

On the convolution of the k -Lucas sequences

Abstract

In this paper, we study the iterated convolution of the k -Lucas sequences in a similar form that the iterated convolution of the k -Fibonacci sequences [1].

A particular case is for the self-convolution of these sequences. Moreover the generating functions of all this convolved sequences, we find the recurrence relation between the terms of the resulting sequences.

Keywords: k -Fibonacci and k -Lucas numbers. Convolution. Generating function. Recurrence relation.

MSC2000: 11C20; 11B39; 15A36.

1 Introduction

In [6], the convolved Fibonacci sequences are defined in the form

$$F_n^{(r)} = \sum_{j=0}^n F_j F_{n-j}^{(r-1)}$$

with initial condition $F_n^{(0)} = F_n$ and where F_n are the classical Fibonacci numbers.

The aim of this paper consists of extending this concept to case of the k -Lucas numbers.

Definition 1 For any integer number $k \geq 1$, the k -Fibonacci sequence, say $\{F_{k,n}\}_{n \in \mathbf{N}}$ is defined recurrently by:

$$F_{k,0} = 0, F_{k,1} = 1, \text{ and } F_{k,n+1} = k F_{k,n} + F_{k,n-1} \text{ for } n \geq 1.$$

Characteristic equation from the definition is $r^2 = k r + 1$ whose solutions are $\sigma_1 = \frac{k + \sqrt{k^2 + 4}}{2}$ and $\sigma_2 = \frac{k - \sqrt{k^2 + 4}}{2}$, that verify $\sigma_1 \cdot \sigma_2 = -1$, $\sigma_1 + \sigma_2 = k$, $\sigma_1 - \sigma_2 = \sqrt{k^2 + 4}$, $\sigma^2 = k \sigma + 1$, $\sigma_1 > 0$, $\sigma_2 < 0$.

For the properties of the k -Fibonacci numbers, see [3,4]. In particular, the

Binet Identity is $F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$.

Generating function of the k -Fibonacci numbers is $f(x) = \frac{x}{1 - kx - x^2}$.

Finally, we define the k -Fibonacci numbers of negative index as $F_{k,-n} = (-1)^{n+1} F_{k,n}$

Definition 2 For any integer number $k \geq 1$, the k -Lucas sequence, say $\{L_{k,n}\}_{n \in \mathbf{N}}$, is defined recurrently by [2]:

$$L_{k,0} = 2, L_{k,1} = k, \text{ and } L_{k,n+1} = k L_{k,n} + L_{k,n-1} \text{ for } n \geq 1.$$

The Binet Identity for the k -Lucas numbers is $L_{k,n} = \sigma_1^n + \sigma_2^n$.

The k -Lucas numbers are related to the k -Fibonacci numbers by the relation

$L_{k,n} = F_{k,n-1} + F_{k,n+1}$. From this relation, it is easy to prove that

$$F_{k,n} = \frac{L_{k,n+1} + L_{k,n-1}}{k^2 + 4}.$$

Generating function of the k -Lucas numbers is $f(x) = \frac{2 - kx}{1 - kx - x^2}$.

Moreover, $L_{k,-n} = (-1)^n L_{k,n}$.

1.1 Convolved k -Fibonacci sequences

A convolved k -Fibonacci sequence [1] is obtained applying a convolution operation to the k -Fibonacci sequence one or more times. Specifically, define

$$F_{k,n}^{(0)} = F_{k,n} \text{ and}$$

$$F_{k,n}^{(r)} = \sum_{j=0}^n F_{k,j} F_{k,n-j}^{(r-1)} \tag{1}$$

In particular, for $k = 1$ and the classical Fibonacci numbers, $F_{k,n}^{(r)} = F_n^{(r)}$.

For $k = 2$ and the Pell numbers, $F_{2,n}^{(r)} = P_n^{(r)}$.

For $k = 3, 4, \dots$, no convolved k -Fibonacci sequence is indexed in OEIS except for $r = 0$.

By induction, we can prove the following identities for the elements $F_{k,n}^{(r)}$ of the convolved k -Fibonacci sequence: $F_{k,n}^{(r)} = 0$ for $0 \leq n \leq r$ and

$$F_{k,r+p+1}^{(r)} = \sum_{j=0}^p \binom{p-j}{j} \binom{r+p-j}{p-j} k^{p-2j}.$$

If $r = 0$, this formula becomes Formula (1) to obtain the k -Fibonacci numbers.

