Original Research Article

Equal Sums of Four Even Powers

Abstract

Let u, v, w, z, k, m and I be any integers such that $k = z - w = w - v = v - u$. The study of integer I for which $I = u^{2n} + v^{2n} + w^{2n} + z^{2n} = k^2 + m^2 + r^2$ is not known. This study is therefore, set to partially overcome this challenge by introducing new formula relating sums of four even powers as an exact sum of three squares.

Keywords: Keyword, Sums of Three Squares, Sums of Four Even powers.

1 Introduction

The study of integer decomposition into sums of powers is an area that has received much attention since the advent of cryptography . Perhaps, this is because of the fact that integer factorization has direct application in cryptography. Most researchers who attempted the representations of integers as as a sum of powers has met their effort with very minimal success. For recent work on polynomial equations on sums of powers see [1,3,5,7,8,9,10,11,12,13,15] and for detailed recap on integer sums of two square studies the reader may refer to [2,4,6,14,16]. In most of this studies, the literature on integer representation as a sum of four powers is still hardily available. Moreover, documented results on the relationship between sums of four powers and and sums of three squares is not known. This study is therefore, set to introduce and develop the formula $I = u^{2n} + v^{2n} + w^{2n} + z^{2n} = k^2 + m^2 + n^2$ which has also has geometrical application in field of Geometry.

[1]

2 Main Results

Theorem 2.1. *$I = u^2 + v^2 + w^2 + z^2 = k^2 + m^2 + n^2$ has solution in integers if $v = u + k, w = u + 2k, z = u + 3k$.*

Proof. Suppose $v = u + k, w = u + 2k, z = u + 3k$. Then, $u^2 + v^2 + w^2 + z^2 = u^2 + (u + k)^2 + (u + 2k)^2 + (u + 3k)^2 = u^2 + u^2 + 2uk + k^2 + u^2 + 4uk + 4k^2 + u^2 + 6uk + 9k^2 = 4u^2 + 12uk + 14k^2 = k^2 + 4u^2 + 12uk + 9k^2 + 4k^2 = k^2 + (2u + 3k)^2 + (2k)^2 \cdots (**)$. Letting $m = 2u + 3k$ and $n = 2k$ we obtain $I = u^2 + v^2 + w^2 + z^2 = k^2 + m^2 + n^2$ completing the proof. This clearly shows that sums of four squares can be suppressed as a sum of three squares. □

Application of theorem 2.1.
Suppose a farmer has four square plots and will wish to share them among his three children so that each child can get a squares plot . How does the farmer go about it?. The answer to this question easily follows from theorem 2.1. This theorem can also be extended to other real life application.

Theorem 2.2. *$I = u^4 + v^4 + w^4 + z^4 = k^2 + m^2 + n^2$ has solution in integers if $v^2 = u^2 + k, w^2 = u^2 + 2k, z^2 = u^2 + 3k$.*

Proof. Suppose $v^2 = u^2 + k, w^2 = u^2 + 2k, z^2 = u^2 + 3k$. Then, $u^4 + v^4 + w^4 + z^4 = u^4 + (u^2 + k)^2 + (u^2 + 2k)^2 + (u^2 + 3k)^2 = 4u^4 + 14k^2 + 12ku^2 = k^2 + (2u^2 + 3k)^2 + 4k^2$. Put $m = 2u^2 + 3k$ and $n = 2k$ we have $I = u^4 + v^4 + w^4 + z^4 = k^2 + m^2 + n^2$ concluding the proof. □

The result on theorem 2.2 shows that sums of four fourth powers can be suppressed as a sum of three squares.

Theorem 2.3. *$I = u^6 + v^6 + w^6 + z^6 = k^2 + m^2 + n^2$ has solution in integers if $v^2 = u^2 + k, w^2 = u^2 + 2k, z^2 = u^2 + 3k$.*

Proof. Suppose $v^2 = u^3 + k, w^2 = u^3 + 2k, z^2 = u^3 + 3k$. Then, $u^6 + v^6 + w^6 + z^6 = u^6 + (u^3 + k)^2 + (u^3 + 2k)^2 + (u^3 + 3k)^2 = 4u^6 + 12ku^3 + 14k^2 = k^2 + (2u^3 + 3k)^2 + 4k^2$. Put $m = (2u^3 + 3k)$ and $n = 2k$ so that $I = u^6 + v^6 + w^6 + z^6 = k^2 + m^2 + n^2$ concluding the proof. □

The result on theorem 2.3 shows that sums of four sixth powers powers can be suppressed as a sum of three squares.

