\frac{V_m}{V_m{}^*}\right) + \frac{1}{(\mu+\upsilon_m+\boldsymbol{\varrho_m})}E_m + \frac{1}{\upsilon_m}I_m.$$

$k_1 > 0, k_2 > 0, k_3 > 0.$

$$S_n{}^* = \frac{\Lambda_n}{\mu_n}, S^* = \frac{\Lambda(1-\varepsilon)(\mu+\omega_m)}{\mu(\mu+\alpha_m+\omega_m)}, V_m{}^* = \frac{\Lambda\alpha_m(1-\varepsilon)}{\mu(\mu+\alpha_m+\omega_m)}.$$

To verify that $V > 0$ for all

$$(S_n, E_n, I_n, S, V_m, E_m, I_m) \neq \left(\frac{\Lambda_n}{\mu_n}, 0, 0, \frac{\Lambda(1-\varepsilon)(\mu+\omega_m)}{\mu(\mu+\alpha_m+\omega_m)}, \frac{\Lambda\alpha_m(1-\varepsilon)}{\mu(\mu+\alpha_m+\omega_m)}, 0, 0\right),$$

We observe that

$$k_1\left(S_n - S_n{}^* - S_n{}^*\ln\frac{S_n}{S_n{}^*}\right) > 0, k_2\left(S - S^* - S^*\ln\frac{S}{S^*}\right) > 0, k_3\left(V_m - V_m{}^* - V_m{}^*\ln\frac{V_m}{V_m{}^*}\right) > 0,$$

since the function $g(x) = x - 1 - \ln x$ achieves a global minimum at $x = 1$ and $g(1) = 0$. Hence $g(x) > 0$ for all $x > 0$ and $x \neq 1$. $V$is radially unbounded.

$$\frac{d}{dt}V = k_1\left(1 - \frac{S_n{}^*}{S_n}\right)S_n{}' + \frac{1}{(\mu_n+\varrho_n+\upsilon_n)}E_n{}' + \frac{1}{\upsilon_n}I_n{}' + k_2\left(1-\frac{S^*}{S}\right)S + k_3\left(1-\frac{V_m{}^*}{V_m}\right) + \frac{1}{(\mu+\upsilon_m+\varrho_m)}E_m{}' + \frac{1}{\upsilon_m}I_m{}'.$$

$$\begin{aligned}\frac{d}{dt}V = {} & k_1\left(1-\frac{S_n{}^*}{S_n}\right)[\Lambda_n - (\mu_n+\lambda_n)S_n] \\ & + \frac{1}{(\mu_n+\varrho_n+\upsilon_n)}[\lambda_n S_n - (\mu_n+\varrho_n+\upsilon_n)E_n] \\ & + \frac{1}{\upsilon_n}[\upsilon_n E_n - (\mu_n+d_n+\rho_n)I_n] + k_2\left(1-\frac{S^*}{S}\right)S \\ & + k_3\left(1-\frac{V_m{}^*}{V_m}\right)[\alpha_m S - (\mu+\omega_m)V_m] \\ & + \frac{1}{(\mu+\upsilon_m+\boldsymbol{\varrho_m})}[\lambda_m S - (\mu+\upsilon_m+\boldsymbol{\varrho_m})E_m] \\ & + \frac{1}{\upsilon_m}[\upsilon_m E_m - (\mu+d_m+\rho_m)I_m].\end{aligned}$$

$$
\begin{aligned}
&\frac{dV}{dt} \\
&= \frac{\Lambda(\mu^2 k_2 + ((k_2 + k_3)\alpha_m + \omega_m k_2)\mu + k_3\alpha_m\omega_m)}{\mu(\mu + \alpha_m + \omega_m)} + k_2(1-\epsilon)\Lambda + 2k_1\Lambda_n \\
&+ \frac{k_2\Lambda\omega_m\alpha_m}{\mu(\mu + \alpha_m + \omega_m)} \\
&+ \frac{(\mu^2\Lambda_n k_1 + (\mu_n\Lambda k_2 + k_1\Lambda_n\alpha_m + k_1\Lambda_n\omega_m)\mu + \omega_m k_2\Lambda\mu_n(1-\delta_m))c_n\beta_n I_n}{\mu_n(S_n + E_n + I_n)(\mu + \alpha_m + \omega_m)\mu} \\
&+ \frac{k_2\Lambda R_{\{\backslash\text{tt }0\}_m}(\mu + v_m + \varrho_m)(\mu + d_m + \rho_m)I_m}{\mu v_m(S + V_m + E_m + I_m)} - \frac{k_1{\Lambda_n}^2}{\mu_n S_n} \\
&- \frac{k_2\Lambda\delta_m\beta_n c_n I_n}{(\mu + \alpha_m + \omega_m)(S_n + E_n + I_n)} - k_1 S_n\mu_n \\
&- \frac{I_n(-1 + k_2(\mu + v_m + \varrho_m))(1-\delta_m)c_n S\beta_n}{(\mu + v_m + \varrho_m)(S_n + E_n + I_n)} - \frac{k_3\Lambda{\alpha_m}^2 S}{\mu(\mu + \alpha_m + \omega_m)V_m} \\
&- \frac{I_n(\mu_n + d_n + \rho_n)}{v_n} - \frac{k_2\Lambda\omega_m V_m(\mu + \omega_m)}{\mu(\mu + \alpha_m + \omega_m)S} - V_m((-k_2 + k_3)\omega_m + \mu k_3) \\
&- \frac{I_m(\mu + d_m + \rho_m)S(-1 + k_2(\mu + v_m + \varrho_m))R_{\{\backslash\text{tt }0\}_m}(\mu + \alpha_m + \omega_m)}{v_m(\mu + \omega_m)(S + V_m + E_m + I_m)} \\
&- \frac{c_n\beta_n I_n S_n(-1 + k_1(\mu_n + \varrho_n + v_n))}{(\mu_n + \varrho_n + v_n)(S_n + E_n + I_n)} - \frac{I_m(\mu + d_m + \rho_m)}{v_m} - \frac{k_2\Lambda^2(1-\epsilon)(\mu + \omega_m)}{\mu(\mu + \alpha_m + \omega_m)S} \\
&- ((k_2 - k_3)\alpha_m + \mu k_2)S
\end{aligned}
$$

If we let $k_1 = \frac{1}{\mu_n + \varrho_n + v_n}$, $k_2 = k_3 = \frac{1}{\mu + v_m + \varrho_m}$, then the derivative simplifies to:

$$
\begin{aligned}
\frac{dV}{dt} = &\frac{(\mu^2 + (2\alpha_m + \omega_m)\mu + \alpha_m\omega_m)\Lambda}{(\mu + v_m + \varrho_m)\mu(\mu + \alpha_m + \omega_m)} + \frac{(1-\epsilon)\Lambda}{\mu + v_m + \varrho_m} + 2\frac{\Lambda_n}{\mu_n + \varrho_n + v_n} \\
&+ \frac{\Lambda\omega_m\alpha_m}{(\mu + v_m + \varrho_m)\mu(\mu + \alpha_m + \omega_m)} \\
&+ \frac{c_n\beta_n I_n}{\mu_n(S_n + E_n + I_n)(\mu + \alpha_m + \omega_m)\mu}\Big(\frac{\Lambda_n\mu^2}{\mu_n + \varrho_n + v_n} + \Big(\frac{\mu_n\Lambda}{\mu + v_m + \varrho_m} \\
&+ \frac{\Lambda_n\alpha_m}{\mu_n + \varrho_n + v_n} + \frac{\Lambda_n\omega_m}{\mu_n + \varrho_n + v_n}\Big)\mu + \frac{\Lambda\omega_m\mu_n(1-\delta_m)}{\mu + v_m + \varrho_m}\Big) \\
&+ \frac{\Lambda(\mu + d_m + \rho_m)I_m}{\mu v_m(S + V_m + E_m + I_m)} + \frac{\Lambda(R_{0_m} - 1)(\mu + d_m + \rho_m)I_m}{\mu v_m(S + V_m + E_m + I_m)} \\
&- \frac{S_n\mu_n}{\mu_n + \varrho_n + v_n} - \frac{I_m(\mu + d_m + \rho_m)}{v_m} \\
&- \frac{\Lambda{\alpha_m}^2 S}{(\mu + v_m + \varrho_m)\mu(\mu + \alpha_m + \omega_m)V_m} \\
&- \frac{\Lambda\omega_m V_m(\mu + \omega_m)}{(\mu + v_m + \varrho_m)\mu(\mu + \alpha_m + \omega_m)S} - \frac{V_m\mu}{\mu + v_m + \varrho_m} \\
&- \frac{\Lambda^2(1-\epsilon)(\mu + \omega_m)}{(\mu + v_m + \varrho_m)\mu(\mu + \alpha_m + \omega_m)S} - \frac{\mu S}{\mu + v_m + \varrho_m} \\
&- \frac{I_n(\mu_n + d_n + \rho_n)}{v_n} - \frac{\Lambda\delta_m\beta_n c_n I_n}{(\mu + v_m + \varrho_m)(\mu + \alpha_m + \omega_m)(S_n + E_n + I_n)}.
\end{aligned}
$$

