On the convolution of the k–Lucas sequences

Abstract

In this paper, we study the iterated convolution of the k–Lucas sequences in a form similar to the iterated convolution of the k–Fibonacci sequences [5].

A particular case is for the self-convolution of these sequences. Moreover the generating functions of all this convolved sequences, we find the recurrence relation between the terms of the resulting sequences.

Keywords: k–Fibonacci and k–Lucas numbers. Convolution. Generating function. Recurrence relation.

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1 Introduction

In [9], the convolved Fibonacci sequences are defined in the form

$$F_n^{(r)} = \sum_{j=0}^{n} F_j F_{n-j}^{(r-1)}$$

with initial condition $F_n^{(0)} = F_n$ and where F_n are the classical Fibonacci numbers.

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The aim of this paper consists of extending this concept to case of the k–Lucas numbers.

Definition 1 *For any integer number $k \geq 1$, the k–Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbf{N}}$ is defined recurrently by:*

$$F_{k,0} = 0,\ \ F_{k,1} = 1,\ \ \textit{and}\ F_{k,n+1} = k\,F_{k,n} + F_{k,n-1}\ \textit{for}\ n \geq 1.$$

The characteristic equation from the definition is $r^2 = k\,r + 1$ whose solutions are $\sigma_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2 = \frac{k-\sqrt{k^2+4}}{2}$, that verify $\sigma_1 \cdot \sigma_2 = -1$, $\sigma_1 + \sigma_2 = k$, $\sigma_1 - \sigma_2 = \sqrt{k^2+4}$, $\sigma^2 = k\,\sigma + 1$, $\sigma_1 > 0$, $\sigma_2 < 0$.

For the properties of the k–Fibonacci numbers, see [7,8]. In particular, the Binet Identity is $F_{k,n} = \dfrac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$.
The generating function of the k–Fibonacci numbers is $f(x) = \dfrac{x}{1 - k\,x - x^2}$.
Finally, we define the k–Fibonacci numbers of negative index as $F_{k,-n} = (-1)^{n+1}F_{k,n}$

Definition 2 *For any integer number $k \geq 1$, the k–Lucas sequence, say $\{L_{k,n}\}_{n\in\mathbf{N}}$, is defined recurrently by* [6]*:*

$$L_{k,0} = 2,\ \ L_{k,1} = k,\ \ \textit{and}\ L_{k,n+1} = k\,L_{k,n} + L_{k,n-1}\ \textit{for}\ n \geq 1.$$

The Binet Identity for the k–Lucas numbers is $L_{k,n} = \sigma_1^n + \sigma_2^n$.
The k–Lucas numbers are related to the k–Fibonacci numbers by the relation $L_{k,n} = F_{k,n-1} + F_{k,n+1}$.
From this relation, it is easy to prove that $F_{k,n} = \dfrac{L_{k,n+1} + L_{k,n-1}}{k^2+4}$.
Since the recurrence relation indicated in the definition is the same for the k–Fibonacci numbers as for the k–Lucas numbers, the denominator of the

generating function is the same for both. The numerator does vary since it depends on the initial conditions: (0, 1) for the k–Fibonacci numbers and (2, k) for the k–Lucas. Then the generating function of the k–Lucas numbers is [2] $f(x) = \dfrac{2 - k\,x}{1 - k\,x - x^2}$.
Moreover, $L_{k,-n} = (-1)^n L_{k,n}$.

1.1 Convolved k–Fibonacci sequences

A convolved k–Fibonacci sequence [5] is obtained by applying a convolution operation to the k–Fibonacci sequence one or more times. Specifically, define $F^{(0)}_{k,n} = F_{k,n}$ and

$$F^{(r)}_{k,n} = \sum_{j=0}^{n} F_{k,j} F^{(r-1)}_{k,n-j} \tag{1}$$

In particular, for $k = 1$ the classical Fibonacci numbers are $F^{(r)}_{k,n} = F^{(r)}_{n}$.
For $k = 2$ we have the Pell numbers, $F^{(r)}_{2,n} = P^{(r)}_{n}$.
For $k = 3, 4, \ldots$, no convolved k–Fibonacci sequence is indexed in OEIS except for $r = 0$.
By induction [?] , we have proven the following identities for the elements $F^{(r)}_{k,n}$ of the convolved k–Fibonacci sequence: $F^{(r)}_{k,n} = 0$ for $0 \leq n \leq r$ and
$F^{(r)}_{k,r+p+1} = \sum_{j=0}^{p} \binom{p-j}{j} \binom{r+p-j}{p-j} k^{p-2j}$.
If $r = 0$, this formula becomes Formula (1) to obtain the k–Fibonacci numbers.