1.2 Recurrence relation

Convolved k -Fibonacci sequences verify the following recurrence relation:

for $n, r \geq 1$, $F_{k,n+1}^{(r)} = k F_{k,n}^{(r)} + F_{k,n-1}^{(r)} + F_{k,n}^{(r-1)}$ and this formula was proven by induction.

1.3 Convolved k -Fibonacci numbers and the Fibonacci polynomials

Sequences $F_{k,n}^{(r)}$ are related to the Fibonacci polynomials by the relation

$$F_{k,n}^{(r)} = \frac{1}{r!} \frac{d^r}{dk^r} F_{k,n}, \text{ where } \frac{d^r}{dk^r} F_{k,n} \text{ is the derivative of order } r \text{ with respect to } k$$

of the k -Fibonacci numbers of Definition 1.

1.4 Generating function

In [7], formula (2.2.3), the following theorem is proven:

If $f(x)$ and $g(x)$ are the respective generating functions of the sequences u_n and v_n , then $f(x) \cdot g(x)$ is the generating function of the convolution of these sequences.

So, and taking into account the generating function of the k -Fibonacci numbers is $f(x) = \frac{x}{1 - kx - x^2}$, the generating function of the convolved k -Fibonacci sequences is

$$f(x, r) = \left(\frac{x}{1 - kx - x^2} \right)^{r+1} \tag{2}$$

As a special case, $F_{k,n}^{(1)}$ is the self-convolution of the $F_{k,n}$ numbers.

Sometimes, the convolution of the sequences $U = \{u_n\}$ and $V = \{v_n\}$ is represented as $U \odot V = \{u_n \cdot v_n\}$, so $F_{k,n}^{(1)}$ can be $F_{k,n} \odot F_{k,n}$.

Theorem 3 *The self-convolution of the k -Fibonacci numbers verifies*

$$F_{k,n}^{(1)} = \sum_{j=0}^n F_{k,j} F_{k,n-j} = \frac{n L_{k,n} - k F_{k,n}}{k^2 + 4} \tag{3}$$

For instance, for the Pell numbers $F_{2,n} = P_n$ it is $P_n^{(1)} = \frac{(n-1)P_n + n P_{n-1}}{4}$.

Theorem 4 *The self-convolution of the k -Fibonacci numbers verifies the recurrence relation*

$$F_{k,n+1}^{(1)} = 2k F_{k,n}^{(1)} - (k^2 - 2) F_{k,n-1}^{(1)} - 2k F_{k,n-2}^{(1)} - F_{k,n-3}^{(1)} \tag{4}$$

A way to find these recurrences if we take into account that the denominator of the generating function corresponds to the recurrence relation between the terms of the sequence.

The expansion of the respective denominators leads us to the recurrence relation of the corresponding sequence without more than changing x^p by $F_{k,n-p}^{(r)}$. So, from Equation (2), and taking into account $1 = x^0 = F_{k,n}^{(0)}$:

$$\begin{aligned}
 r = 0 &\rightarrow 1 - kx - x^2 = 0 \rightarrow 1 = kx + x^2 \\
 &\rightarrow F_{k,n}^{(0)} = kF_{k,n-1}^{(0)} + F_{k,n-2}^{(0)} \quad (F_{k,n} = kF_{k,n-1} + F_{k,n-2}) \\
 r = 1 &\rightarrow (1 - kx - x^2)^2 = 0 \rightarrow 1 - 2kx + (k^2 - 2)x^2 + 2kx^3 + x^4 = 0 \\
 &\rightarrow 1 = 2kx - (k^2 - 2)x^2 - 2kx^3 - x^4 \\
 &\rightarrow F_{k,n}^{(1)} = 2kF_{k,n-1}^{(1)} - (k^2 - 2)F_{k,n-2}^{(1)} - 2kF_{k,n-3}^{(1)} - F_{k,n-4}^{(1)} \\
 r = 2 &\rightarrow (1 - kx - x^2)^3 = 0 \rightarrow 1 - 3kx - (3 - 3k^2)x^2 + (6k - k^3)x^3 = 0 \\
 &\quad + (3 - 3k^2)x^4 - 3kx^5 + x^6 = 0 \\
 &\rightarrow 1 = 3kx + (3 - 3k^2)x^2 - (6k - k^3)x^3 - (3 - 3k^2)x^4 + 3kx^5 - x^6 \\
 &\rightarrow F_{k,n}^{(2)} = 3kF_{k,n-1}^{(2)} + (3 - 3k^2)F_{k,n-2}^{(2)} - (6k - k^3)F_{k,n-3}^{(2)} \\
 &\quad - (3 - 3k^2)F_{k,n-4}^{(2)} + 3kF_{k,n-5}^{(2)} - F_{k,n-6}^{(2)}
 \end{aligned}$$

