# On a generalization of the Padovan numbers

**Abstract**

This paper studies an extension of the classical Padovan sequence and that contains this as a particular case. Some very interesting formulas are found for the sum of these new sequences, for the sum of their squares as well as their self–convolution.

*Key words:* Padovan numbers, Generating function, Self-convolution,

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## 1 Introduction

In this section we remember the Padovan numbers and study some of the results obtained for them that we will later adapt to our new numbers.

### *1.1 The Padovan sequence*

The *Padovan sequence* [3] is the integer sequence $P(n)$ defined by the recurrence relation $P(n) = P(n-2) + P(n-3)$ with initial values $P(0) = P(1) =$

$P(2) = 1$. The first values of $\{P(n)\}$ are $\{1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, \ldots\}$. This sequence is indexed in the OEIS [1] as A000931.

The Padovan sequence is named after Richard Padovan who attributed its discovery to Dutch architect Hans van der Laan in his 1994 essay *Dom. Hans van der Laan : Modern Primitive*. The sequence was described by Ian Stewart in his Scientific American column Mathematical Recreations in June 1996.

### *1.2 Recurrence relations*

The Padovan sequence also satisfies the recurrence relations

$$P(n) = P(n-1) + P(n-5) \tag{1}$$
$$P(n) = P(n-2) + P(n-4) + P(n-8) \tag{2}$$

Proof of the Formula (1):

$$\begin{aligned} P(n) &= P(n-2) + P(n-3) = P(n-2) + (P(n-1) - P(n-4)) \\ &= P(n-2) + P(n-1) - (P(n-2) - P(n-5)) \\ &= P(n-1) + P(n-5) \end{aligned}$$

Proof of the Formula (2):

$$\begin{aligned} P(n) &= P(n-2) + P(n-3) = P(n-2) + (P(n-1) - P(n-4)) \\ &= P(n-2) + P(n-1) - (P(n-6) + P(n-7)) \\ &= P(n-2) + (P(n-1) - P(n-3)) + P(n-5) - P(n-7) \\ &= P(n-2) + (P(n-4) + P(n-5)) - P(n-7) \\ &= 2P(n-2) - P(n-7) \end{aligned}$$

### *1.3 Sum of the Padovan numbers*

The sum of the first $n$ terms in the Padovan sequence is

$$\sum_{j=0}^{n} P(j) = P(n+5) - 2 \tag{3}$$

*Proof.*

$$\begin{aligned} S_n &= \sum_{j=0}^{n} P(j) = P(n+5) - 2 \\ &\rightarrow S_{n+1} = \sum_{j=0}^{n+1} P(j) = \sum_{j=0}^{n} P(j) + P(n+1) = P(n+5) - 2 + P(n+1) \\ &= P(n+3) + P(n+2) - 2 + P(n+1) = P(n+3) - 2 + P(n+4) \\ &= P(n+6) - 2 = P((n+1)+5) - 2 = S_{n+1} \end{aligned}$$

The formulas for the sum of the even or the odd Padovan numbers can also be proven by induction: $\sum_{j=0}^{n} P(2j) = P(2n+3) - 1$ and $\sum_{j=0}^{n} P(2j+1) = P(2n+4) - 1$

### *1.4 Two formulas for the sum of the squares of the Padovan numbers*

**Theorem 1 (First formula)** *The sum of the squares of the Padovan numbers is*

$$S(n) = \sum_{j=0}^{n} P(j)^2 = 2P(n)P(n+1) - P(n-2)^2 \tag{4}$$

*Proof by induction.*
$\sum_{j=0}^{3} P(j)^2 = 1 + 1 + 1 + 4 = 7$ and $2P(3)P(4) - P(1)^2 = 2 \cdot 2 \cdot 2 - 1^1 = 7$
Let us suppose the formula is true up to "n". Then

$$\begin{aligned}
&\sum_{j=0}^{n+1} P(j)^2 = \sum_{j=0}^{n} P(j)^2 + P(n+1)^2 \\
&= 2P(n)P(n+1) - P(n-2)^2 + P(n+1)^2 \\
&= 2P(n)P(n+1) - (P(n+1) - P(n-1))^2 + P(n+1)^2 \\
&= 2P(n)P(n+1) - P(n+1)^2 + 2P(n-1)P(n+1) \\
&\quad -P(n-1)^2 + P(n+1)^2 \\
&= 2P(n+1)(P(n) + P(n-1)) - P(n-1)^2 \\
&= 2P(n+1)P(n+2) - P(n-1)^2 = S(n+1)
\end{aligned}$$