1.2 Recurrence relation

Convolved k–Fibonacci sequences verify the following recurrence relation: for $n, r \geq 1$, $F^{(r)}_{k,n+1} = k\,F^{(r)}_{k,n} + F^{(r)}_{k,n-1} + F^{(r-1)}_{k,n}$ and this formula was proven by induction [**?**].

1.3 Convolved k–Fibonacci numbers and the Fibonacci polynomials

Sequences $F_{k,n}^{(r)}$ are related to the Fibonacci polynomials by the relation $F_{k,n}^{(r)} = \frac{1}{r!}\frac{d^r}{dk^r}F_{k,n}$, where $\frac{d^r}{dk^r}F_{k,n}$ is the derivative of order r with respect to k of the k–Fibonacci numbers of Definition 1.

1.4 Generating function

In [12], formula (2.2.3), the following theorem is proven:

If $f(x)$ and $g(x)$ are the respective generating functions of the sequences u_n and v_n, then $f(x) \cdot g(x)$ is the generating function of the convolution of these sequences.

So, by taking into account the generating function of the k–Fibonacci numbers is $f(x) = \frac{x}{1 - k\,x - x^2}$, the generating function of the convolved k–Fibonacci sequences is

$$f(x,k,r) = \left(\frac{x}{1 - k\,x - x^2}\right)^{r+1} \tag{2}$$

As a special case, $F_{k,n}^{(1)}$ is the self-convolution of the $F_{k,n}$ numbers.

Sometimes, the convolution of the sequences $U = \{u_n\}$ and $V = \{v_n\}$ is represented as $U \otimes V = \{u_n \otimes v_n\}$, so $F_{k,n}^{(1)}$ can be $F_{k,n} \otimes F_{k,n}$.

Theorem 1 *The self-convolution of the k–Fibonacci numbers verifies*

$$F_{k,n}^{(1)} = \sum_{j=0}^{n} F_{k,j}F_{k,n-j} = \frac{n\,L_{k,n} - k\,F_{k,n}}{k^2+4} \tag{3}$$

For instance, for the Pell numbers $F_{2,n} = P_n$ it is $P_n^{(1)} = \dfrac{(n-1)P_n + n\,P_{n-1}}{4}$.

Theorem 2 *The self-convolution of the k–Fibonacci numbers verifies the recurrence relation*

$$F_{k,n+1}^{(1)} = 2k\,F_{k,n}^{(1)} - (k^2-2)F_{k,n-1}^{(1)} - 2k\,F_{k,n-2}^{(1)} - F_{k,n-3}^{(1)} \tag{4}$$

A way to find these recurrences if we take into account that the denominator of the generating function corresponds to the recurrence relation between the terms of the sequence.

The expansion of the respective denominators leads us to the recurrence relation of the corresponding sequence by simply changing x^p by $F_{k,n-p}^{(r)}$. So, from Equation (2), and taking into account $1 = x^0 = F_{k,n}^{(0)}$:

$$\begin{aligned}
r = 0 &\rightarrow 1 - k\,x - x^2 = 0 \rightarrow 1 = k\,x + x^2\\
&\rightarrow F_{k,n}^{(0)} = k\,F_{k,n-1}^{(0)} + F_{k,n-2}^{(0)} \quad (F_{k,n} = k\,F_{k,n-1} + F_{k,n-2})\\
r = 1 &\rightarrow (1 - k\,x - x^2)^2 0 \rightarrow = 1 - 2k + (k^2-2)x^2 + 2kx^3 + x^4 = 0\\
&\rightarrow 1 = 2k - (k^2-2)x^2 - 2kx^3 - x^4\\
&\rightarrow F_{k,n}^{(1)} = 2k\,F_{k,n-1}^{(1)} - (k^2-2)F_{k,n-2}^{(1)} - 2k\,F_{k,n-3}^{(1)} - F_{k,n-4}^{(1)}\\
r = 2 &\rightarrow (1 - k\,x - x^2)^3 = 0 \rightarrow 1 - 3kx - (3-3k^2)x^2 + (6k-k^3)x^3 = 0\\
&\qquad\qquad\qquad +(3-3k^2)x^4 - 3k\,x^5 + x^6 = 0\\
&\rightarrow 1 = 3k\,x + (3-3k^2)x^2 - (6k-k^3)x^3 - (3-3k^2)x^4 + 3k\,x^5 - x^6\\
&\rightarrow F_{k,n}^{(2)} = 3k\,F_{k,n-1}^{(2)} + (3-3k^2)F_{k,n-2}^{(2)} - (6k-k^3)F_{k,n-3}^{(2)}\\
&\qquad\qquad\qquad -(3-3k^2)F_{k,n-4}^{(2)} + 3k\,F_{k,n-5}^{(2)} - F_{k,n-6}^{(2)}
\end{aligned}$$