Observe that $\frac{dV}{dt} < 0$with $R_{0_m} < 1$, Therefore, by Lyapunov's theorem, $\mathbb{E}_{0_m}$ is globally asymptotically stable.■

Theorem 6:
Assume $R_{0_h} < 1$. Then $\mathbb{E}_{0_h}$ is globally stable.

Proof:

$$V = k\left(S - S^* - S^* \ln\frac{S}{S^*}\right) + \frac{1}{(\mu+\rho_1)} I_h + \frac{1}{\rho_1} A_h.$$

To show that $V > 0$ for all $(S, I_h, A_h) \neq \left(\frac{(1-\varepsilon)\Lambda}{\mu}, 0,0\right)$, we observe that $k\left(S - S^* - S^* \ln\frac{S}{S^*}\right) > 0$.

V is radially unbounded.

$$\frac{d}{dt}V = k\left(1 - \frac{S^*}{S}\right)S' + \frac{1}{(\mu+\rho_1)} {I_h}' + \frac{1}{\rho_1} {A_h}'.$$

$$\frac{d}{dt}V = k\left(1 - \frac{S^*}{S}\right)\big((1-\varepsilon)\Lambda + \omega_m V_m - (\mu+\lambda_h)S\big) + \frac{1}{(\mu+\rho_1)}\left(\varepsilon\Lambda + \lambda_h S - (\mu+\rho_1)I_h\right) + \frac{1}{\rho_1}\left(\rho_1 I_h - (\mu+d_h)A_h\right).$$

$$\begin{aligned}
\frac{dV}{dt} = {} & 2k\Lambda(1-\epsilon) + \frac{\epsilon\Lambda}{\mu+\rho_1} + \frac{\phi_1(\mu+d_h)(\mu+\rho_1)k\Lambda(1-\epsilon)}{\big((\mu+d_h)\phi_1+\phi_3\rho_1\big)\mu}\frac{I_h}{(\mathrm{S}+I_h+A_h)} \\
& + 2\frac{(\mu+d_h)\phi_3\left(\frac{1}{2}+\delta_h(\mu+\rho_1)k\right)}{(1-\delta_h)\big((\mu+d_h)\phi_1+\phi_3\rho_1\big)}\frac{SA_h}{(\mathrm{S}+I_h+A_h)} \\
& + \frac{(\mu+d_h)\phi_1}{\big((\mu+d_h)\phi_1+\phi_3\rho_1\big)}\frac{SI_h}{(\mathrm{S}+I_h+A_h)} \\
& + \frac{k\Lambda(\mu+\rho_1)(\mu+d_h)\phi_3(1-\epsilon)}{\big((\mu+d_h)\phi_1+\phi_3\rho_1\big)\mu}\frac{A_h}{(\mathrm{S}+I_h+A_h)} \\
& - \frac{\big(R_{0_h}-1\big)(\mu+d_h)\phi_1(k\mu+k\rho_1)}{\big((\mu+d_h)\phi_1+\phi_3\rho_1\big)}\frac{SI_h}{(\mathrm{S}+I_h+A_h)} \\
& - \frac{\big(R_{0_h}-1\big)(\mu+d_h)\phi_1}{\big((\mu+d_h)\phi_1+\phi_3\rho_1\big)}\frac{SI_h}{(\mathrm{S}+I_h+A_h)} \\
& + \frac{k\Lambda\big(R_{0_h}-1\big)(\mu+\rho_1)(\mu+d_h)\phi_3(1-\epsilon)}{\big((\mu+d_h)\phi_1+\phi_3\rho_1\big)\mu}\frac{A_h}{(\mathrm{S}+I_h+A_h)} \\
& + 2\frac{(\mu+d_h)\big(R_{0_h}-1\big)\phi_3\left(\frac{1}{2}+\delta_h(\mu+\rho_1)k\right)}{(1-\delta_h)\big((\mu+d_h)\phi_1+\phi_3\rho_1\big)}\frac{SA_h}{(\mathrm{S}+I_h+A_h)} \\
& + \frac{\phi_1\big(R_{0_h}-1\big)(\mu+d_h)(\mu+\rho_1)k\Lambda(1-\epsilon)}{\big((\mu+d_h)\phi_1+\phi_3\rho_1\big)\mu}\frac{I_h}{(\mathrm{S}+I_h+A_h)} \\
& - \frac{(\mu+d_h)\phi_1(k\mu+k\rho_1-1)}{\big((\mu+d_h)\phi_1+\phi_3\rho_1\big)}\frac{SI_h}{(\mathrm{S}+I_h+A_h)} - \frac{(\mu+d_h)}{\rho_1}A_h \\
& - \frac{k\Lambda^2(\epsilon-1)^2}{\mu S} - k\mu S.
\end{aligned}$$

For any $k > 0$ and $R_{0_h} < 1$, we observe that $\frac{dV}{dt}$ is negative. Therefore, by Lyapunov's theorem, $\mathbb{E}_{0_h}$ is globally stable.■

Theorem 7 - Krasovkii-LaSalle Theorem (Maia, 2015)

Consider the system $x^{'} = f(x)$, where x^* is an equilibrium, that is, $f(x^*) = 0$. Suppose there exists a positive definite continuously differentiable function $V: \mathbf{R}^{\boldsymbol{n}} \to \mathbf{R}^{\boldsymbol{n}}$on the entire space and radially unblounded and that it satisfies

$$V^{'}(x) \leq 0 \quad \text{for all } t \text{ and all } x \in \mathbf{R}^{\boldsymbol{n}}.$$

Define the invariant set

$$\mathcal{I} = \{x \in \mathbf{R}^{\boldsymbol{n}} | V^{'}(x) = 0\}.$$

If $\mathcal{I}$ contains only the equilibrium x^*, then the equilibrium x^* is globally stable.

III. DEE'S Global Stability

Theorem 8 (DEE's global stability of the monkeypox sub-model)

Assume $R_{0_m} > 1$. Then the monkeypox sub-model'sDEE is globally asymptotically stable.

