A LOGICAL PROOF OF THE POLIGNAC'S CONJECTUREBASED ON PARTITIONS OF AN EVEN NUMBER OF A NEW FORMULATION

Abstract

Polignac's Conjecture, proposed by Alphonse de Polignac in the 19th century, is a captivating hypothesis that extends the notion of twin primes to a broader context. It posits that for any even positive integer $2k$, there exist infinitely many pairs of consecutive prime numbers whose difference is$2k$. This conjecture is a natural generalization of the Twin Prime Conjecture, which focuses solely on pairs of primes differing by two.The conjecture has significant implications for our understanding of the distribution of prime numbers and the nature of their gaps and its exploration serves as a testament to the enduring fascination and mystery surrounding prime numbers and their properties. However, despite extensive efforts by mathematicians over the years, Polignac's Conjecture remains unproven, standing as one of the many unsolved problems in number theory. Thisstudy utilizes a set of all odd partitions generated from an even number of a new formulation, and we show that from this set of all pairs of odd numbers, there exist proper subsets containing infinitely many pairs of prime numbers whose differences is a fixed even gap. Finally, using these results and the facts that the difference of any two prime numbers is even and there exist infinitely many prime numbers, a logical proof of the Polignac's Conjecture is provided.

Keywords: Polignac's Conjecture, Even numbers, Odd numbers, Prime numbers,

1. Introduction

In the realm of number theory, the quest to uncover the mysteries of prime numbers has long been a central focus of mathematical inquiry. Among the intriguing conjectures that have emerged over the centuries, Polignac's Conjecture stands as a compelling hypothesis proposed by Alphonse de Polignac in the 19th century [1]. This conjecture extends the notion of twin primes to a broader context, suggesting that for any even positive integer 2k, there exist infinitely many pairs of consecutive prime numbers whose difference is precisely 2k [2]. In essence, Polignac's Conjecture provides a framework for understanding the distribution of prime numbers with respect to their gaps, offering tantalizing insights into the patterns underlying these fundamental elements of number theory.

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Polignac's Conjecture originates from the long-standing fascination with twin primes, which are pairs of primes that differ by two and have intrigued mathematicians for centuries[3]. By generalizing this concept to arbitrary even differences, Polignac's Conjecture broadens the scope of inquiry, prompting exploration into the regularities and irregularities of prime number pairs[18]. The conjecture not only deepens our understanding of prime number theory but also raises profound questions about the nature of prime gaps and the distribution of primes along the number line [4].

Despite its elegance and intuitive appeal, Polignac's Conjecture remains an enigma, resisting conclusive proof or disproof[19]. Over the years, mathematicians have pursued various strategies and techniques in attempts to unravel the mysteries of Polignac's Conjecture, but the conjecture continues to elude resolution[5]. Its elusive nature underscores the complexity and richness of prime number theory, challenging mathematicians to push the boundaries of their understanding and explore new avenues of inquiry. In this study, we utilize allodd partitions of an even number of a new formulationof the form $(P_1+P_2)+(P_2-P_1)^n$[6], to show that there exist infinitely many prime pairs for every even number $2j \leq (P_1+P_2)+(P_2-P_1)^n$ whose difference equals the even number$2j$. Using these results and the facts that there exist infinitely many prime numbers and the difference between any two prime numbers is even, a logicalproof of Polignac's Conjecture is provided.

2. Preliminaries

Polignac's Conjecture asserts the existence of infinitely many pairs of consecutive primes with prescribed even differences and the conjecture hints at underlying structures within the sequence of prime numbers [7]. This paper combines the results of partitions of an even number of a new formulation [6,8] and the results from the following theorem 1 and theorem 2 that shows that,

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there exist infinitely many prime numbers and the difference of two odd umbers is even respectively. Using these results, we present a logical proof of the Polignac's conjecture.

Theorem 1(Euclid's theorem)

There are infinitely many prime numbers [9,17].

Proof

Given the list of prime numbers $p_1, p_2, \ldots, p_n$, the number $N = (p_1 * p_2 * p_3 * \cdots * p_n) + 1$ must contain a prime factor not among the primes used in its construction. To see this, notice p_1 does not divide N since it leaves a remainder of 1 (or alternatively N/p_1 is clearly not an integer). Similarly, the other pi's does not divide N. We therefore conclude that any finite list of primes is not complete, and therefore there must be infinitely many primes [10,17].