There are necessarily $2(r+1)$ initial conditions for these relations.

Corollary 1 *For the classical Fibonacci numbers ($k = 1$), the respective relations are*

$$\begin{aligned}
F_n^{(0)} &= F_n = F_{n-1} + F_{n-2}\\
F_n^{(1)} &= F_n \otimes F_n = 2F_{n-1}^{(1)} + F_{n-2}^{(1)} - F_{n-3}^{(1)} - F_{n-4}^{(1)}
\end{aligned}$$

$$F_n^{(2)} = F_n \otimes F_n^{(1)} = 3F_{n-1}^{(2)} - 5F_{n-3}^{(2)} + 3F_{n-5}^{(2)} + F_{n-6}^{(2)}$$

2 On the convolution of the k–Lucas sequences

The convolution of Fibonacci and Lucas numbers has been studied by many authors. The convolution of the k–Fibonacci numbers has also been studied, some of the results of which have been presented in the previous section. We want to apply the results obtained in that case to the k–Lucas sequences.

The definition of the convolved k–Lucas sequences is similar to definition of the convolution of the k–Fibonacci numbers (1):

Definition 3 *With the initial condition* $L_{k,n}^{(0)} = L_{k,n}$*, the convolved* k*–Lucas sequences is defined as* $L_{k,n}^{(r)} = \sum\limits_{j=0}^{n} L_{k,j} L_{k,n-j}^{(r-1)}$

Taking into account that $L_{k,-n} = (-1)^{n+1} L_{k,n}$, it is also $L_{k,-n}^{(r)} = (-1)^{n+1} L_{k,n}^{(r)}$. Moreover, $L_{k,0}^{(r)} = 2^{r+1}$, $L_{k,1}^{(r)} = 2^r (r+1)k$, $L_{k,2}^{(r)} = 2^{r-2}(r+1)(r+4)k^2 + 2^{r+1}(r+1)$.

From the definition,

$$\begin{aligned}
L_k^{(0)} &= L_k = \{L_{k,n}\} = \{2, k, 2+k^2, 3k+k^3, 2+4k^2+k^4, 5k+5k^3+k^5, \ldots\} \\
L_k^{(1)} &= L_k \otimes L_k = \{L_{k,n}^{(1)}\} = \{4, 4k, 8+5k^2, 16k+6k^3, 12+26k^2+7k^4, \ldots\} \\
L_k^{(2)} &= \{L_{k,n}^{(2)}\} = \{8, 12k, 24+18k^2, 60k+25k^3, 48+114k^2+33k^4, \ldots\} \\
L_k^{(3)} &= \{L_{k,n}^{(3)}\} = \{16, 32k, 64+56k^2, 192k+88k^3, 160+416k^2+129k^4, \ldots\} \\
L_k^{(4)} &= \{L_{k,n}^{(4)}\} = \{32, 80k, 160+160k^2, 560k+280k^3, \ldots\}
\end{aligned}$$

For $k = 1$ the classical Lucas sequences [1]

$$\begin{aligned}
L^{(0)} &= L = \{L_n\} = \{2, 1, 3, 4, 7, 11, 18, 29, \ldots\} \\
L^{(1)} &= L \otimes L = \{L_n^{(1)}\} = \{4, 4, 13, 22, 45, 82, 152, 274, \ldots\} \\
L^{(2)} &= \{L_n^{(2)}\} = \{8, 12, 42, 85, 195, 399, 816, 1611, \ldots\}
\end{aligned}$$

$$L^{(3)} = \{L_n^{(3)}\} = \{16, 32, 120, 280, 705, 1588, 3526, 7520, \ldots\}$$
$$L^{(4)} = \{L_n^{(4)}\} = \{32, 80, 320, 840, 2290, 5601, 13355, 30470, \ldots\}$$