Proof:

$$\begin{aligned} V = {} & k_1\left(S_n - {S_n}^* - {S_n}^* \ln \frac{S_n}{{S_n}^*}\right) + k_2\left(E_n - {E_n}^* - {E_n}^* \ln \frac{E_n}{{E_n}^*}\right) \\ & + k_3\left(I_n - {I_n}^* - {I_n}^* \ln \frac{I_n}{{I_n}^*}\right) + +k_4\left(S - S^* - S^* \ln \frac{S}{S^*}\right) \\ & + k_5\left(V_m - {V_m}^* - {V_m}^* \ln \frac{V_m}{{V_m}^*}\right) + k_6\left(E_m - {E_m}^* - {E_m}^* \ln \frac{E_m}{{E_m}^*}\right) \\ & + k_7\left(I_m - {I_m}^* - {I_m}^* \ln \frac{I_m}{{I_m}^*}\right). \end{aligned}$$

We notice that

$$k_1\left(S_n - {S_n}^* - {S_n}^* \ln \frac{S_n}{{S_n}^*}\right) > 0, \qquad k_2\left(E_n - {E_n}^* - {E_n}^* \ln \frac{E_n}{{E_n}^*}\right) > 0,$$

$$k_3\left(I_n - {I_n}^* - {I_n}^* \ln \frac{I_n}{{I_n}^*}\right) > 0, \qquad k_4\left(S - S^* - S^* \ln \frac{S}{S^*}\right) > 0,$$

$$k_5\left(V_m - {V_m}^* - {V_m}^* \ln \frac{V_m}{{V_m}^*}\right) > 0, \qquad k_6\left(E_m - {E_m}^* - {E_m}^* \ln \frac{E_m}{{E_m}^*}\right) > 0,$$

$$k_7\left(I_m - {I_m}^* - {I_m}^* \ln \frac{I_m}{{I_m}^*}\right) > 0.$$

$V$is radially unbounded.

$$\begin{aligned} \frac{d}{dt}V = {} & k_1\left(1 - \frac{{S_n}^*}{S_n}\right){S_n}^{'} + k_2\left(1 - \frac{{E_n}^*}{E_n}\right){E_n}^{'} + k_3\left(1 - \frac{{I_n}^*}{I_n}\right){I_n}^{'} + k_4\left(1 - \frac{S^*}{S}\right)S^{'} \\ & + k_5\left(1 - \frac{{V_m}^*}{V_m}\right){V_m}^{'} + k_6\left(1 - \frac{{E_m}^*}{E_m}\right){E_m}^{'} + k_7\left(1 - \frac{{I_m}^*}{I_m}\right){I_m}^{'}. \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}V = {} & k_1\left(1 - \frac{{S_n}^*}{S_n}\right)(\Lambda_n - (\mu_n + \lambda_n)S_n) + k_2\left(1 - \frac{{E_n}^*}{E_n}\right)(\lambda_n S_n \\ & - (\mu_n + \varrho_n + v_n)E_n) + k_3\left(1 - \frac{{I_n}^*}{I_n}\right)(v_n E_n - (\mu_n + d_n + \rho_n)I_n) \\ & + k_4\left(1 - \frac{S^*}{S}\right)((1 - \varepsilon)\Lambda + \boldsymbol{\omega_m} V_m + \boldsymbol{\zeta} R_m - (\mu + \lambda_m + \alpha_m)S) \\ & + k_5\left(1 - \frac{{V_m}^*}{V_m}\right)(\alpha_m S - (\mu + \omega_m)V_m) + k_6\left(1 - \frac{{E_m}^*}{E_m}\right)(\lambda_m S \\ & - (\mu + v_m + \boldsymbol{\varrho_m})E_m) + k_7\left(1 - \frac{{I_m}^*}{I_m}\right)(v_m E_m - (\mu + d_m + \rho_m)I_m). \end{aligned}$$

$\therefore \quad \frac{d}{dt}V = A - B;$

where

$$A = k_4 L_7 L_9 \mathrm{S}^* - (R_{0_m} - 1)(L_{11} k_6 \delta_m + L_7 L_9 k_4) S + k_6 L_{11} (1 - \delta_m) \frac{{E_m}^* S}{E_m} + k_1 \Lambda_n + k_2 L_2 + k_3 L_4 + k_4 \alpha_m \mathrm{S}^* + k_5 \alpha_m S + k_6 L_8 S + k_7 v_m E_m + k_6 L_8 \delta_m \frac{{E_m}^* S}{E_m} + k_1 \mu_n {S_n}^* + k_1 L_1 {S_n}^* + k_6 L_{11} R_{0_m} S + k_6 L_{12} \frac{{E_m}^*}{E_m} + k_7 L_{13} \frac{{I_m}^*}{I_m} + k_5 L_{10} \frac{{V_m}^*}{V_m} + k_3 L_5 \frac{{I_n}^*}{I_n} + k_4 L_7 L_8 \mathrm{S}^* + k_2 L_3 \frac{{E_n}^*}{E_n} + k_4 L_6;$$

$$B = k_1 \Lambda_n \frac{{S_n}^*}{S_n} + k_2 L_2 \frac{{E_n}^*}{E_n} + k_3 L_4 \frac{{I_n}^*}{I_n} + k_4 L_7 L_8 S + k_4 L_6 \frac{\mathrm{S}^*}{S} + k_6 L_8 \delta_m S + k_1 \mu_n S_n + k_1 L_1 S_n + k_4 \alpha_m S + k_2 L_3 + k_3 L_5 + k_5 L_{10} + k_6 L_{12} + k_7 L_{13} + k_5 \alpha_m \frac{{V_m}^* S}{V_m} + k_6 L_8 \frac{{E_m}^* S}{E_m} + k_7 v_m \frac{{I_m}^* E_m}{I_m} - k_6 L_{11} (R_{0_m} - 1)(1 - \delta_m) \frac{{E_m}^* S}{E_m} + (L_{11} k_6 \delta_m + L_7 L_9 k_4) S - k_4 L_7 L_9 (R_{0_m} - 1) S^*;$$

$L_1 = \beta_n c_n \dfrac{I_n}{S_n + E_n + I_n}$; $L_2 = \beta_n c_n \dfrac{I_n S_n}{S_n + E_n + I_n}$; $L_3 = (\mu_n + \varrho_n + v_n) E_n$; $L_4 = v_n E_n$;
$L_5 = (\mu_n + d_n + \rho_n) I_n$; $L_6 = (1 - \epsilon)\Lambda + \omega_m V_m$; $L_7 = \mu + 1 - \delta_m$;
$L_8 = \beta_n c_n \dfrac{I_n}{S_n + E_n + I_n}$;
$L_9 = \dfrac{(\mu + \alpha_m + \omega_m)(\mu + v_m + \varrho_m)(\mu + d_m + \rho_m)}{v_m (1 - \delta_m)(\mu + \omega_m)} \dfrac{I_m}{(S + V_m + E_m + I_m)}$;
$L_{10} = (\mu + \omega_m) V_m$;
$L_{11} = \dfrac{(\mu + \alpha_m + \omega_m)(\mu + v_m + \varrho_m)(\mu + d_m + \rho_m)}{v_m (1 - \delta_m)(\mu + \omega_m)} \dfrac{I_m}{(S + V_m + E_m + I_m)}$;
$L_{12} = (\mu + v_m + \varrho_m) E_m$; $L_{13} = (\mu + d_m + \rho_m) I_m$.

We apply the Krasovkii-LaSalle theorem. We consider where the Lyapunov function is equal to zero:

$$\mathcal{J} = \{x \in \mathbf{R}^{\boldsymbol{n}} | V'(x) = 0\}.$$

For any $k_1, k_2, k_3, k_4, k_5, k_6, k_7$ and $R_{0_m} > 1$we observe that $\frac{d}{dt} V \leq 0$ if $A < B$, and $\frac{d}{dt} V = 0$ iff $S_n = {S_n}^*, E_n = {E_n}^*$, $I_n = {I_n}^*$, $S = S^*$, $V_m = {V_m}^*$, $E_m = {E_m}^*$, $I_m = {I_m}^*$. Hence, the set $\mathcal{J}$ consists of the singleton $({S_n}^*, {E_n}^*, {I_n}^*, S^*, {V_m}^*, {E_m}^*, {I_m}^*)$ and the DEE is globally asymptotically stable. ■

Theorem 9 (DEE's global stability of the HIV-AIDS sub-model)

Assume $R_{0_m} > 1$. Then the HIV-AIDS sub-model'sDEE is globally asymptotically stable.