Theorem 2.4. *$I = u^8 + v^8 + w^8 + z^8 = k^2 + m^2 + n^2$ has solution in integers if $v^2 = (u^4 + k)^2, w^2 = (u^4 + 2k)^2, z^2 = (u^4 + 3k)^2$.*

Proof. Suppose $v^2 = u^4 + k, w^2 = u^4 + 2k, z^2 = u^4 + 3k$. Then, $u^8 + v^8 + w^8 + z^8 = u^8 + (u^4 + k)^2 + (u^4 + 2k)^2 + (u^4 + 3k)^2 = 4u^8 + 12ku^4 + 14k^2 = k^2 + (2u^4 + 3k)^2 + 4k^2$. Put $m = (2u^4 + 3k)$ and $n = 2k$ so that $I = u^8 + v^8 + w^8 + z^8 = k^2 + m^2 + n^2$ establishing the proof. □

The result on theorem 2.4 shows that sums of four eighth powers can be suppressed as a sum of three squares.

Theorem 2.5. *$I = u^{2n} + v^{2n} + w^{2n} + z^{2m} = k^2 + m^2 + n^2$ has solution in integers if $v^2 = u^n + k, w^2 = u^n + 2k, z^2 = u^n + 3k$.*

Proof. Suppose $v^2 = u^n + k, w^2 = u^n + 2k, z^2 = u^n + 3k$. Then, $u^{2n} + v^{2n} + w^{2n} + z^{2n} = u^{2n} + (u^n + k)^2 + (u^n + 2k)^2 + (u^n + 3k)^2 = 4u^{2n} + 14k^2 + 12ku^n = k^2 + (2u^n + 3k)^2 + 4k^2$. Put $m = (2u^n + 3k)$ and $n = 2k$ so that $I = u^{2n} + v^{2n} + w^{2n} + z^{2m} = k^2 + m^2 + n^2$ establishing the proof. □

The result on theorem 2.3 shows that sums of four even powers powers can be suppressed as a sum of three squares.

2.1 Some Examples

table 1: In this subsection, we provide some examples to argument our results in Theorem 2.1.

u^2	v^2	w^2	z^2	$u^2 + v^2 + w^2 + z^2 = I = k^2 + m^2 + n^2$	k^2	m^2	n^2
1	4	9	16	30	1	25	4
4	9	16	25	54	1	49	4
1	9	25	49	84	4	64	16
4	25	64	121	214	9	169	36
9	49	121	225	404	16	324	64
25	100	225	400	750	25	625	100
4	64	196	400	664	36	484	144
16	121	324	625	1086	49	841	196
49	225	529	961	1764	64	1444	256

table 2 : In this subsection, we provide some examples to argument our results in Theorem 2.2.

u^4	v^4	w^4	z^4	$u^4+v^4+w^4+z^4=I=k^2+m^2+n^2$	k^2	m^2	n^2
1	4	9	16	30	1	25	4
16	25	36	49	126	1	121	4
81	121	169	225	596	4	576	16
256	400	576	784	2018	16	1986	16
625	784	961	1156	3526	9	3481	36
16	49	100	169	334	9	225	36
1	36	121	256	441	25	289	100
81	144	225	324	774	9	729	36
256	324	400	484	1464	4	1444	16

table 3 : In this subsection, we provide some examples to argument our results in Theorem 2.3.

u^6	v^6	w^6	z^6	$u^6+v^6+w^6+z^6=I=k^2+m^2+n^2$	k^2	m^2	n^2
5^6	8^6	7^6	8^6	442074	1	442069	4
6^6	7^6	8^6	9^6	957890	1	957885	4
9^6	11^6	13^6	15^6	18520436	4	18520416	16
14^6	17^6	20^6	23^6	243702994	9	243702949	36
19^6	23^6	27^6	31^6	1470005940	16	1470005860	64
25^6	30^6	35^6	40^6	6907406250	36	6907406070	144
26^6	32^6	38^6	44^6	$1.1649...*10^{10}$	36	$1.1649...*10^{10}$	144
32^6	39^6	46^6	53^6	$3.6231...*10^{10}$	49	$3.6231...*10^{10}$	196
39^6	47^6	55^6	63^6	$1.0450...*10^{11}$	64	$1.0450...*10^{11}$	256

table 4 : In this subsection, we provide some examples to argument our results in Theorem 2.4.