The classical Lucas sequence $L = \{2, 1, 3, 4, 7, 11, 18, 29, \ldots\}$ is indexed in the OEIS [10]: A000032 and the self-convolution $L^{(1)} = L \otimes L$ as A099924. No other of these convolutions are indexed in the OEIS.

Theorem 3 *First convolution of the k–Lucas numbers is called the self convolution of these numbers and verify the formula [4]*

$$L_{k,n}^{(1)} = L_{k,n} \otimes L_{k,n} = \sum_{j=0}^{n} L_{k,j} L_{k,n-j} = (n+1)L_{k,n} + 2F_{k,n+1} \tag{5}$$

Proof.

$$\begin{aligned}
L_{k,n}^{(1)} &= \sum_{j=0}^{n} L_{k,j} L_{k,n-j} = \sum_{j=0}^{n} \left(\sigma_1^j + \sigma_2^j\right)\left(\sigma_1^{n-j} + \sigma_2^{n-j}\right) \\
&= \sum_{j=0}^{n} \left(\sigma_1^n + \sigma_2^n + \sigma_1^n \left(\frac{\sigma_2}{\sigma_1}\right)^j + \sigma_2^n \left(\frac{\sigma_1}{\sigma_2}\right)^j\right) \\
&= (n+1)L_{k,n} + \sigma_1^n \sum_{j=0}^{n} \left(\frac{\sigma_2}{\sigma_1}\right)^j + \sigma_2^n \sum_{j=0}^{n} \left(\frac{\sigma_1}{\sigma_2}\right)^j \\
&= (n+1)L_{k,n} + \sigma_1^n \frac{\left(\frac{\sigma_2}{\sigma_1}\right)^{n+1} - 1}{\frac{\sigma_2}{\sigma_1} - 1} + \sigma_2^n \frac{\left(\frac{\sigma_1}{\sigma_2}\right)^{n+1} - 1}{\frac{\sigma_1}{\sigma_2} - 1} \\
&= (n+1)L_{k,n} + \frac{\sigma_2^{n+1} - \sigma_1^{n+1}}{\sigma_2 - \sigma_1} + \frac{\sigma_1^{n+1} - \sigma_2^{n+1}}{\sigma_1 - \sigma_2} \\
\sum_{j=0}^{n} L_{k,j} L_{k,n-j} &= (n+1)L_{k,n} + 2F_{k,n+1}
\end{aligned}$$

Because the denominator of the generating function of the k–Lucas numbers and the k–Fibonacci numbers is the same, the recurrence relation between the terms of the sequences $F_{k,n}^{(r)}$ and $L_{k,n}^{(r)}$ is the same for both convolved sequences. For instance, the terms of the self-convolution of the k–Lucas numbers verify the relation $L_{k,n+1}^{(1)} = 2k\, L_{k,n}^{(1)} - (k^2 - 2)L_{k,n-1}^{(1)} - 2k\, L_{k,n-2}^{(1)} - L_{k,n-3}^{(1)}$ that, in the classical case ($k = 1$) takes the form $L_n^{(1)} = 2L_{n-1}^{(1)} + L_{n-2}^{(1)} - 2L_{n-3}^{(1)} - L_{n-4}^{(1)}$ with initial conditions $L_0^{(1)} = 4$, $L_1^{(1)} = 4$, $L_2^{(1)} = 13$ and $L_3^{(1)} = 22$

Theorem 4 *The convolved k–Lucas numbers verify the recurrence relation*

$$L_{k,n+1}^{(r)} = k\, L_{k,n}^{(r)} + L_{k,n-1}^{(r)} + \sum_{i=0}^{r-1} \left(L_{k,n+1}^{(i)} + L_{k,n-1}^{(i)} \right)$$