Proof:

$$V = k_1 \left(S - S^* - S^* \ln \frac{S}{S^*} \right) + k_2 \left(I_h - {I_h}^* - {I_h}^* \ln \frac{I_h}{{I_h}^*} \right) + k_3 \left(A_h - {A_h}^* - {A_h}^* \ln \frac{A_h}{{A_h}^*} \right).$$

To showthat$V > 0$, we notice that

$$k_1\left(S-S^*-S^*\ln\frac{S}{S^*}\right)>0,\qquad k_2\left(I_h-{I_h}^*-{I_h}^*\ln\frac{I_h}{{I_h}^*}\right)>0,$$
$$k_3\left(A_h-{A_h}^*-{A_h}^*\ln\frac{A_h}{{A_h}^*}\right)>0.$$
V is radially unbounded.
$$\frac{d}{dt}V=k_1\left(1-\frac{S^*}{S}\right)S'+k_2\left(1-\frac{{I_h}^*}{I_h}\right){I_h}'+k_3\left(1-\frac{{A_h}^*}{A_h}\right){A_h}'.$$
$$\frac{d}{dt}V=k_1\left(1-\frac{S^*}{S}\right)((1-\varepsilon)\Lambda-(\mu+\lambda_h)S)+k_2\left(1-\frac{{I_h}^*}{I_h}\right)(\boldsymbol{\varepsilon}\Lambda+\lambda_hS$$
$$-(\mu+\rho_1)I_h)+k_3\left(1-\frac{{A_h}^*}{A_h}\right)(\rho_1I_h-(\mu+d_h)A_h).$$
$$\therefore\quad \frac{d}{dt}V=A-B,$$
where
$$A=(1-\epsilon)\Lambda k_1+\mu k_1S^*-\frac{(R_{0_h}-1)(\mu+\rho_1)(\mu+d_h)(A_h\phi_3+I_h\phi_1)k_1S}{((\mu+d_h)\phi_1+\phi_3\rho_1)(S+I_h+A_h)}$$
$$-\frac{(R_{0_h}-1)(\mu+\rho_1)(\mu+d_h)(A_h\phi_3+I_h\phi_1)Sk_2{I_h}^*}{((\mu+d_h)\phi_1+\phi_3\rho_1)(S+I_h+A_h)I_h}$$
$$+\frac{(\mu+\rho_1)(\mu+d_h)(A_h\phi_3+I_h\phi_1)k_1S^*}{((\mu+d_h)\phi_1+\phi_3\rho_1)(S+I_h+A_h)}+\varepsilon\Lambda k_2$$
$$+\frac{(\mu+\rho_1)(\mu+d_h)(A_h\phi_3+I_h\phi_1)Sk_2}{((\mu+d_h)\phi_1+\phi_3\rho_1)(S+I_h+A_h)}+(\mu+\rho_1)k_2{I_h}^*+k_3{A_h}^*d_h$$
$$+k_3{A_h}^*\mu+k_3I_h\rho_1$$
$$B=\frac{(1-\epsilon)\Lambda k_1S^*}{S}+\mu k_1S+\frac{(\mu+\rho_1)(\mu+d_h)(A_h\phi_3+I_h\phi_1)k_1S}{((\mu+d_h)\phi_1+\phi_3\rho_1)(S+I_h+A_h)}$$
$$-\frac{\left(1-\delta_h\left(R_{0_h}-1\right)\right)(\mu+\rho_1)(\mu+d_h)(A_h\phi_3+I_h\phi_1)k_1S^*}{\left((\mu+d_h)\phi_1+\phi_3\rho_1\right)(1-\delta_h)(S+I_h+A_h)}$$
$$-\frac{(R_{0_h}-1)(\mu+\rho_1)(\mu+d_h)(A_h\phi_3+I_h\phi_1)Sk_2}{((\mu+d_h)\phi_1+\phi_3\rho_1)(S+I_h+A_h)}+\frac{\varepsilon\Lambda k_2{I_h}^*}{I_h}$$
$$+\frac{(\mu+\rho_1)(\mu+d_h)(A_h\phi_3+I_h\phi_1)Sk_2{I_h}^*}{((\mu+d_h)\phi_1+\phi_3\rho_1)(S+I_h+A_h)I_h}+(\mu+\rho_1)I_hk_2$$
$$+\frac{k_3{A_h}^*I_h\rho_1}{A_h}+k_3A_hd_h+k_3A_h\mu$$
Again, we also apply the Krasovkii-LaSalle theorem. We consider where the Lyapunov function is equal to zero:
$$\mathcal{J}=\{x\in\mathbf{R}^{\boldsymbol{n}}|V'(x)=0\}.$$
For any k_1,k_2,k_3 and $R_{0_h}>1$we observe that $\frac{d}{dt}V\le 0$if $A<B$, and $\frac{d}{dt}V=0$ iff $S=S^*,I_h={I_h}^*,SA_h={A_h}^*$. Hence, the set $\mathcal{J}$ consists of the singleton $(S^*,{I_h}^*,{A_h}^*)$ and the DEE is globally asymptotically stable. ■

4. Numerical Simulations

The following numerical simulations are presented in order to show the effects of certain parameters on different compartments. We magnified some parts of the plots for a better observation.

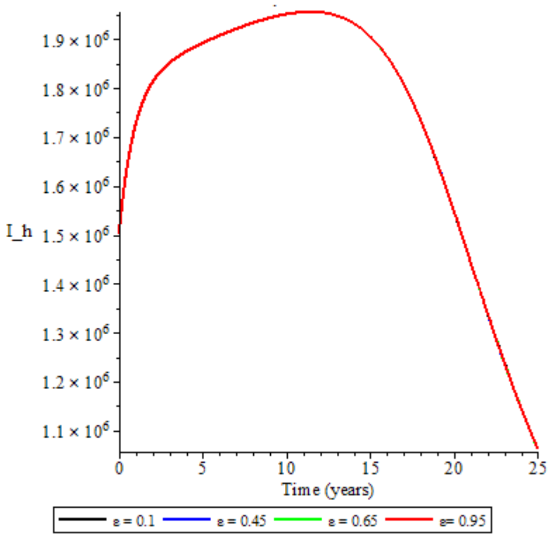

Fig. 2a: Effect of ε on I_h.

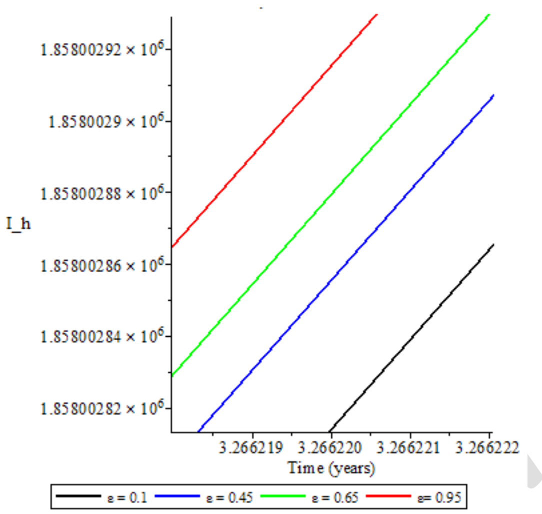

Fig. 2b: Effect of ε on I_h (magnified view).

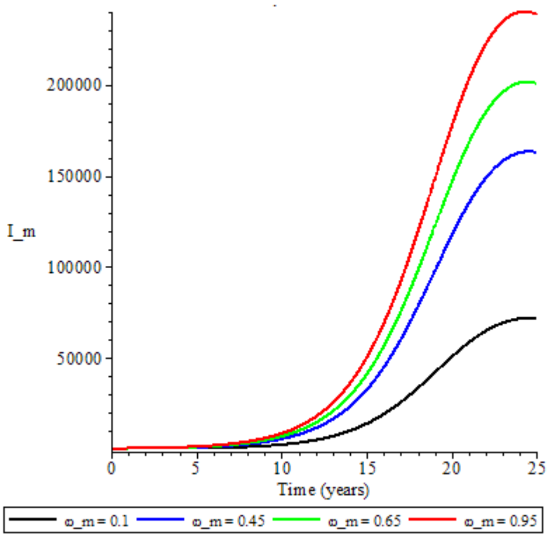

Fig. 3a: Effect of ω_m on I_m.

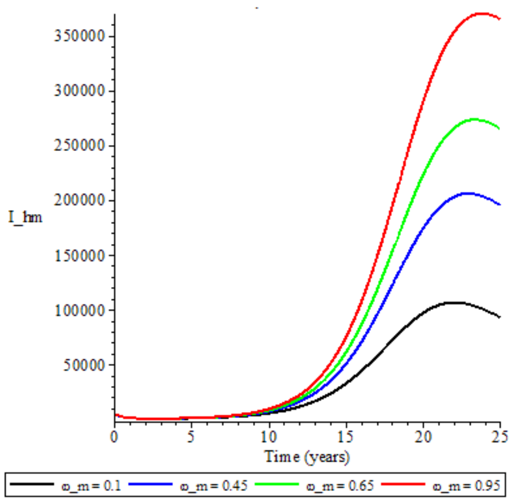

Fig. 3b: Effect of ω_m on I_{hm}.