**Theorem 2 (Second formula)** *The sum of the squares of the Padovan numbers is*

$$\sum_{j=0}^{n} P(j)^2 = P(n+2)^2 - P(n-1)^2 - P(n-3)^2$$

From Equation (4):

$$\begin{aligned}
S(n) &= 2P(n+1)P(n+2) - P(n-1)^2 \\
&= 2P(n+1)[P(n) + P(n-1)] - P(n-1)^2 \\
&= 2P(n)P(n+1) + 2P(n-1)P(n+1) - P(n-1)^2 \\
&= [P(n+1) + P(n)]^2 - P(n+1)^2 - P(n)^2 - P(n-1)^2 \\
&\quad +2P(n-1)P(n+1) \\
&= [P(n+1) + P(n)]^2 - P(n)^2 - [P(n+1) - P(n)]^2 \\
&= P(n+3)^2 - P(n^2 - P(n-2)^2
\end{aligned}$$

**Definition 1** *The negative Padovan numbers are deefined as*

$P(-n) = P(-n+3) - P(-n+1).$

As a consequence, it is

$$\begin{aligned}
P(-1) &= P(2) - P(1) = 1 - 1 = 0 \\
P(-2) &= P(1) - P(-1) = 1 \\
P(-3) &= P(0) - P(-2) = 1 - 1 = 0 \\
P(-4) &= P(-1) - P(-3) = 0 \\
&\dots
\end{aligned}$$

In this way the following table is obtained:

| n | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| P(n) | -1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 7 |

**Theorem 3** *Generating matrix of the Padovan numbers is [6]* $Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ *because it is verified that* $Q^n = \begin{pmatrix} P(n-5) & P(n-3) & P(n-4) \\ P(n-4) & P(n-2) & P(n-3) \\ P(n-3) & P(n-1) & P(n-2) \end{pmatrix}$

This theorem will be proven by induction.
For $n = 2$ it is $Q^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$
Applying the previous formula and taking into account the negative Padovan numbers
$Q^2 = \begin{pmatrix} P(-3) & P(-1) & P(-2) \\ P(-2) & P(0) & P(-1) \\ P(-1) & P(0) & P(1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$
Assuming that the formula is true for the power $n$, let us show that it is also true for $n + 1$:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} P(n-5) & P(n-3) & P(n-4) \\ P(n-4) & P(n-2) & P(n-3) \\ P(n-3) & P(n-1) & P(n-2) \end{pmatrix}$$
$$= \begin{pmatrix} P(n-4) & P(n-2) & P(n-3) \\ P(n-3) & P(n-1) & P(n-2) \\ P(n-5)+P(n-4) & P(n-3)+P(n-2) & P(n-4)+P(n-3) \end{pmatrix}$$
$$= \begin{pmatrix} P(n-4) & P(n-2) & P(n-3) \\ P(n-3) & P(n-1) & P(n-2) \\ P(n-2) & P(n) & P(n-1) \end{pmatrix} = Q^{n+1}$$

as we wanted to prove.

A change in numbering allows us to present the previous matrix in its most common form

$$\begin{pmatrix} P(n+4) & P(n+2) & P(n+3) \\ P(n+3) & P(n+1) & P(n+2) \\ P(n+2) & P(n) & P(n+1) \end{pmatrix} \tag{5}$$

## 2 Generalized $k$–Padovan sequence

The goal of this article is to study a generalization of the Padovan sequence that contains the classical one as a particular case.

**Definition 2** *Let $k \in N - \{0\}$ be. We define the generalized Padovan sequence of parameter $k$ or $k$–Padovan sequence to the sequence defined by the recurrence relation $P_k(n) = k\, P_k(n-2) + P_k(n-3)$ with initial conditions $P_k(0) = P_k(1) = P_k(2) = 1$*

Then, the first elements of the $k$–Padovan sequence are
$P_k = \{1, 1, 1, k+1, k+1, k^2+k+1, k^2+2k+1, k^3+k^2+2k+1, \ldots\}$
Only for $k = 1$ and $k = 2$ the sequences obtained are indexed in the OEIS.
Characteristic equation of this recurrence relation is $r^3 - k\, r - 1 = 0$.
For $k = 1$, the classical Padovan sequence already studied in the previous section is obtained. The characteristic equation is $x^3 - x - 1 = 0$ admits only one real solution $\Psi = \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} = 1.324718$ while the other two are complex. This value of $\Psi$ is called plastic number (or plastic ratio or plastic constant or silver number).
It is easy to proove that the limit of the quotient is the plastic number [2]:

$$\lim_{n\to\infty} \frac{P(n+1)}{P(n)} = \lim_{n\to\infty} \frac{P(n-1)+P(n-2)}{P(n)} = \lim_{n\to\infty} \frac{\frac{P(n-1)}{P(n-2)}+1}{\frac{P(n)}{P(n-1)}\frac{P(n-1)}{P(n-2)}}$$

$$\text{If } \lim_{n\to\infty} \frac{P(n+1)}{P(n)} = \lim_{n\to\infty} \cdot \frac{P(n)}{P(n-1)} = \lim_{n\to\infty} \frac{P(n-1)}{P(n-2)} = L$$

$$\to L = \frac{L+1}{L \cdot L} \to L^3 - L - 1 = 0 \mapsto \psi$$

### 2.1 *On the characteristic roots*

Characteristic equation associated with the recurrence relation of the $k$–Padovan sequence is $r^3 - k\,r - 1 = 0$. Applying the results obtained in [5], the discriminant associated with this equation is $\Delta = 4k^3 - 27$ and the equation has three different real solutions if $\Delta < 0$, while it has one real and two complex if $\Delta > 0$. Therefore, $\Delta > 0 \rightarrow 4k^3 - 27 > 0 \rightarrow k > \frac{3}{\sqrt[3]{4}} = 1.88988 \rightarrow k \geq 2$ since $k \in N - \{0\}$.

Galois theory allows proving that when the three roots are real, and none is rational (casus irreducibilis), one cannot express the roots in terms of real radicals. Nevertheless, purely real expressions of the solutions may be obtained using trigonometric functions, specifically in terms of cosines [4]. In short:

- $\forall k \in N - \{0\}$ there is always a real root.
- For $k = 1$ there are other two complex roots and are the only complex characteristic roots for any value of $k$.
- There is only an integer root $r = -1$ for $k = 2$.
- If $k > 2$, the three roots are irrational and can be calculated by mean of the formula $r_m = 2\sqrt{\frac{k}{3}}\,cos\left[\frac{1}{3}\,arccos\left(\frac{3}{2k}\sqrt{\frac{3}{k}}\right) - \frac{2\pi\,m}{3}\right]$ for $m = 0,\ 1,\ 2$ [4].

**Example 1** *Find the characteristic roots for* $k = 3$

For $k = 3$, the preceding formula is $r_m = 2\,cos\left[\frac{1}{3}\,arc\,cos\left(\frac{1}{2}\right) - \frac{2\pi\,m}{3}\right]$. Then

(1) $m = 0 \rightarrow r_0 = 2cos\left(\frac{1}{3}\frac{\pi}{3}\right) = 1.87939$

(2) $m = 1 \rightarrow r_1 = 2cos\left(\frac{1}{3}\frac{\pi}{3} - \frac{2\pi}{3}\right) = 2\,cos\left(-\frac{5\pi}{9}\right) = -0.347296$

(3) $m = 2 \rightarrow r_2 = 2cos\left(\frac{1}{3}\frac{\pi}{3} - \frac{2\pi}{3}\right) = 2\,cos\left(-\frac{5\pi}{9}\right) = 2cos\left(\frac{1}{3}\frac{\pi}{3} - \frac{4\pi}{3}\right) = 2cos\left(-\frac{11\pi}{9}\right) = -1.53209$

**Example 2** *Find the characteristic roots for $k = 4$*

Similarly, for $k = 4$, the roots verify the formula
$r_m = 2\sqrt{\frac{4}{3}}\,cos\left[\frac{1}{3}\,arccos\left(\frac{3}{8}\sqrt{\frac{3}{4}}\right) - \frac{2\pi\, m}{3}\right]$ and therefore
$r_{\{0,1,2\}} = \{2.11491, -0.254102, -1.86081\}$