Proof. For $r = 1$, from Equation (5):

$$\begin{aligned} L_{k,n+1}^{(1)} &= (n+2)L_{k,n+1} + 2F_{k,n+2} \\ &= (n+2)(k\, L_{k,n} + L_{k,n-1}) + 2\,(k\, F_{k,n+1} + F_{k,n}) \\ &= k\,((n+1)L_{k,n} + 2F_{k,n+1}) + (n\, L_{k,n-1} + 2F_{k,n}) \\ &\quad + (k\, L_{k,n} + L_{k,n-1}) + L_{k,n-1} = k\, L_{k,n}^{(1)} + L_{k,n-1}^{(1)} + L_{k,n+1}^{(0)} + L_{k,n-1}^{(0)} \end{aligned}$$

Let us suppose the formula is true until r. Then

$$\begin{aligned} L_{k,n+1}^{(r+1)} &= \sum_{j=0}^{n+1} L_{k,j} L_{k,n+1-j}^{(r)} \\ &= \sum_{j=0}^{n+1} L_{k,j} \left[k\, L_{k,n-j}^{(r)} + L_{k,n-1-j}^{(r)} + \sum_{i=0}^{r-1} \left(L_{k,n+1-j}^{(i)} + L_{k,n-1-j}^{(i)} \right) \right] \\ &= k \sum_{j=0}^{n+1} L_{k,j} L_{k,n-j}^{(r)} + \sum_{j=0}^{n+1} L_{k,j} L_{k,n-1-j}^{(r)} \\ &\quad + \sum_{i=0}^{r-1} \sum_{j=0}^{n+1} L_{k,j} L_{k,n+1-j}^{(i)} + \sum_{i=0}^{r-1} \sum_{j=0}^{n+1} L_{k,j} L_{k,n-1-j}^{(i)} \\ &= k \sum_{j=0}^{n} L_{k,j} L_{k,n-j}^{(r)} + \sum_{j=0}^{n-1} L_{k,j} L_{k,n-1-j}^{(r)} \\ &\quad + \sum_{i=0}^{r-1} L_{k,n+1}^{(i+1)} + \sum_{i=0}^{r-1} L_{k,n-1}^{(i+1)} \\ &= k\, L_{k,n}^{(r+1)} + L_{k,n-1}^{(r+1)} + \sum_{i=0}^{r} \left(L_{k,n+1}^{(i)} + L_{k,n-1}^{(i)} \right) \end{aligned}$$

Both self-convolutions of the k–Fibonacci and k–Lucas numbers are related to each other by mean of the equation $L_k^{(1)} + (k^2+4)F_k^{(1)} = 2(n+1)L_{k,n}$. That is $\sum_{j=0}^{n} L_{k,j}L_{k,n-j} + (k^2+4)\sum_{j=0}^{n} F_{k,j}F_{k,n-j} = 2(n+1)L_{k,n}$.

3 Second convolved k–Lucas sequence

After studying the self-convolution of the k–Lucas sequence, this section is dedicated to studying the second convolution, that is, $L_k^{(2)} = \left\{L_{k,n}^{(2)}\right\}$.

Theorem 5 *Second convolved k–Lucas sequence verify the relation*

$$L_{k,n}^{(2)} = \frac{n+2}{2}\left((n+1)L_{k,n} + 6F_{k,n+1}\right)$$

Proof. Self convolution of the k–Lucas sequence is $L_k \otimes L_k = \left\{\sum_{j=0}^{n} L_{k,j}L_{k,n-j}\right\}$ where $\sum_{j=0}^{n} L_{k,j}L_{k,n-j} = (n+1)L_{k,n} + 2F_{k,n+1}$. Besides, $\sum_{j=0}^{n} F_{k,j}L_{k,n-j} = (n+1)F_{k,n}$ and therefore $\sum_{j=0}^{n} L_{k,j}F_{k,n+1-j} = (n+2)F_{k,n+1}$.

Finally, $\sum_{j=0}^{n} j\, r^j = \dfrac{r^{n+1}\left(n(r-1)-1\right)+r}{(r-1)^2}$.