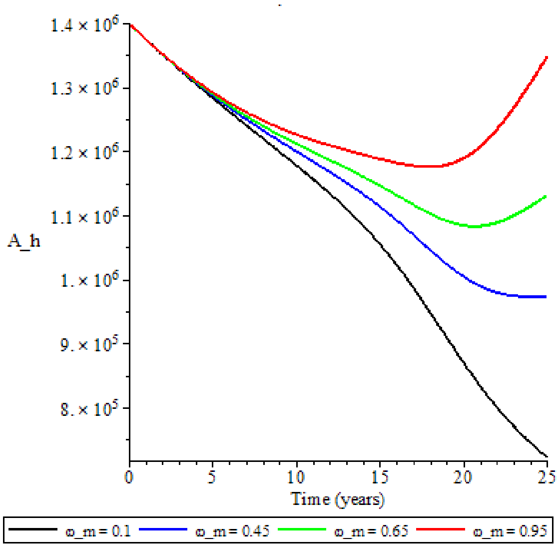

Fig. 3c: Effect of ω_m on A_h.

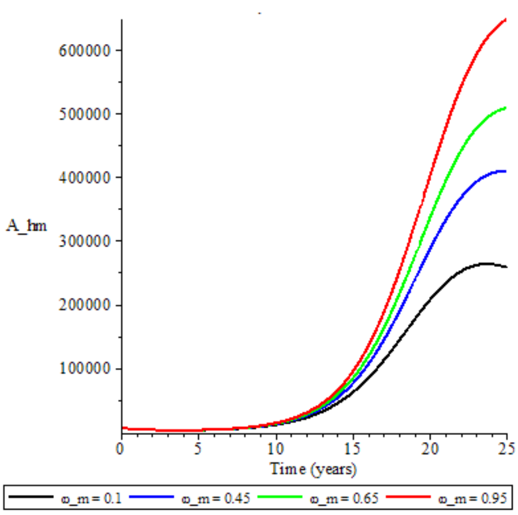

Fig. 3d: Effect of ω_m on A_{hm}.

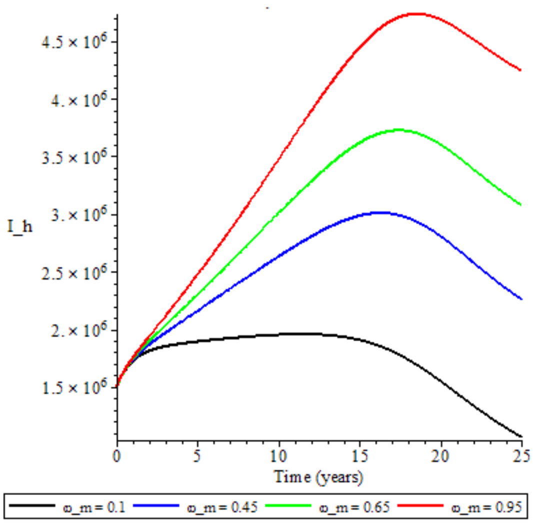

Fig. 3e: Effect of ω_m on I_h.

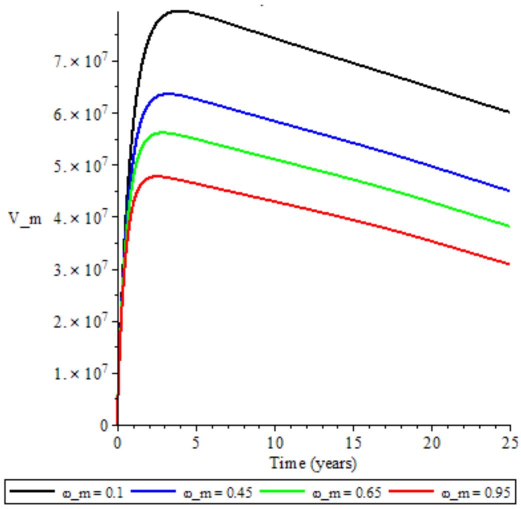

Fig. 3f: Effect of ω_m on V_m.

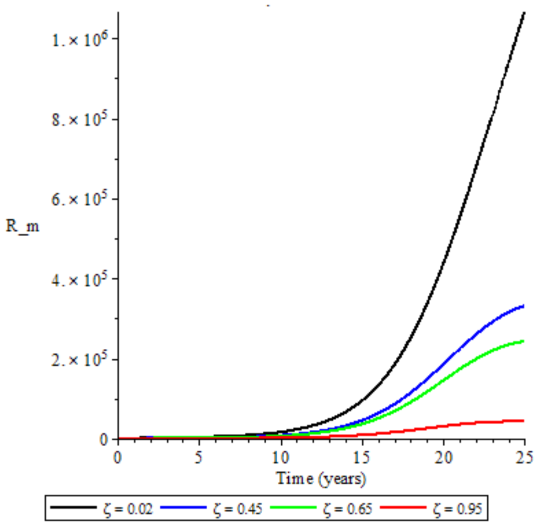

Fig. 4a: Effect of ζ on R_m.

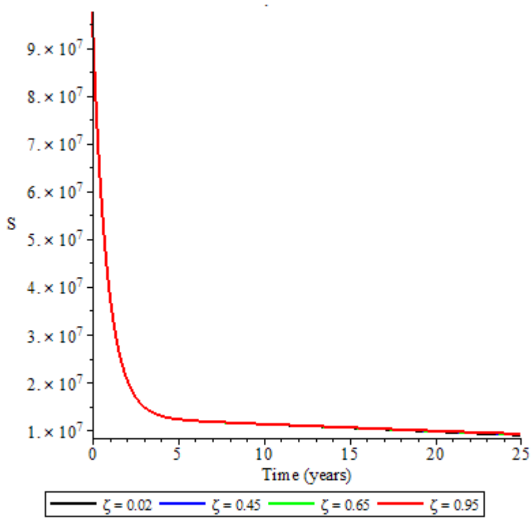

Fig. 4b: Effect of ζ on S.

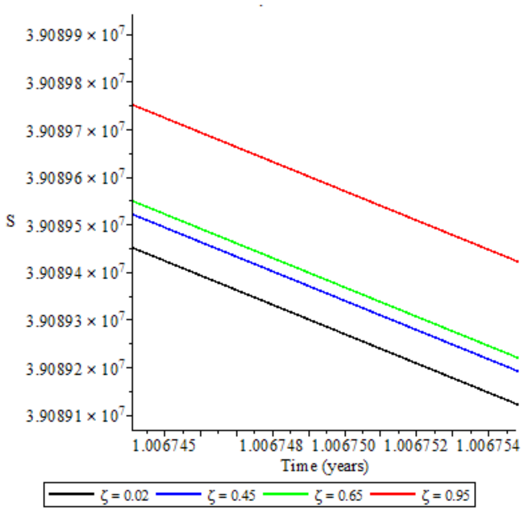

Fig. 4c: Effect of ζ on S (magnified view).

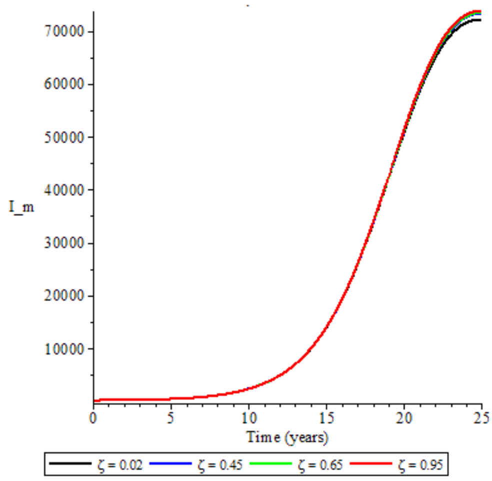

Fig. 4d: Effect of ζ on I_m.