### *2.2 Sum of the k–Padovan sequences*

Given the $k$–Padovan sequence $P_k = \{1, 1, 1, k+1, k+1, k^2+k+1, k^2+2k+1, k^3+k^2+2k+1, \ldots\}$ the sum sequence of its first "n" terms is
$S_k = \{S_k(n)\} = \{1, 2, 3, k+4, 2k+5, k^2+3k+6, 2k^2+5k+7, \ldots\}$
For $n \geq 4$, the terms of this sum sequence verify the recurrence relation $S_k(n) = S_k(n-1) + k\,S_k(n-2) - (k-1)S_k(n-3) - S_k(n-4)$ with initial conditions $S_k(0) = 1$, $S_k(1) = 2$, $S_k(2) = 3$, $S_k(3) = k+4$
Its characteristic equation is $r^4 - r^3 - k\,r^2 + (k-1)r + 1 = 0$ and its factorization is $(r-1)(r^3 - k\,r - 1) = 0$. Obviously, an integer root is $r = 1$ and the factor $r^3 - k\,r - 1 = 0$ had been studied in the preceding subsection. Therefore, the general term of each of these sequences has the form $S_k(n) = C_1 + C_2 r_2 + C_3 r_3 + C_4 r_4$ with the preceding conditions. Each of the roots is calculated in the way indicated in the previous subsection. To find the constants $C_i$, any mathematical program that allows the resolution of a 4x4 system must be used.
Curiously, for $k$ =3 the characteristic equation reduces to the third degree equation $r^3 - 3r - 1 = 0$.

**Example 3** *Find the recurrence relation for the sums $\{S_4(n)\}$.*

First characteristic root is $= 1$ and the other three roots have been found in Exemple 2: $\{2.11491, -0.254102, -1.86081\}$.

— Then $S_4(n) = C_1 + C_2(2.11491)^n + C_3(-0.254102)^n + C_4(-1.86081)^n$ For $n = 0, 1, 2, 3$ and initial conditions $S_4(0) = 1$, $S_4(1) = 2$, $S_4(2) = 3$ and $S_4(4) = 8$, we solve the linear system and find the recurrence relation $S_4(n) = 0.250005 + 0.722587(2.11491)^n + 0.169782(-0.254102)^n + (-0.142374)(-1.86081)^n$

### *2.3 On the $P_2(n)$ sequence*

Taking into account that $r^3 - 2r - 1 = 0$ is the only equation that has an integer root ($r = -1$), the sequence $P_2$ constitutes a special case of the $k-$ Padovan sequences. This sequence is for $k = 2$

$P_2 = \{1, 1, 1, 3, 3, 7, 9, 17, 25, 43, 67, 111, 177, 289, 465, 755, \ldots\}$: A066983 inthe OPEIS and is called the Pell-Padovan sequence.

Characteristic equation of the recurrence relation for $k = 2$ is $r^3 - 2\,r - 1 = 0$ and its solutions are $-1$, $\frac{1-\sqrt{5}}{2}$ y $\frac{1+\sqrt{5}}{2}$.

From these characteristic roots it is possible to find Binnet's formula to find the general term of the sequence. This must be of the form

$$P_2(n) = C_1(-1)^n + C_2\left(\frac{1+\sqrt{5}}{2}\right)^n + C_3\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

For $n = 0, 1, 2$ the following system is obtained:

$$\begin{aligned}
n = 0 &\rightarrow P_2(0) = C_1 + C_2 + C_3 = 1 \\
n = 1 &\rightarrow P_2(1) = -C_1 + \frac{1+\sqrt{5}}{2}C_2 + \frac{1-\sqrt{5}}{2}C_3 = 1 \\
n = 2 &\rightarrow P_2(2) = C_1 + \left(\frac{1+\sqrt{5}}{2}\right)^2 C_2 + \left(\frac{1-\sqrt{5}}{2}\right)^2 C_3 = 1
\end{aligned}$$

The solution of this system is $C_1 = -1,\ C_2 = 1 - \frac{1}{\sqrt{5}},\ C_3 = 1 + \frac{1}{\sqrt{5}}$ and so $P_2(n) = -(-1)^n + \left(1 - \frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(1 + \frac{3}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$.

This last formula can be writen as $P_2(n) = -(-1)^n + 2\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}}\right)$, being $\alpha = \frac{1+\sqrt{5}}{2}$ the Golden Ratio and $\beta = -\frac{1}{\alpha}$. So $P_n = 2F(n) - \frac{1-(-1)^n}{2}$, where $F(n)$ is the Fibonacci number of order $n$.