Then

$$\begin{aligned}
L_{k,n}^{(2)} &= \sum_{j=0}^{n} L_{k,j}L_{k,n-j}^{(1)} = \sum_{j=0}^{n} L_{k,j}\left((n+1-j)L_{k,n-j} + 2F_{k,n+1-j}\right)\\
&= (n+1)\sum_{j=0}^{n} L_{k,j}L_{k,n-j} - \sum_{j=0}^{n} j\, L_{k,j}L_{k,n-j} + 2\sum_{j=0}^{n} L_{k,j}F_{k,n+1-j}\\
&= (n+1)\left((n+1)L_{k,n} + 2F_{k,n+1}\right) + 2(n+2)F_{k,n+1} - C\\
C &= \sum_{j=0}^{n} j\left(\sigma_1^j + \sigma_2^j\right)\left(\sigma_1^{n-j} + \sigma_2^{n-j}\right)\\
&= \sum_{j=0}^{n}\left(\sigma_1^n + \sigma_2^n + \sigma_1^n\left(\frac{\sigma_2}{\sigma_1}\right)^j + \sigma_2^n\left(\frac{\sigma_1}{\sigma_2}\right)^j\right)\\
&= L_{k,n}\sum_{j=0}^{n} j + \sigma_1^n\sum_{j=0}^{n} j\left(\frac{\sigma_2}{\sigma_1}\right)^j + \sigma_2^n\sum_{j=0}^{n} j\left(\frac{\sigma_1}{\sigma_2}\right)^j
\end{aligned}$$

$$\begin{aligned}
&= \frac{n(n+1)}{2} L_{k,n} + \sigma_1^n \; \frac{\left(\frac{\sigma_2}{\sigma_1}\right)^{n+1}\left(n\,\frac{\sigma_2}{\sigma_1} - n - 1\right) + \frac{\sigma_2}{\sigma_1}}{\left(\frac{\sigma_2}{\sigma_1} - 1\right)^2} \\
&+\sigma_2^n \; \frac{\left(\frac{\sigma_1}{\sigma_2}\right)^{n+1}\left(n\,\frac{\sigma_1}{\sigma_2} - n - 1\right) + \frac{\sigma_1}{\sigma_2}}{\left(\frac{\sigma_1}{\sigma_2} - 1\right)^2} \\
&= \frac{n(n+1)}{2} L_{k,n} - \frac{1}{(\sigma_1 - \sigma_2)^2}\left(\frac{n\,\sigma_2^{n+1} + n\sigma_2^{n-1} + \sigma_2^{n-1}}{\sigma_1} - \sigma_1^n\right) \\
&- \frac{1}{(\sigma_2 - \sigma_1)^2}\left(\frac{n\,\sigma_1^{n+1} + n\sigma_1^{n-1} + \sigma_1^{n-1}}{\sigma_2} - \sigma_2^n\right) \\
&= \frac{n(n+1)}{2} L_{k,n} - \frac{1}{(\sigma_1 - \sigma_2)^2}\left(-n\sigma_2^{n+2} - \sigma_1^n - (n+1)\sigma_2^n\right) \\
&- \frac{1}{(\sigma_1 - \sigma_2)^2}\left(-n\sigma_1^{n+2} - \sigma_2^n - (n+1)\sigma_1^n\right) \\
&= \frac{n(n+1)}{2} L_{k,n} + \frac{1}{k^2+4}\left(n\,L_{k,n+2} + n\,L_{k,n}\right) \\
&= \frac{n(n+1)}{2} L_{k,n} + n\,F_{k,n+1} \\
L_{k,n}^{(2)} &= (n+1)\left((n+1)L_{k,n} + 2F_{k,n+1}\right) + 2(n+2)F_{k,n+1} \\
&-\frac{n(n+1)}{2} L_{k,n} + n\,F_{k,n+1} \\
&= \frac{(n+1)(n+2)}{2} L_{k,n} + (3n+6)F_{k,n+1} \rightarrow \\
&\rightarrow L_{k,n}^{(2)} = \frac{n+2}{2}\left((n+1)L_{k,n} + 6F_{k,n+1}\right)
\end{aligned}$$

The second convolved classical Lucas sequence verifies the recurrence relation $L_n^{(2)} = 3L_{n-1}^{(2)} - 5L_{n-3}^{(2)} + 3L_{n-5}^{(2)} + L_{n-6}^{(2)}$ with initial conditions $L_0^{(2)} = 8$, $L_1^{(2)} = 12$, $L_2^{(2)} = 42$, $L_3^{(2)} = 85$, $L_4^{(2)} = 195$, $L_5^{(2)} = 399$.