There are necessary $2(r + 1)$ initial conditions for these relations.

Corollary 1 *For the classical Fibonacci numbers ($k = 1$), the respective relations are*

$$\begin{aligned}
 F_n^{(0)} &= F_n = F_{n-1} + F_{n-2} \\
 F_n^{(1)} &= F_n \otimes F_n = 2F_{n-1}^{(1)} + F_{n-2}^{(1)} - F_{n-3}^{(1)} - F_{n-4}^{(1)} \\
 F_n^{(2)} &= F_n \otimes F_n^{(1)} = 3F_{n-1}^{(2)} - 5F_{n-3}^{(2)} + 3F_{n-5}^{(2)} + F_{n-6}^{(2)}
 \end{aligned}$$

2 On the convolution of the k -Lucas sequences

We will dedicate this section to apply the preceding results to the case of the k -Lucas numbers.

Definition of the convolved k -Lucas sequences is similar to definition of the convolution of the k -Fibonacci numbers (1):

Definition 1 *With the initial condition $L_{k,n}^{(0)} = L_{k,n}$, the convolved k -Lucas sequences is defined as $L_{k,n}^{(r)} = \sum_{j=0}^n L_{k,j} L_{k,n-j}^{(r-1)}$*

Taking into account that $L_{k,-n} = (-1)^{n+1} L_{k,n}$, it is also $L_{k,-n}^{(r)} = (-1)^{n+1} L_{k,n}^{(r)}$.
 Moreover, $L_{k,0}^{(r)} = 2^{r+1}$, $L_{k,1}^{(r)} = 2^r(r+1)k$, $L_{k,2}^{(r)} = 2^{r-2}(r+1)(r+4)k^2 + 2^{r+1}(r+1)$.

From the definition,

$$\begin{aligned} L_k^{(0)} &= \{L_{k,n}\} = \{2, k, 2 + k^2, 3k + k^3, 2 + 4k^2 + k^4, 5k + 5k^3 + k^5, \dots\} \\ L_k^{(1)} &= \{L_{k,n}^{(1)}\} = \{4, 4k, 8 + 5k^2, 16k + 6k^3, 12 + 26k^2 + 7k^4, \dots\} \\ L_k^{(2)} &= \{L_{k,n}^{(2)}\} = \{8, 12k, 24 + 18k^2, 60k + 25k^3, 48 + 114k^2 + 33k^4, \dots\} \\ L_k^{(3)} &= \{L_{k,n}^{(3)}\} = \{16, 32k, 64 + 56k^2, 192k + 88k^3, 160 + 416k^2 + 129k^4, \dots\} \\ L_k^{(4)} &= \{L_{k,n}^{(4)}\} = \{32, 80k, 160 + 160k^2, 560k + 280k^3, \dots\} \end{aligned}$$

For $k = 1$ and the classical Lucas sequences

$$\begin{aligned} L_k^{(0)} &= \{L_{k,n}\} = \{2, 1, 3, 4, 7, 11, 18, 29, \dots\} \\ L_k^{(1)} &= \{L_{k,n}^{(1)}\} = \{4, 4, 13, 22, 45, 82, 152, 274, \dots\} \\ L_k^{(2)} &= \{L_{k,n}^{(2)}\} = \{8, 12, 42, 85, 195, 399, 816, 1611, \dots\} \\ L_k^{(3)} &= \{L_{k,n}^{(3)}\} = \{16, 32, 120, 280, 705, 1588, 3526, 7520, \dots\} \\ L_k^{(4)} &= \{L_{k,n}^{(4)}\} = \{32, 80, 320, 840, 2290, 5601, 13355, 30470, \dots\} \end{aligned}$$

Classical Lucas sequence $L = \{2, 1, 3, 4, 7, 11, 18, 29, \dots\}$ is indexed in the OEIS [5]: A000032 and the self-convolution $L^{(1)} = L \otimes L$ as A099924. No other of these convolutions are indexed in the OEIS.