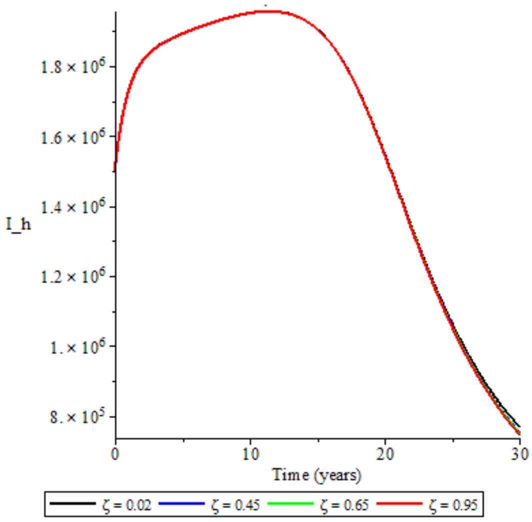

Fig. 4e: Effect of ζ *on* I_h.

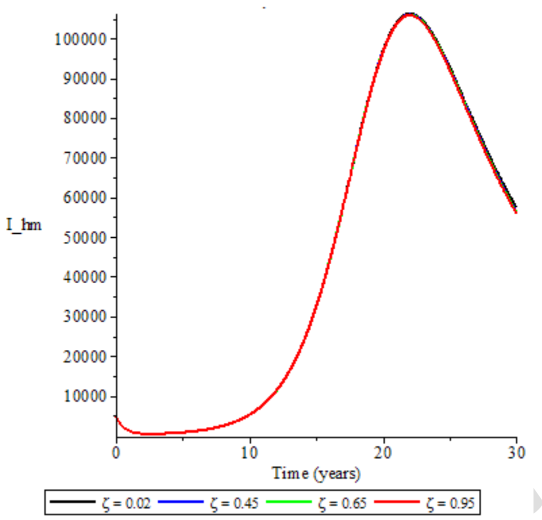

Fig. 4f: Effect of ζ *on* I_{hm}.

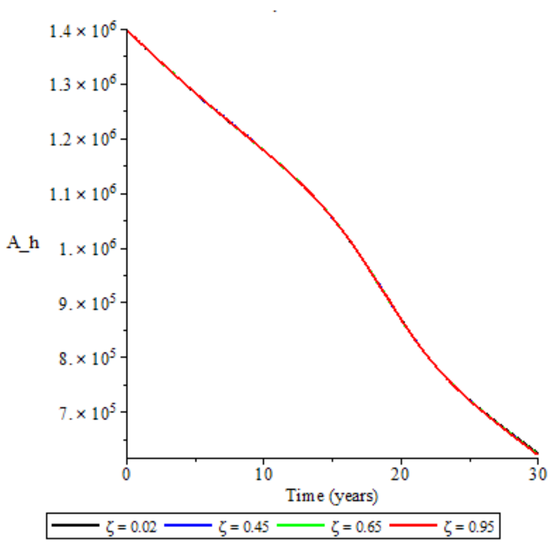

Fig. 4g: Effect of ζ *on* A_h.

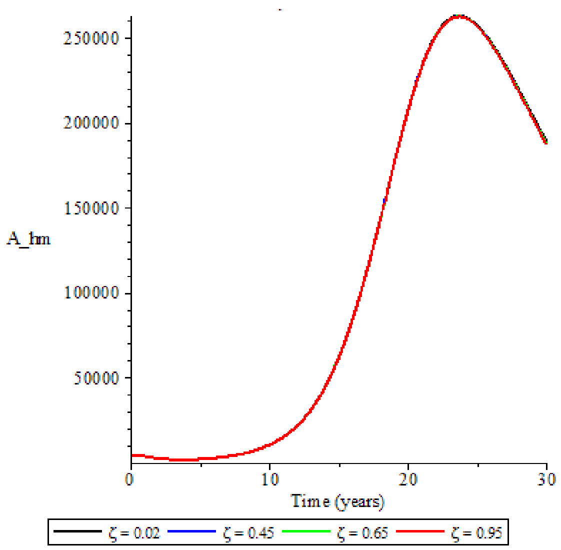

Fig. 4h: Effect of ζ *on* A_{hm}.

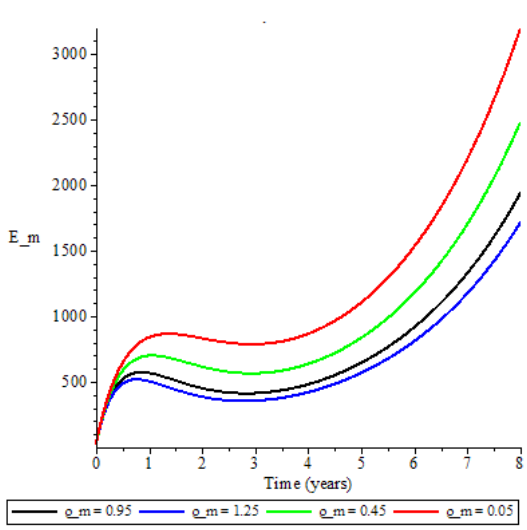

Fig. 5a: Effect of ϱ_m *on* E_m.

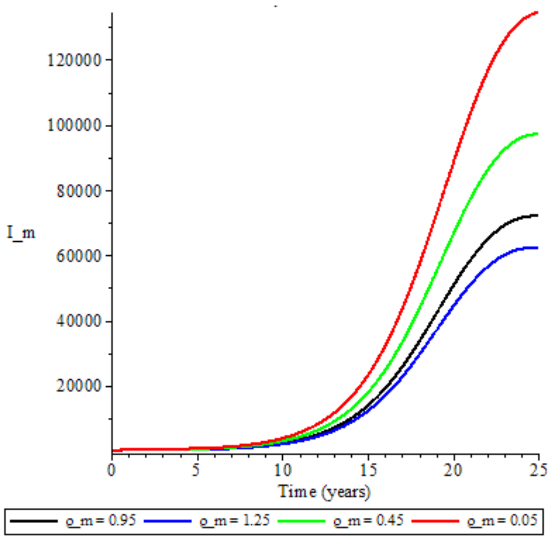

Fig. 5b: Effect of ϱ_m *on* I_m.

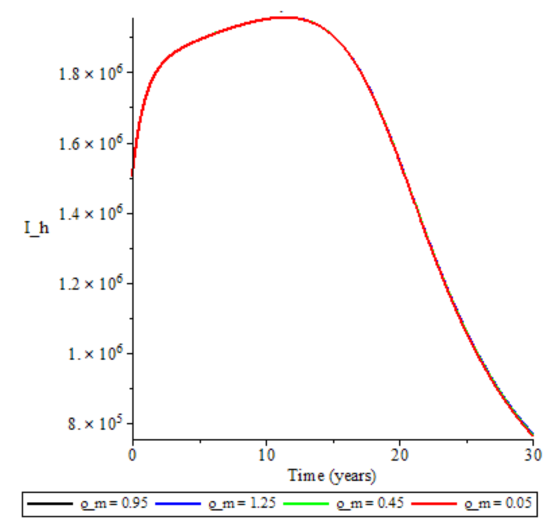

Fig. 5c: Effect of ϱ_m *on* I_h.

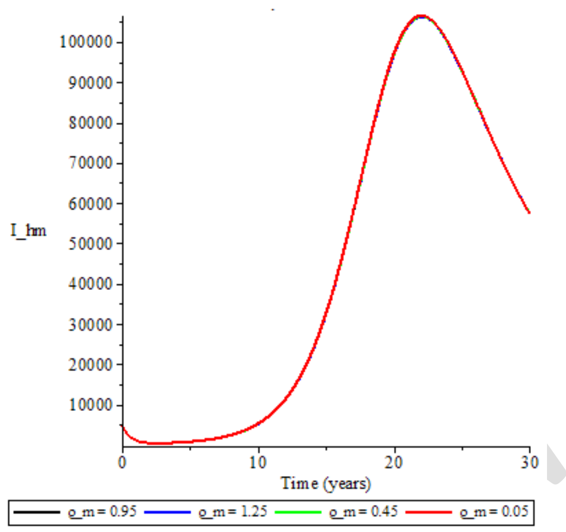

Fig.5d: Effect of ϱ_m *on* I_{hm}.