3.1 Generating function

It is well known that if $f(x)$ and $g(x)$ are the generating functions of the numerical sequences $A = \{a_n\}$ and $B = \{b_n\}$ respectively, then $f(x) \cdot g(x)$ is the generating function of the convolution of both sequences $A \otimes B$. Therefore, since $l_k(x) = \dfrac{2 - k\,x}{1 - k\,x - x^2}$ is the generating function of the k–Lucas sequence, the generating function of the convolution $L_k^{(r)}$ is the function $l_k(x)^r$. Therefore taking into account that $l(k,x) = \dfrac{2 - k\,x}{1 - k\,x - x^2}$ is the generating function of the k–Lucas numbers, $l(k,x)^2$ is the generating function of their self convolution $L_k \otimes L_k$ and $l(k,x)^3$ that of the second convolution $L_k^{(2)}$ [11].

It is evident that developing this function in series becomes more and more complicated as the value of r increases. The solution is to use a Mathematics program that solves the problem. For example, if using Mathematica©, the small program for the self convolution could have the following form:

- $f[k\_, x\_] := \dfrac{2 - k * x}{1 - k * x - x^2}$
- $r = 2$
- Expand[CoefficientList[Series[f[k, x]^r, x, 0, 10], x]].

When executing the program, the coefficients of the series expansion of $l(x,k,r)$ are obtained, that is, the numerical sequence $L_k(n)$ dependent on the value of k.

If $\%/.k \to a$ is then sorted, the corresponding numerical sequence is obtained, being $a = 1, 2, 3 \ldots$.

You can also directly obtain the numerical sequences for $k = 1, 2, 3 \ldots$, using the command Table[CoefficientList[Series[f[k, x]^r, x, 0, 10], x]],k,3] instead of the previous order.

Conclusions

In this paper, the iterated convolution of the k–Lucas numbers has been studied in a general way and then the first and second convolutions have been studied more specifically. Subsequently, a special dedication has been made to the case of classical Lucas numbers as well as to those of Lucas-Pell. More information on convolutions of sets of sequences defined by linear recurrence relations such as those of the Fibonacci or Lucas form can be found in [3] or [13].

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References

[1] P. Barry, *On integer-sequence-based constructions on generalized Pascal triangles*, Journal of Integer Sequences, 9, Article 06.2.4 (2006). https://cs.uwaterloo.ca/journals/JIS/VOL9/Barry/barry91.pdf

[2] A. Benjamin, *The Lucas triangle recounted*, Applications of Fibonacci Numbers, 11 (William Webb, ed.) Kluwer Academic Publishes (2008)

[3] Dresden, G.; Wang, Y. *A General Convolution Identity*, https://dresden.academic.wlu.edu/files/2021/08/Universal.pdf 2021, 0, 1–14.

[4] M.S. El Naschie, *Notes on superstings and the infinite sums of Fibonacci and Lucas numbers*, Chaos, Solitons & Fractals, 12 (10) (2001), 1937–1940

[5] S. Falcon, *Convolved k–Fibonacci sequences*, Int. Journal of Inn. in Sci. & Math. (IJISM), February 2024.

[6] S. Falcon, *On the k–Lucas numbers*, Int. J. Contemp. Math. Sciences, 6(21), 2011, 1039–1050

[7] S. Falcon, A. Plaza, *On the Fibonacci k-numbers*, Chaos, Solit. & Fract. 320(5), (2007) 1615–24.

[8] S. Falcon, A. Plaza, *The k-Fibonacci sequence and the Pascal 2-triangle*, Chaos, Solit. & Fract. 33(1), (2007) 38–49.

[9] V.E.Hoggat, JR. , M. Bicknell-Johnson, *Fibonacci convolution sequences*, The Fibonacci Quarterly 15(2) (1977), 117–122.

[10] N.J. Sloane, *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org/

[11] N. Robbins, *Vieta's triangular array and a related family of polynomials*, International J. Mayth & Math. Sci. 14(2) (1991), 239–244.

[12] Herbert S. Wilf, *Generating-functionology*, http://www.math.upenn.edu/wilf/DownldGF.html, (1994)

[13] Ye, X.; Zhang, Z. *A common generalization of convolved generalized Fibonacci and Lucas polynomials and its applications*, Appl. Math. Comput. 2017, 306, 31–37.