Theorem 2

The difference of two odd numbers is even.

Proof

Let $m = 2k_1 + 1$ and $n = 2k_2 + 1$ $for\ all\ k_1\ and\ k_2 \in Z$, where Z is the set of all integers. We have $m - n = 2k_1 + 1 - (2k_2 + 1) = 2k_1 - 2k_2 = 2(k_1 - k_2)$.Let $k = k_1 - k_2$, then we have $m - n = 2k$, where $k \in Z$ which implies that $m - n$ is even. Therefore, the difference of two odd integers is even [11].

Definition 1

A combination is a mathematical technique that determines the number of possible arrangements in a collection of items where the order of the selection does not matter. In combinations, you can select the items in any order [12].

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2.1 Partitioning an even number of the new formulation into all pairs of odd numbers

Using the new formulation of a set of even numbers as $(P_1+P_2)+(P_2-P_1)^n$ [8], it has been shown that it is always possible to partition any even number into all pairs of odd numbers using the following algorithm [6]:

Let P be the set of all prime numbers, ℕ be the set of all natural numbers and O the set of all odd numbers.

Step 1 : Let P_1 and $P_2 \in P$,then $(P_1+P_2)+(P_2-P_1)^n$ is even, $\forall n \in \mathbb{N}$, and $p_2 > p_1$ [8].

Step 2: Let d be even and belong to the half-open interval $[1, [\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n]]$.

Step 3: Let z_i , y_i and i belong to the set of odd numbers in the half-open interval $[1, [\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n]]$.

With p_1, p_2, d and z_i, we partition $(p_1+p_2)+(p_2-p_1)^n$ as follows:

Partition 1: $((P_1+P_2)+(P_2-P_1)^n)$ - $(d+z_1)=y_1$

Partition 2: $((p_1+p_2)+(p_2-p_1)^n)$- $(d+z_3)=y_3$

Partition 3: $((p_1+p_2)+(p_2-p_1)^n)$- $(d+z_5)=y_5$

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Partition i: $((p_1+p_2)+(p_2-p_1)^n)$- $(d+z_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)})=y_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)}$

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The set of pairs $(d+z_1, y_1), (d+z_3, y_3), (d+z_5, y_5), \ldots,$

$(d+z_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)}, y_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)})$ of odd numbers are all partitions of the even number $(p_1+p_2)+(p_2-p_1)^n$. Since prime numbers greater than 2 are subsets of odd numbers, from this set of pairs of odd numbers, the possibility is that, there exist at least one pair of primes [13]. We are to show that when an even number $(p_1+p_2)+(p_2-p_1)^n$ is partitioned using anymultipleof $d$and the set of odd numbers belonging to the half-open interval [1, [$\frac{1}{2}$($(p_1+p_2)+(p_2-p_1)^n$]], into a set say A containing all pairs of odd numbers, that is, A = { $(d+z_1, y_1), (d+z_3, y_3), (d+z_5, y_5), \ldots, (d+z_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)}, y_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)})$ },

then for every even number $d$in the interval [1, [$\frac{1}{2}$($(p_1+p_2)+(p_2-p_1)^n$]], there exist a proper subset say $B \subset A$containing all prime numbers less than $(p_1+p_2)+(p_2-p_1)^n$ with at least a pair of prime numbers for each value of d in the interval $(1, (p_1+p_2)+(p_2-p_1)^n)$ whose difference is d.

Corollary 1

Let $p_1 and p_2 \in P$, where P is the set of all primes such that $d = p_2 - p_1 > 0$ and let $z_i \in 1 \leq O \leq \frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)$ be the set of odd numbers for $i \in 1 \leq O$ $\leq (\frac{1}{2}(p_1+p_2)+(p_2-p_1)^n)$, then any multiple of d in the range $\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n) < d < (p_1+p_2)+(p_2-p_1)^n$ can be used to generate the same set of pairs of odd numbers or a subset of pairs of odd numbers whose sum is $(p_1+p_2)+(p_2-p_1)^n$. For the set of values of d in the open interval

$[\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n),(p_1+p_2)+(p_2-p_1)^n]$, it is expected that as the value of d gets closer to $(p_1+p_2)+(p_2-p_1)^n$,the pairs of odd numbers whose sum is $(p_1+p_2)+(p_2-p_1)^n$ reduces significantly[6].