Moreover, $P_2(n)$ verify the recurrence relation $P_2(n+1) = P_2(n) + P_2(n-1) - \eta$ where $\eta = \frac{1-(-1)^n}{2}$

Finally, the sum of the $P_2$–Padovan numbers is $S_2(n) = 2F(n) - \eta$

### *2.4 Negative k–Padovan numbers*

As with any sequence defined by a recurrence relation, $k$–Padovan numbers $P_k(n)$ for $n < 0$ can be defined by rewriting the recurrence relation as $P_k(n) = P_k(n+3) - k\,P_k(n+1)$. Then

$$
\begin{aligned}
P_k(-1) &= P_k(2) - k\,P_k(0) = 1 - k \\
P_k(-2) &= P_k(1) - k\,P_k(-1) = 1 - k(1-k) = k^2 - k + 1 \\
P_k(-3) &= P_k(0) - k\,P_k(-2) = 1 - k(k^2 - k + 1 = -k^3 + k^2 - k + 1 \\
P_k(-4) &= P_k(-1) - P_k(-3) = k^4 - k^3 + k^2 - 2k + 1 \\
&\ \ \cdots
\end{aligned}
$$

As a similar form than in the classical Padovan numbers, it is

| n | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|
| $P_k(n)$ | $k^2 - k + 1$ | $1 - k$ | 1 | 1 | 1 | $k + 1$ | $k + 1$ | $k^2 + k + 1$ |

**Theorem 4 (Generating function)** *Generating function of the k–Padovan numbers is* $p(k, x) = \dfrac{1 + x + (1-k)x^2}{1 - k\,x^2 - x^3}$

*Proof.* Taking into account the recurrence relation $r^3 = k\,r + 1$ of the Definition

$$
\begin{aligned}
p(k,x) &= \sum_{j=0}^{\infty} P_k(n)x^n = P_k(0) + P_k(1)x + P_k(2)x^2 + P_k(3)x^3 + \cdots \\
&\quad +P_k(n-1)x^{n-1} + P_k(n)x^n + \cdots \\
k\,x^2 p(k,x) &= \qquad k\,P_k(0)x^2 + k\,P_k(1)x^3 + k\,P_k(2)x^3 + \cdots \\
&\quad +k\,P_k(n-3)x^{n-1} + k\,P_k n - 2x^n + \cdots \\
x^3 p(k,x) &= \qquad\qquad P_k(0)x^3 + P_k(1)x^4 + \cdots \\
&\quad +P_k(n-4)x^{n-1} + P_k(n-3)x^n + \cdots \\
\hline
&\quad p(k,x)(1 - k\,x^2 - x^3) = P_k(0) + P_k(1)x + (P_k(2) - k\,P_k(0))x^2 \\
&\rightarrow p(k,x) = \frac{1 + x + (1-k)x^2}{1 - k\,x^2 - x^3}
\end{aligned}
$$

As a practical application, some infinite sums can be found. For example, if $x = \frac{1}{r}$:

$$\sum_{j=0}^{n} \frac{P_k j}{r^n} = \frac{1 + \frac{k}{r} + \frac{k^2-k}{r^2}}{1 - \frac{k}{r^2} - \frac{1}{r^3}} = \frac{r^3 + k\,r^2 + (k^2-k)r}{r^3 - k\,r - 1}$$

As a particular case of the latter, if $k = 1$ and $r = 2$, then $\displaystyle\sum_{n=0}^{\infty} \frac{P(n)}{2^n} = \frac{12}{5}$

### 2.5 Self-convolution of the $k$–Padovan sequence

Self-convolution of the $k$–Padovan sequence is $C(k,n) = \sum_{j=0}^{n} P(k,j)P(k,n-j)$

For $k = 1$, the classical Padovan sequence is $\{1, 1, 1, 2, 2, 3, 4, 5, 7, 9, \ldots$ and its self-convolution is generates the sequence $\{1, 2, 3, 6, 9, 14, 22, 32, 48, 70, 101, \ldots\}$.

For $k = 2$, the Pell-Padovan sequence is $\{1, 1, 1, 3, 3, 7, 9, 17, 25, 43, 67, \ldots\}$ so its self-convolution is $\{1, 2, 3, 8, 13, 26, 47, 84, 153, 266, \ldots\}$.

Self-convolution of the $k$–Padovan sequences verify the recurrence relation $C(k,n) = 2k\,C(k,n-2) + 2C(k,n-3) - k^2C(k,n-4) - 2k\,C(k,n-5) - C(k,n-6)$. Then, for the classical Padovan sequence it is $C(n) = 2C(n-2) + 2C(n-3) - C(n-4) - 2C(n-5) - C(n-6)$. And in similar form for the Pell-Padovan sequence.

## Conclusion

We have recalled the Padovan numbers and proven some of their properties. Next, this concept has been generalized by means of a parameter $k$ and some of the properties of the new numbers have been proven. The generating function of this new sequence has been found and has been particularized for the classical Padovan sequence as well as for that the Pell-Padovan.
We keep doors open for future research on this topic.

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