Theorem 5 *First convolution of the k -Lucas numbers is called the self con-*

volution of these numbers and verify the formula

$$L_{k,n}^{(1)} = L_{k,n} \otimes L_{k,n} = \sum_{j=0}^n L_{k,j} L_{k,n-j} = (n+1)L_{k,n} + 2F_{k,n+1} \quad (5)$$

Proof.

$$\begin{aligned} L_{k,n}^{(1)} &= \sum_{j=0}^n L_{k,j} L_{k,n-j} = \sum_{j=0}^n (\sigma_1^j + \sigma_2^j) (\sigma_1^{n-j} + \sigma_2^{n-j}) \\ &= \sum_{j=0}^n \left(\sigma_1^n + \sigma_2^n + \sigma_1^n \left(\frac{\sigma_2}{\sigma_1} \right)^j + \sigma_2^n \left(\frac{\sigma_1}{\sigma_2} \right)^j \right) \\ &= (n+1)L_{k,n} + \sigma_1^n \sum_{j=0}^n \left(\frac{\sigma_2}{\sigma_1} \right)^j + \sigma_2^n \sum_{j=0}^n \left(\frac{\sigma_1}{\sigma_2} \right)^j \\ &= (n+1)L_{k,n} + \sigma_1^n \frac{\left(\frac{\sigma_2}{\sigma_1} \right)^{n+1} - 1}{\frac{\sigma_2}{\sigma_1} - 1} + \sigma_2^n \frac{\left(\frac{\sigma_1}{\sigma_2} \right)^{n+1} - 1}{\frac{\sigma_1}{\sigma_2} - 1} \\ &= (n+1)L_{k,n} + \frac{\sigma_2^{n+1} - \sigma_1^{n+1}}{\sigma_2 - \sigma_1} + \frac{\sigma_1^{n+1} - \sigma_2^{n+1}}{\sigma_1 - \sigma_2} \\ \sum_{j=0}^n L_{k,j} L_{k,n-j} &= (n+1)L_{k,n} + 2F_{k,n+1} \end{aligned}$$

Because the denominator of the generating function of the k -Lucas numbers and the k -Fibonacci numbers is the same, the recurrence relation between the terms of the sequences $F_{k,n}^{(r)}$ and $L_{k,n}^{(r)}$ is the same for both convolved sequences. For instance, the terms of the self-convolution of the k -Lucas numbers verify the relation $L_{k,n+1}^{(1)} = 2k L_{k,n}^{(1)} - (k^2 - 2)L_{k,n-1}^{(1)} - 2k L_{k,n-2}^{(1)} - L_{k,n-3}^{(1)}$ that, in the classical case ($k = 1$) takes the form $L_n^{(1)} = 2L_{n-1}^{(1)} + L_{n-2}^{(1)} - 2L_{n-3}^{(1)} - L_{n-4}^{(1)}$ with initial conditions $L_0^{(1)} = 4$, $L_1^{(1)} = 4$, $L_2^{(1)} = 13$ and $L_3^{(1)} = 22$

Theorem 6 *Convolved k -Lucas numbers verify the recurrence relation*

$$L_{k,n+1}^{(r)} = k L_{k,n}^{(r)} + L_{k,n-1}^{(r)} + \sum_{i=0}^{r-1} \left(L_{k,n+1}^{(i)} + L_{k,n-1}^{(i)} \right)$$

Proof. For $r = 1$, from Equation (5):

$$\begin{aligned}
 L_{k,n+1}^{(1)} &= (n+2)L_{k,n+1} + 2F_{k,n+2} \\
 &= (n+2)(kL_{k,n} + L_{k,n-1}) + 2(kF_{k,n+1} + F_{k,n}) \\
 &= k((n+1)L_{k,n} + 2F_{k,n+1}) + (nL_{k,n-1} + 2F_{k,n}) \\
 &\quad + (kL_{k,n} + L_{k,n-1}) + L_{k,n-1} = kL_{k,n}^{(1)} + L_{k,n-1}^{(1)} + L_{k,n+1}^{(0)} + L_{k,n-1}^{(0)}
 \end{aligned}$$