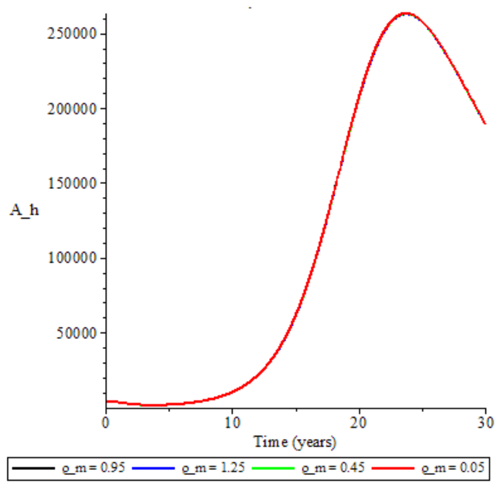

Fig. 5e: Effect of ϱ_m *on* A_h.

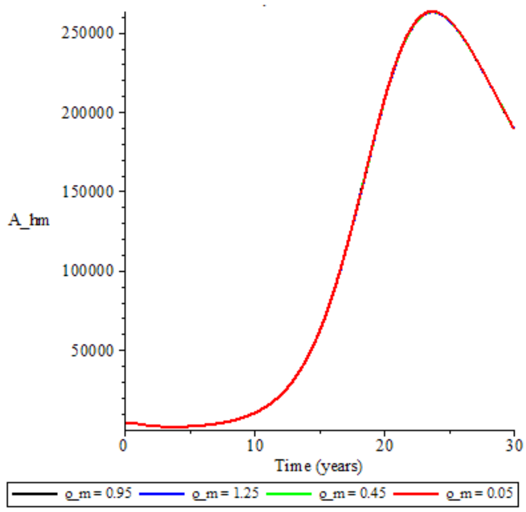

Fig. 5f: Effect of ϱ_m *on* A_{hm}.

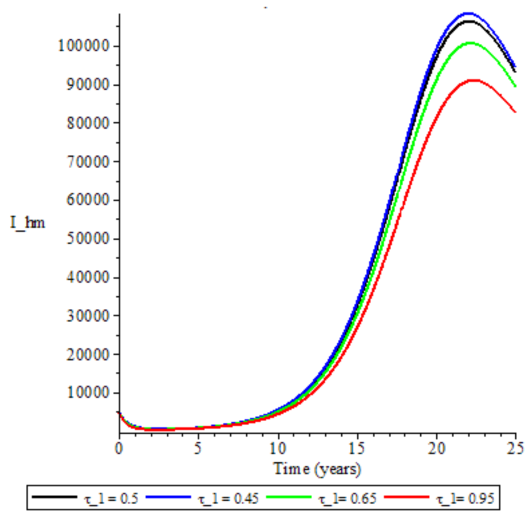

Fig. 6a: Effect of τ_1 *on* I_{hm}.

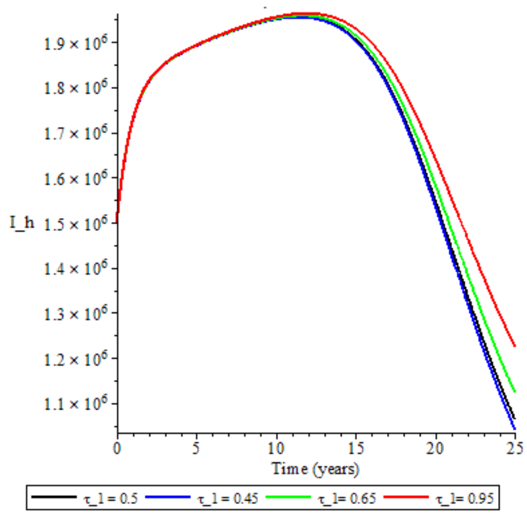

Fig. 6b: Effect of τ_1 *on* I_h.

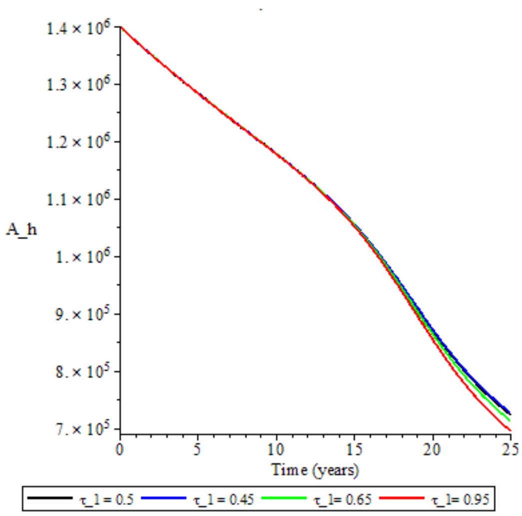

Fig. 6c: Effect of τ_1 on A_h.

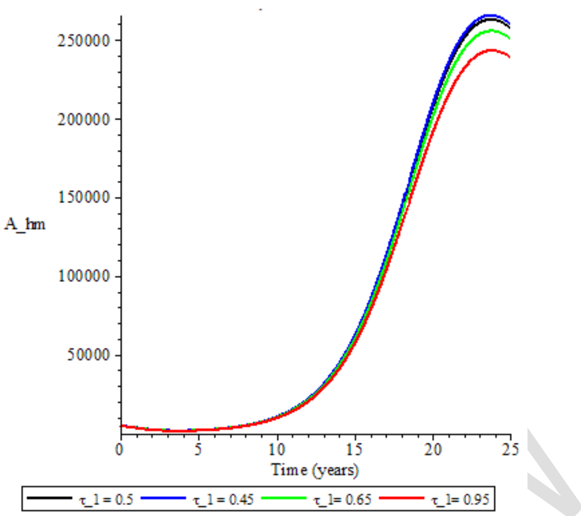

Fig. 6d: Effect of τ_1 on A_{hm}

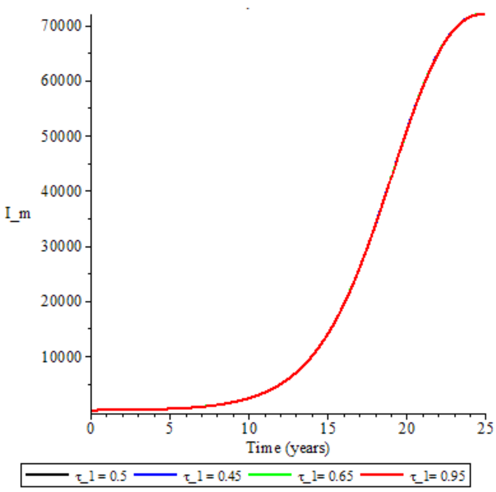

Fig. 6e: Effect of τ_1 on I_m.

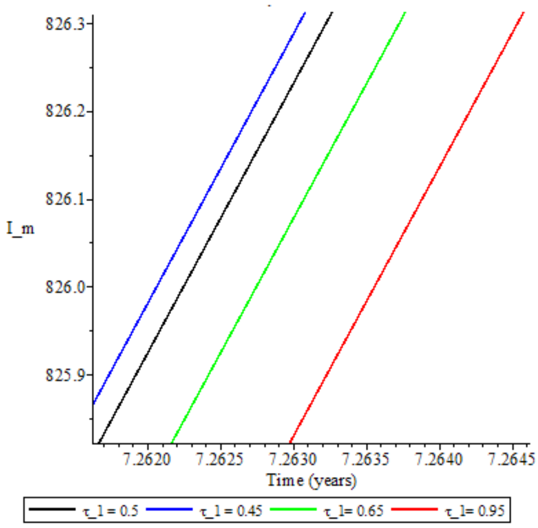

Fig. 6f: Effect of τ_1 on I_m (magnified view)

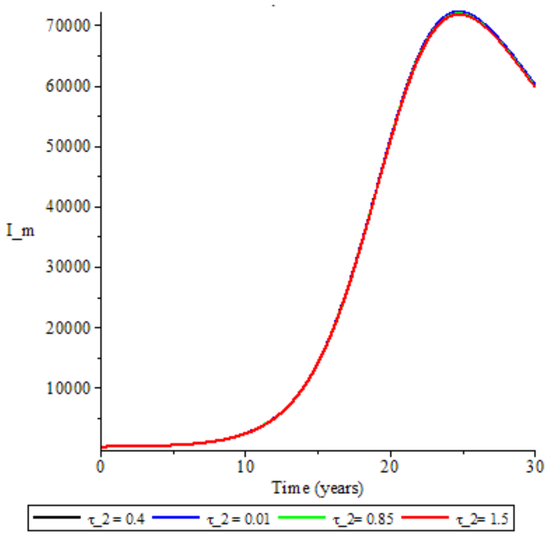

Fig. 7a: Effect of τ_2 on I_m.