2.2 Exploringthe connection between partitions involving odd numbers and Polignac's Conjecture

2.2.1 Statement of the Polignac's Conjecture

For any positive even number $2k,$ there are infinitely many prime gaps of size $2k$. In other words: There are infinitely many cases of two consecutive prime numbers with difference $2k$[15].

Remark 1

In general, in order to obtain all odd partitions associated to the even number $(p_1+p_2)+(p_2-p_1)^n$ as in section 2.1, we can consider during the partitioning of the even number $(p_1+p_2)+(p_2-p_1)^n$, all the multiples of $d$and the set of odd numbers z_i in the interval $(1,(p_1+p_2)+(p_2-p_1)^n)$ that generates positive pairs of odd numbers as follows:

Example 1

Step 1: Let $p_1=13$, $p_2=23$and $n=1$,then

$(p_1+p_2)+(p_2-p_1)^1=(13+23)+(23-13)^1=36+10=46$ is even.

Remark 2

The multiples of $d$belonging to the open interval(1, 46)are

$\{2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34,36,38,40,42,44\}$ and the odd numbers in the same closed-open interval are [1,3,5,7,9,11,13,15,17,19,21,23,25,27,29,31,33,35,37,39,41,43

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,45). In this example 1, we let $d = 4$ and we partition 46 for all odd numbers as follows:

Step 2:set $d = 4$

Step 3: Take odd numbers in the range $1 \le O < (46) \rightarrow \{1,3,5,7,9,11,13,15,17,19,21,23,25,$

$27,29,31,33,35,37,39,41,43,45\}$

Then we partition 46 as follows:

I. $46-(4+1)=41$

II. $46-(4+3)=39$

III. $46-(4+5)=37$

IV. $46-(4+7)=35$

V. $46-(4+9)=33$

VI. $46-(4+11)=31$

VII. $46-(4+13)=29$

VIII. $46-(4+15)=27$

IX. $46-(4+17)=25$

X. $46-(4+19)=23$

XI. $46-(4+21)=21$

XII. $46-(4+23)=19$

XIII. $46-(4+25)=17$

XIV. $46-(4+27)=15$

XV. $46-(4+29)=13$

XVI. $46-(4+31)=11$

XVII. $46-(4+33)=9$

XVIII. $46-(4+35)=7$

XIX. $46-(4+37)=5$

XX. $46-(4+39)=3$

XXI. $46-(4+41)=1$

XXII. $46-(4+43)=-1$

XXIII. $46-(4+45)=-3$

The partitions of 46 are therefore: $((4+1),41),((4+3),39),((4+5),37),((4+7),35),((4+9),33),((4+11),31),((4+13),29),((4+15),27),((4+17),25),((4+$

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$19), 23), ((4 + 21), 21), ((4 + 23), 19), ((4 + 29), 13), ((4 + 31), 11), ((4 + 33), 9), ((4 + 35), 7), ((4 + 37), 5), ((4 + 39), 3), ((4 + 41), 1), ((4 + 43), -1)$ *and* $((4 + 45), -3)$.

Therefore, the set A containing all the positivepairs of odd numbers is the set A = $\{(5,41), (7,39), (9,37), (11,35), (13,33), (15,31), (17,29), (19,27), (21,25), (23,23), (25,21), (27,19), (29,17), (31,15), (33,13), (35,11), (37,9), (39,7), (41,5), (43,3), (45,1)\}$. Since prime numbers greater than 2 are subsets of odd numbers, It follows that there exist a proper subset $B \subset A$containing all prime numbers arranged in ascending order. Notice that in this case the proper subset B ={3,5,7,11,13,17,19,23,29,31,37,41,43}.

What is left now is to show that there exist pairs of prime numbers p and q in the proper subset Bwhose difference is even for each multiple of d in the set {2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34,36,38,40,42,44} .It is also important to note that the largest closest value of d that has been used in generating the sets B_i for all $\forall\, i \in 2n$ (as below) is the value $d_{largest}$ = 43- $d \geq 3$, where 43 is the largest prime number in the set $B$and d is the even number that is closest to 46, which is $d = 40$.This helps to eliminate pairs of odd numberscontaining 1 and the negative pairs ofodd numbers sincePolignac's Conjecture deals with positive prime numbers.