u^8	v^8	w^8	z^8	$u^8+v^8+w^8+z^8=I=k^2+m^2+n^2$	k^2	m^2	n^2
5^8	6^8	7^8	8^8	24612258	1	2461253	4
8^8	9^8	10^8	11^8	374182813	1	374182813	4
17^8	19^8	21^8	23^8	$1.4009...*10^{11}$	4	$1.4009...*10^{11}$	16
32^8	36^8	40^8	44^8	$2.4522...*10^{13}$	16	$2.4522...*10^{13}$	16
37^8	40^8	43^8	46^8	$3.8293...*10^{13}$	9	$3.8293...*10^{13}$	36
16^8	19^8	22^8	25^8	$2.2874...*10^{11}$	9	$2.2874...*10^{11}$	36
21^8	26^8	31^8	36^8	$3.9206...*10^{12}$	25	$3.9206...*10^{12}$	100
21^8	24^8	27^8	30^8	$1.0864...*10^{12}$	9	$1.0864...*10^{12}$	36
24^8	26^8	28^8	30^8	$1.3525...*10^{12}$	4	$1.3525...*10^{12}$	16

Conjecture 1. *$I = u^{2n}+v^{2n}+w^{2n}+z^{2m} = k^2+m^2+n^2$ is impossible if $k \neq z-w \neq w-v \neq v-u$ for some integer k.*

3 Conclusion

This research, has contributed to problem of integer representation of sums of four powers as a sum of three squares. However,the problem of integer representation as a sum of powers is still a very long standing problem. We therefore, encourage other researchers to devote there attention in this field of research.

References

[1] Amir .F., Pooya .M., Rahim. F., (2012). A Simple Method to Solve Quartic Equations. *Australian Journal of Basic and Applied Sciences*, 6(6): 331-336, ISSN ,1991-8178.

[2] Bombieri. E., Bourgain. J., (2015) A problem on sums of two squares, *Internatinal Mathematics Research* , (11):3343-3407.

[3] Cavallo, A. (2019). Galois groups of symmetric sextic trinomials. arXiv:1902.00965v1 [math.GR]. https://arxiv.org/abs/1902.00965.

[4] David A., (2016) A partition-theoretic proof of Fermat's two squares theorem.*Discrete Mathematics* ,339:4:1410–1411, DOI:10.1016/j.disc.2015.12.002.

[5] Giorgos P. Kouropoulos., (2021). A combined methodology for approximate estimation of the roots of the general sextic polynomial equation. *Research Square* , DOI: https//doi.org/10.21203/rs.3.rs-882192/v2.

[6] Lao H., (2022).Some Formulae For Integer Sums of Two Squares.*Journal of Advances in Mathematics and Computer Science* , 37(4): 53-57, Article no.JAMCS.87824,ISSN: 2456-9968.DOI: 10.9734/JAMCS/2022/v37i430448.

[7] Lao H., (2024).Radical Solution of Some Higher Degree Equation Via Radicals.*Journal of Advances in Mathematics and Computer Science* ,volume 39(3):20-28,Article no.JAMCS.113540ISSN: 2456-9968, DOI: 10.9734/JAMCS/2024/v39i31872

[8] Lao H.,Zachary K., Kinyanjui J. (2023).Some Generalized Formula For Sums of Cube.*Journal of Advances in Mathematics and Computer Science* , 37(4): 53-57, Article no.JAMCS.87824,ISSN: 2456-9968, DOI: 10.9734/JAMCS/2023/v38i81789.

[9] Lao. H., Maurice. O, Michael . O., (2023).On The Sum of Three Square Formula,*Science Mundi* ,Vol 3(Iss.1), pp.111-120, Science Mundi ISSN:2788-5844 http://sciencemundi.net. DOI: https://doi.org/10.51867/scimundi.3.1.11

[10] Kimtai B., Lao, H., (2023). *On Generalized Sum of Six, Seven and Nine Cube*. Mundi,Vol 3(1),pp.135-142, Science Mundi ISSN:2788-5844 http://sciencemundi.net DOI: https://doi.org/10.51867/scimundi.3.1.14

[11] Mochimaru, Y. (2005). Solution of Sextic Equations. International Journal of Pure and Applied Mathematics, 23(4), 575-583. https://ijpam.eu/contents/2005-23-4/9/9.

[12] Najman .F., (2010). On The Diophantine Equation $x^4 \pm y^4 = iz^2$ in Gaussian Integers. *Amer. Math. Monthly*, 117(7), 637-641.

[13] Najman .F., (2011).Torsion of elliptic curves over quadratic cyclotomic fields *Math. J. Okayama Univ.*. , 53, 75-82.

[14] Par Y. , (2016).Waring-Golbach problem. Two squares and Higher Powers. *Journal de Theorie des Nombres* ,791-810.

[15] Ruffini, P. (1799). Teoria Generale delle Equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto [General Theory of equations, in which the algebraic solution of general equations of degree higher than four is proven impossible]. Book on Demand Ltd. ISBN: 978-5519056762

[16] Tignol, J. P. (2001). Galois' Theory of Algebraic Equations. World Scientific, Louvain, Belgium. DOI: 10.1142/4628 .