Let us suppose the formula is true until r . Then

$$\begin{aligned}
 L_{k,n+1}^{(r+1)} &= \sum_{j=0}^{n+1} L_{k,j} L_{k,n+1-j}^{(r)} \\
 &= \sum_{j=0}^{n+1} L_{k,j} \left[kL_{k,n-j}^{(r)} + L_{k,n-1-j}^{(r)} + \sum_{i=0}^{r-1} (L_{k,n+1-j}^{(i)} + L_{k,n-1-j}^{(i)}) \right] \\
 &= k \sum_{j=0}^{n+1} L_{k,j} L_{k,n-j}^{(r)} + \sum_{j=0}^{n+1} L_{k,j} L_{k,n-1-j}^{(r)} \\
 &\quad + \sum_{i=0}^{r-1} \sum_{j=0}^{n+1} L_{k,j} L_{k,n+1-j}^{(i)} + \sum_{i=0}^{r-1} \sum_{j=0}^{n+1} L_{k,j} L_{k,n-1-j}^{(i)} \\
 &= k \sum_{j=0}^n L_{k,j} L_{k,n-j}^{(r)} + \sum_{j=0}^{n-1} L_{k,j} L_{k,n-1-j}^{(r)} \\
 &\quad + \sum_{i=0}^{r-1} L_{k,n+1}^{(i+1)} + \sum_{i=0}^{r-1} L_{k,n-1}^{(i+1)} \\
 &= kL_{k,n}^{(r+1)} + L_{k,n-1}^{(r+1)} + \sum_{i=0}^r (L_{k,n+1}^{(i)} + L_{k,n-1}^{(i)})
 \end{aligned}$$

Both self-convolutions of the k -Fibonacci and k -Lucas numbers are related to each other by mean of the equation $L_k^{(1)} + (k^2 + 4)F_k^{(1)} = 2(n+1)L_{k,n}$. That is $\sum_{j=0}^n L_{k,j} L_{k,n-j} + (k^2 + 4) \sum_{j=0}^n F_{k,j} F_{k,n-j} = 2(n+1)L_{k,n}$.

3 Second convolved k -Lucas sequence

After studying the self-convolution of the k -Lucas sequence, this section is dedicated to studying the second convolution, that is, $L_k^{(2)} = \{L_{k,n}^{(2)}\}$.

Theorem 7 *Second convolved k -Lucas sequence verify the relation*

$$L_{k,n}^{(2)} = \frac{n+2}{2} ((n+1)L_{k,n} + 6F_{k,n+1})$$

Proof. Self convolution of the k -Lucas sequence is $L_k \otimes L_k = \left\{ \sum_{j=0}^n L_{k,j} L_{k,n-j} \right\}$

where $\sum_{j=0}^n L_{k,j} L_{k,n-j} = (n+1)L_{k,n} + 2F_{k,n+1}$. Besides, $\sum_{j=0}^n F_{k,j} L_{k,n-j} = (n+1)F_{k,n}$ and therefore $\sum_{j=0}^n L_{k,j} F_{k,n+1-j} = (n+2)F_{k,n+1}$.

Finally, $\sum_{j=0}^n j r^j = \frac{r^{n+1}(n(r-1)-1)+r}{(r-1)^2}$.