For all the partitions of the even number 46, the following are the pairs of primes for each of the multiples of $d$in the set {2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34,36,38,40,42,44}whose difference is d:

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I. For $d_2 = 2$, the set $B_2 = \{(5,3),(7,5),(13,11),(19,17),(31,29),(43,41)\}$containspairs of consecutive prime numbers less than 46 whose difference is 2.

The prime number pairs within this set are consistent with the Twin Prime Conjecture which posits that there are infinitely many pairs of prime numbers that differ by 2. [16]

II. For $d_4 = 4$, the set $B_4 = \{(7,3),(11,7),(17,13),(23,19),(41,37)\}$containspairs of prime numbers less than 46 whose difference is 4.

III. For $d_6 = 6$, the set $B_6 = \{(11,5),(13,7),(17,11),(19,13),(23,17),(37,31),(43,37)\}$containspairs of prime numbers less than 46 whose difference is 6.

IV. For $d_8 = 8$, the set $B_8 = \{(11,3),(13,5),(17,11),(19,11),(31,23),(37,29)\}$contains pairs of prime numbers less than 46 whose difference is 8.

V. For $d_{10} = 10$, the set $B_{10} = \{(13,3),(17,7),(23,13),(29,19),(41,31)\}$contains pairs of prime numbers less than 46 whose difference is 10.

VI. For $d_{12} = 12$, the set $B_{12} = \{(17,5),(19,7),(23,11),(29,17),(31,19),(41,29),(43,31)\}$contains pairs of prime numbers less than 46 whose difference is 12.

VII. For $d_{14} = 14$, the set $B_{14} = \{(17,3),(19,5),(23,11),(31,17),(37,23),(43,29)\}$ contains pairs of prime numbers less than 46 whose difference is 14.

VIII. For $d_{16} = 16$, the set $B_{16} = \{(19,3),(23,7),(29,13)\}$ contains pairs of prime numbers less than 46 whose difference is 16.

IX. For $d_{18} = 18$, the set $B_{18} = \{(23,5),(29,11),(31,13),(37,19),(41,23)\}$ contains pairs of prime numbers less than 46 whose difference is 18.

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X. For $d_{20} = 20$, the set $B_{20} = \{(23,3), (31,11), (37,17), (43,23)\}$ contains pairs of prime numbers less than 46 whose difference is 20.

XI. For $d_{22} = 22$, the set $B_{22} = \{(29,7), (41,19)\}$ contains pairs of prime numbers less than 46 whose difference is 22.

XII. For $d_{24} = 24$, the set $B_{24} = \{(29,5), (31,7), (37,13), (41,17), (43,19)\}$ contains pairs of prime numbers less than 46 whose difference is 24.

XIII. For $d_{26} = 26$, the set $B_{26} = \{(29,3), (31,5), (37,11), (43,17)\}$ contains pairs of prime numbers less than 46 whose difference is 26.

XIV. For $d_{28} = 28$, the set $B_{28} = \{(31,3), (41,13)\}$ contains pairs of prime numbers less than 46 whose difference is 28.

XV. For $d_{30} = 30$, the set $B_{30} = \{(37,7), (41,11), (43,13)\}$ contains pairs of prime numbers less than 46 whose difference is 30.

XVI. For $d_{32} = 32$, the set $B_{32} = \{(37,5), (43,11)\}$ contains pairs of prime numbers less than 46 whose difference is 32.

XVII. For $d_{34} = 34$, the set $B_{34} = \{(37,3), (41,7)\}$ contains pairs of prime numbers less than 46 whose difference is 34.

XVIII. For $d_{36} = 36$, the set $B_{36} = \{(41,5), (43,7)\}$ contains pairs of prime numbers less than 46 whose difference is 36.

XIX. For $d_{38} = 38$, the set $B_{38} = \{(41,3), (43,5)\}$ contains pairs of prime numbers less than 46 whose difference is 38.

XX. For $d_{40} = 40$, the set $B_{40} = \{(43,3)\}$ contains a pair of prime numbers less than 46 whose difference is 40.

XXI. For $d_{42} = 42$, the set $B_{42} = \{(43, \mathbf{1})\}$ contains a pair that is not prime since 1 is not a prime number.

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XXII. For $d_{44} = 44$, the set $B_{44} = \{(43, \mathbf{-1})\}$ contains a pair that is not prime since -1 is not a prime number.