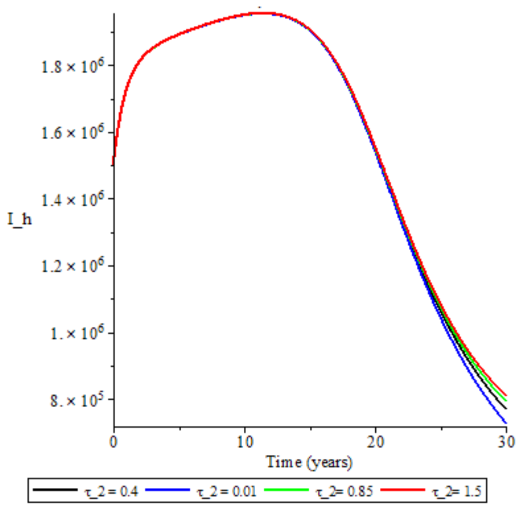

Fig. 7b: Effect of τ_2 on I_h.

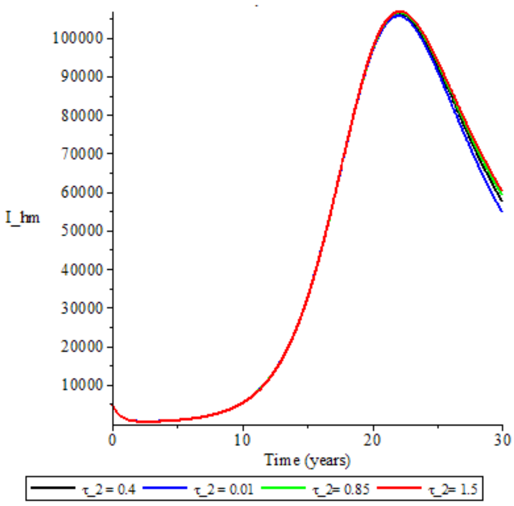

Fig. 7c: Effect of τ_2 on I_{hm}.

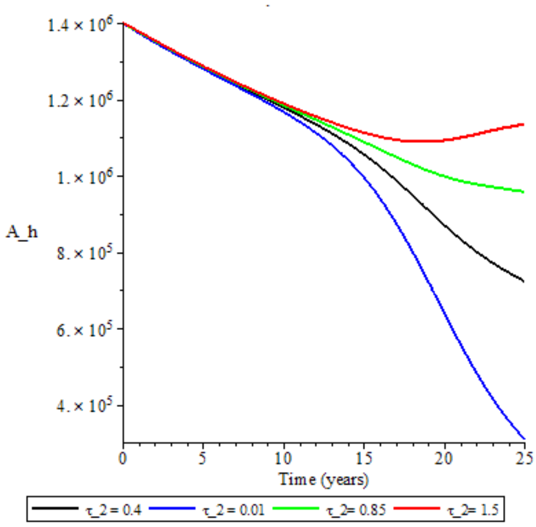

Fig. 7d: Effect of τ_2 on A_h.

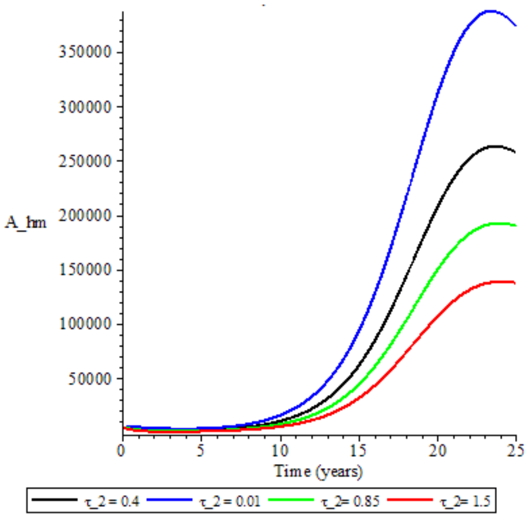

Fig. 7e: Effect of τ_2 on A_{hm}.

5. Discussion and Conclusion

From figure 2a, we observe that the size of the compartment I_h alters with changes in the value of ε, although this is not obvious without a magnification as shown in figure 2b. We observed that when the value of ε is increased, the number of HIV-infected individuals also increased. Figure 3a shows that when the waning rate (ω_m) of the monkeypox vaccine increases, the number of individuals in the compartment (I_m) also increases. Similar results are achieved for the compartments I_{hm}, A_h, A_{hm}, and I_h, as shown in figure 3b, figure 3c, figure 3d and figure 3e respectively. An increase in the value of ω_m reduces the size of the compartment V_m. This is shown in figure3f. Figure 4a reveals that the number of individuals in the compartment $R_m$increases with increase in ζ. It is observed in figure 4c that the number of susceptible humans increases with increasingζ. Figures 4d, 4e, 4f, 4g and 4h reveal that increasing or lowering the value of ζ affects the number of individuals in the compartments I_m, I_h,I_{hm}, A_h and A_{hm} respectively, in a manner to similar compartment S. Figure 5a reveals that lowering the value of ϱ_m causes an increase in the number of individuals in the E_m. Similar result is observed for the class I_m. Observed via some magnifications, figures 5c, 5d, 5e and 5f reveal a similar result for the compartments I_h, I_{hm}, A_h and A_{hm} respectively.

Figure 6a shows that a smaller value of $\tau_1$results in a higher population size for the compartment I_{hm}. Figure 6b shows that a higher value of τ_1 results in a higher population size of the compartment I_h. We observe from figure 6c that an increase in the value of τ_1 results in a reduced population size for the compartment A_h. A similar result exists for the compartment A_{hm} as shown in figure 6d. Figures 6e and 6f reveal that a higher value of τ_1 results in a lower population size for the class I_m. Figure 7a shows that a higher value of τ_2 results in a lower population size of the compartment I_m. Figure 7b reveals that increasing the value of τ_2 causes an increase in size of I_h. A similar trend is shown for the compartment I_{hm} as given in figure 7c. Figure 7d shows that increasing τ_2 increases the size of A_h, while figure 7e reveals that increasing τ_2 lowers the size of A_{hm}.

In this research, a 13-compartment deterministic model was constructed to analyze the co-infection dynamics of HIV/AIDS and monkeypox. The disease-free equilibrium and the basic reproduction numbers of the sub-models and the co-infection model were determined.Thenon-negativity of solutions and the invariant region were established. The Maple 18 programming software was used to simplify some calculations. The expression for the basic reproduction number of monkeypox infection is given as$R_{0_m} = (\beta_m c_m \theta_1 (1 - \delta_m) v_m) \left(\frac{\mu + \omega_m}{\mu + \alpha_m + \omega_m}\right) \left(\frac{1}{(\mu + v_m + \varrho_m)(\mu + d_m + \rho_m)}\right)$, while that of the HIV/AIDS infection is given as$R_{0_h} = \beta_h c_h (1 - \delta_h)(\phi_1 (d_h + \mu) + \phi_3 \rho_1) \frac{1}{(\mu + \rho_1)(d_h + \mu)}$.Lyapunov functions were employed in the stability analysis. The disease-free equilibriums of the monkeypox and the HIV/AIDS sub-models are globally asymptotically stable.Thus, it has been shown that a 13-compartment model of ordinary differential equations can be formulated to study the dynamics of the co-infection of HIV/AIDS and monkeypox infections.

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