Then

$$\begin{aligned} L_{k,n}^{(2)} &= \sum_{j=0}^n L_{k,j} L_{k,n-j}^{(1)} = \sum_{j=0}^n L_{k,j} ((n+1-j)L_{k,n-j} + 2F_{k,n+1-j}) \\ &= (n+1) \sum_{j=0}^n L_{k,j} L_{k,n-j} - \sum_{j=0}^n j L_{k,j} L_{k,n-j} + 2 \sum_{j=0}^n L_{k,j} F_{k,n+1-j} \\ &= (n+1) ((n+1)L_{k,n} + 2F_{k,n+1}) + 2(n+2)F_{k,n+1} - C \\ C &= \sum_{j=0}^n j (\sigma_1^j + \sigma_2^j) (\sigma_1^{n-j} + \sigma_2^{n-j}) \\ &= \sum_{j=0}^n \left(\sigma_1^n + \sigma_2^n + \sigma_1^n \left(\frac{\sigma_2}{\sigma_1}\right)^j + \sigma_2^n \left(\frac{\sigma_1}{\sigma_2}\right)^j \right) \\ &= L_{k,n} \sum_{j=0}^n j + \sigma_1^n \sum_{j=0}^n j \left(\frac{\sigma_2}{\sigma_1}\right)^j + \sigma_2^n \sum_{j=0}^n j \left(\frac{\sigma_1}{\sigma_2}\right)^j \\ &= \frac{n(n+1)}{2} L_{k,n} + \sigma_1^n \frac{\left(\frac{\sigma_2}{\sigma_1}\right)^{n+1} \left(n \frac{\sigma_2}{\sigma_1} - n - 1\right) + \frac{\sigma_2}{\sigma_1}}{\left(\frac{\sigma_2}{\sigma_1} - 1\right)^2} \\ &\quad + \sigma_2^n \frac{\left(\frac{\sigma_1}{\sigma_2}\right)^{n+1} \left(n \frac{\sigma_1}{\sigma_2} - n - 1\right) + \frac{\sigma_1}{\sigma_2}}{\left(\frac{\sigma_1}{\sigma_2} - 1\right)^2} \end{aligned}$$

$$\begin{aligned}
 &= \frac{n(n+1)}{2} L_{k,n} - \frac{1}{(\sigma_1 - \sigma_2)^2} \left(\frac{n\sigma_2^{n+1} + n\sigma_2^{n-1} + \sigma_2^{n-1}}{\sigma_1} - \sigma_1^n \right) \\
 &\quad - \frac{1}{(\sigma_2 - \sigma_1)^2} \left(\frac{n\sigma_1^{n+1} + n\sigma_1^{n-1} + \sigma_1^{n-1}}{\sigma_2} - \sigma_2^n \right) \\
 &= \frac{n(n+1)}{2} L_{k,n} - \frac{1}{(\sigma_1 - \sigma_2)^2} \left(-n\sigma_2^{n+2} - \sigma_1^n - (n+1)\sigma_2^n \right) \\
 &\quad - \frac{1}{(\sigma_1 - \sigma_2)^2} \left(-n\sigma_1^{n+2} - \sigma_2^n - (n+1)\sigma_1^n \right) \\
 &= \frac{n(n+1)}{2} L_{k,n} + \frac{1}{k^2 + 4} (n L_{k,n+2} + n L_{k,n}) \\
 &= \frac{n(n+1)}{2} L_{k,n} + n F_{k,n+1} \\
 L_{k,n}^{(2)} &= (n+1) ((n+1)L_{k,n} + 2F_{k,n+1}) + 2(n+2)F_{k,n+1} \\
 &\quad - \frac{n(n+1)}{2} L_{k,n} + n F_{k,n+1} \\
 &= \frac{(n+1)(n+2)}{2} L_{k,n} + (3n+6)F_{k,n+1} \rightarrow \\
 &\rightarrow L_{k,n}^{(2)} = \frac{n+2}{2} ((n+1)L_{k,n} + 6F_{k,n+1})
 \end{aligned}$$

Second convolved classical Lucas sequence verifies the recurrence relation

$$\begin{aligned}
 L_n^{(2)} &= 3L_{n-1}^{(2)} - 5L_{n-3}^{(2)} + 3L_{n-5}^{(2)} + L_{n-6}^{(2)} \text{ with initial conditions } L_0^{(2)} = 8, L_1^{(2)} = 12, \\
 L_2^{(2)} &= 42, L_3^{(2)} = 85, L_4^{(2)} = 195, L_5^{(2)} = 399.
 \end{aligned}$$

3.1 Generating function

Taking into account that $l(k, x) = \frac{2 - kx}{1 - kx - x^2}$ is the generating function of the k -Lucas numbers, $l(k, x)^2$ is the generating function of their self convolution $L_k \otimes L_k$ and $l(k, x)^3$ that of the second convolution $L_k^{(2)}$.

Conclusions

In this paper, the iterated convolution of the k -Lucas numbers has been studied in a general way and then the first and second convolutions have been studied more specifically. Subsequently, a special dedication has been made to the case of classical Lucas numbers as well as to those of Lucas-Pell.

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