XXIII. .

Remark 3

We note that for the sets B_i, the largest value of d that generates positive prime pairs is $d = 40$. This explains why we obtain the sets $(43,1)$ for $d_{42} = 42$ in XXI and $(43, -1)$ for $d_{44} = 44$ that do not satisfy Polignac's Conjecture.

According to the Polignac's Conjecture, for every even number k, there are infinitely many pairs of prime numbers p and p' such that $p' - p = k$[14]. Now, we observe that for each of the sets B_i, there exist prime numbers say p and p' such that $p' - p = d_i$, where d_i is a fixed even gap. These results are therefore consistent with the statement of the Polignac's Conjecture.

The sets B_i are finite and therefore contains finite pairs of prime number with the fixed gap d_i. The results from partitioning any even number of the form $(p_1 + p_2) + (p_2 - p_1)^n$ is the set of all pairs of odd numbers whose sum is even and since the sets B_i have been generated from all prime pairs obtained from partitioning 46, it is therefore worth mentioning that the subsequent partitioning of 48,50,52,54,…10000,…1000000,…,100000000 and so on, increments the number of pairs of prime numbers in the sets B_i with the fixed gaps d_i respectively while at the same time increasing the multiples of d in the set $\{2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34,36,38,40,42,44\}$. These results however cannot conclusively confirm that there exist infinitely many such pairs of prime numbers in the sets B_i with the fixed gap d_i. So as to prove the Polignac's Conjecture, we are therefore to show that there exist infinitely many pairs of prime numbers in the sets B_i.

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3. Construction of the proof

Let us consider two finite sets of prime numbers $M = \{p_1, p_2, \ldots, p_L\}$ and $N = \{p_1, p_2, \ldots, p_Z\}$ and the fact that according to Theorem 1, there are infinitely many prime numbers. Therefore, it is expected that for the finite set M, if p_L is the large prime number in the set, then, there is another larger prime number than p_L say $P = (p_1 * p_2 * p_3 \cdots * p_L) + 1$and subsequently, there is another larger prime number than P say $K = (p_1 * p_2 * p_3 \cdots * p_L * \big((p_1 * p_2 * p_3 \cdots * pL + 1\ ) + 1$ and so on. The same argument can also be used with the set N so that the prime number $Q = (p_1 * p_2 * p_3 * \cdots * p_Z) + 1$ is a larger prime number than p_Z and say the prime number $H = (p_1 * p_2 * p_3 * \cdots * p_Z * \big((p_1 * p_2 * p_3 * \cdots * p_Z) + 1)\big) + 1$is larger than Q and so on.This shows we can generate two infinite sets of prime numbers $M^* = \{p_1, p_2, \ldots, p_{L,} P, K, \ldots\}$and $N^* = \{p_1, p_2, \ldots, p_Z, Q, H, \ldots\}$.

From Theorem 2,it has been proven that the difference of any two odd numbers is even and Since prime numbers greater than 2 are subsets of odd numbers, then, the difference of elements of set $M^*$to elements of set $N^*$is even. For the sake of illustration, we consider the differences of elements of both infinite sets M^* and $N^*$for values of prime numbers ≥ 3 (although 2 is a prime number, we exempt it from each of the infinite sets since the difference of 2 with any prime number results to an odd number and not an even number). Therefore, the elements in the infinite setsM^* are: {3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97,..., p_L, $(p_1 * p_2 * p_3 \cdots * p_L) + 1, \boldsymbol{L} = (p_1 * p_2 * p_3 \cdots * p_L * \big((p_1 * p_2 * p_3 \cdots * p_L) + 1\ \big)) + 1\}$. For the infinite sets$M^*$and N^*, we let the prime number $\boldsymbol{L}$be any larger arbitrary prime number. Now ifwe consider the differences of each of the elements in the infinite set N^* with $\boldsymbol{L}$,the differences of each of the elements in the infinite set N^* with $\boldsymbol{L_{-1}}$, where $\boldsymbol{L_{-1}}$ is the second largest arbitrary prime number after $\boldsymbol{L}$, the differences of each of the elements in the infinite set N^* with $\boldsymbol{L_{-2}}$, where $\boldsymbol{L_{-2}}$ is the second arbitrary largest prime number after$\boldsymbol{L_{-1}}$ and so on, we expect that the results of all these differences will be even.

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The following table shows the prime numbers being subtracted from each other and the difference with other primes. The differences follow a pattern based on the subtraction of the prime numbers.

Table 1Polignac's Conjecturetable (excluding 2)

Prime NumbersPairs (M^*, N^*)	**Prime Numbers Being Subtracted**	**Prime Numbers Pairs** (M^*, N^*)	**Prime Numbers Being Subtracted**
3, 5	5 - 3 = 2	5, 7	7 - 5 = 2
3, 7	7 - 3 = 4	5, 11	11 - 5 = 6
3, 11	11 - 3 = 8	5, 13	13 - 5 = 8
3, 13	13 - 3 = 10	5, 17	17 - 5 = 12
3, 17	17 - 3 = 14	5, 19	19 - 5 = 14
. . .	. . .	. . .	. . .
7, 11	11 - 7 = 4	11, 13	13 - 11 = 2

7, 13	13 - 7 = 6	11, 17	17 - 11 = 6
7, 17	17 - 7 = 10	11, 19	19 - 11 = 8
7, 19	19 - 7 = 12	11, 23	23 - 11 = 12
. . .	. . .	. . .	. . .
. . .	. . .	$\boldsymbol{L_{-1}, L_{-2}}$	$\boldsymbol{L_{-1} - L_{-2}}$
. . .	. . .	$\boldsymbol{L, L_{-2}}$	$\boldsymbol{L - L_{-2}}$
. . .	. . .	$\boldsymbol{L, L_{-1}}$	$\boldsymbol{L - L_{-1}}$

From table 1, we observethat the even numbersare obtained from subtracting the prime numbers in the infinite sets $M^*$and N^*. As the process of subtractingthe elements of both infinite sets continues, we eventually get tofinding the differences of very large pairs of arbitrary prime numbers$\boldsymbol{L_{-1} - L_{-2}}$, $\boldsymbol{L - L_{-2}}$ and $\boldsymbol{L - L_{-1}}$whose differences according to theorem 2are even.

Worth noting is that these large prime numbers are chosen arbitrarily, and therefore the differences $\boldsymbol{L_{-1} - L_{-2}}$, $\boldsymbol{L - L_{-2}}$ and $\boldsymbol{L - L_{-1}}$ could be any large arbitrary even numbers. The results in table 1 therefore, confirms that the differences of elements of any two arbitrarily chosen infinites sets M^* and N^* of prime numbers is even.Since an infinite set of prime numbers is not bounded, we expect that eventually as the process of finding the differences of these arbitrary large prime numbers of the two infinite sets continues and we considerall the possible combinations of the differences between all the prime numbers in both infinite sets M^* and N^*, the possibility is that, all the fixed gaps (differences between these prime numbers) will include all even numbers.For instance, in example 1, we notice that all the even numbers less than 46 (except 42 and 44) appear as fixed even gaps with associated prime pairs. Partitioning the even number 48 ensures inclusion of the even numbers 42 and 44 with associated prime primes whose differences are 42 and 44 respectively. In general, by partitioning any arbitrary even number $(p_1 + p_2) + (p_2 - p_1)^n$, we obtain a set of all multiples of $2 \leq (p_1 + p_2) + (p_2 - p_1)^n$ with associated pairs of prime numbers whose differences is each even number in the closed-open interval [2, $(p_1 + p_2) + (p_2 - p_1)^n$) and since there will always be a larger even number than the one in the interval [2, $(p_1 + p_2) + (p_2 - p_1)^n$) , then we can say with certainty that such even numbers with associated prime pairs whose differences is even, will be infinitely many. Using these results, it is therefore possible to proof the Polignac's Conjecture.

4. The Proof of the Polignac's Conjecture

Theorem 3 (The Polignac's Conjecture)

For any positive even number $2k$, there are infinitely many prime gaps of size $2k$.

Proof

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Using a new formulation of set of even numbers $(p_1+p_2)+(p_2-p_1)^n$ [8], it has been shown from example 1 that it is possible to obtain all pairs of odd numbers whose differences are fixed even gaps starting from 2, 4, 6 and so on. The set of all pairs of odd numbers $A=\{$ $(d+z_1, y_1), (d+z_3, y_3), (d+z_5, y_5), ..., (d+z_{((1/2((p_1+p_2)+(p_2-p_1)^n)-1)}, y_{((1/2((p_1+p_2)+(p_2-p_1)^n)-1)})$ $\}$[6] generated from partitioning the even number $(p_1+p_2)+(p_2-p_1)^n$ covers all odd numbers in the range (1, $(p_1+p_2)+(p_2-p_1)^n$). Since prime numbers greater than 2 are subsets of odd numbers, we find that from the set containing all these odd partitions, we obtaina proper subset $B \subset A$ containing all pairs of prime numbers. Since odd numbers are infinitely many, the proper subset $B \subset A$ contains infinitely many prime numbers. From this proper subset, it has further been shown in example 1 that it is possible to generateother proper subsets of pairs of prime numbers whose differences is a fixed even gap d_i for each of the even number in the interval [1,46) with the property that the$d_{largest}$ = 43- $d \geq 3$, where 43 is the largest prime number in the set{3,5,7,11,13,17,19,23,29,31,37,41,43}.

Let us now consider all the partitions of any arbitrary even number $(p_1+p_2)+(p_2-p_1)^n$ and we let the set A contain all the pairs of odd numbers obtained from partitioning $(p_1+p_2)+(p_2-p_1)^n$. Clearly, the Set A contains infinitely many pairs of odd numbers since $(p_1+p_2)+(p_2-p_1)^n$ is any arbitrary even number. From all these odd partitions, then we are certain that there exist a proper subset $B \subset A$ containing infinitely many pairs of prime numbers since prime numbers greater than 2 are subsets of odd numbers and there exist infinitely many prime numbers. From the partitioning, when we further consider the set of all even numbers (multiples of d) in the interval $[1, (p_1+p_2)+(p_2-p_1)^n]$, then there exist proper subsets of B say B_i of pairs of prime numbers such that the differences of the pairs of prime numbers in each

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proper subset $B_{2,} B_4, B_6, \ldots, B_{1000}, \ldots, B_{10000}, \ldots$ and so on are the corresponding even numbers for each $d_i \; \forall i \in 2k$ .The question is however whether there exist infinitely many such pairs of prime numbers in the sets $B_i$whose differences is the corresponding $d_i$since if this is the case, then it would show that for any even number say d in the interval $[1,(p_1+p_2)+(p_2-p_1)^n]$, there exist infinitely many pairs of prime numbers whose difference is d and these results will prove the Polignac's Conjecturewhichposits thatfor any positive even number $2k$, there are infinitely many prime gaps of size $2k$.

The results in table 1 shows that the difference of any two prime numbers is even. So if we consider finding the differences of all prime numbers of two infinite sets of prime numbers with all the possible combinations of the differences of the prime numbers, the result of all these differences will be an infinite set of even numbers. Since we have considered all the possible combinations of the differences of all prime numbers and prime numbers are infinitely many, and the difference of any two prime number is even, we then expect that there would be infinitely many prime number differencesfor every even number.This therefore shows that , for any even number $2k$, there exists a set A of infinitely many pairs of prime numbers and a proper subset $B \subset A$ containing infinitely many pairs of prime numbers. Further, from the subset B, there exist infinitely many proper subsets say $B_i$of pairs of prime numbers such that the differences of all pairs of prime numbers of each proper subset is a fixed even number. The results in table 1 of all the possible combinations of all the differences of prime numbers of two infinite sets of prime numbers ensures that the pairs of prime numbers in each proper subset B_i continues to grow as the even number being partitioned grows bigger and eventually as we get to partitions of a very large arbitrary even number, each proper subset B_i will contain infinitely many pairs of prime numbers whose differencesare the corresponding fixed even gaps for each

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d_i. These results therefore proves the Polignac's Conjecturewhichstates thatfor any positive even number $2k$, there are infinitely many prime gaps of size $2k$.

Conclusion

Using the results in table 1, it has been shown that the difference of any two arbitrary prime numbers will always be even. It has further been shown that when an arbitrary even number is partitioned into all pairs of odd numbers, there exist a proper subset containing all pairs of prime numbers such that from these proper subset, we obtain infinitely many proper subsets containing infinitely many pairs of prime numbers with a fixed even gap. Since the difference of any two infinite sets of prime numbers considering all possible combinations of these differences yield an infinite set of even numbers, these confirms that there exist infinitely many pairs of primes in each proper subset of the set containing all pairs of prime numbers with a fixed even gap. These results therefore proves the Polignac's Conjecturewhichstates thatfor any positive even number $2k$, there are infinitely many prime gaps of size $2k$.

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