A New Polar Representation for Pell and Pell-Lucas Split Quaternions

Abstract

In this paper, we define split quaternions with components including Pell and Pell-Lucas number sequences. By using Binet's formulas of these type split quaternions we introduce a new polar representation for these split quaternions using Cayley-Dickson's notation. This alternative representation, based on two complex number sequences, provides a new perspective on the structure of Pell and Pell-Lucas split quaternions and give a deeper understanding of their geometric interpretations and transformations. Furthermore, some fundamental properties and identities for these type of Pell and Pell-Lucas split quaternions are studied. In further the current paper, it would be valuable to replicate similar approaches polar representationin with Pell and Pell-Lucas Split quaternions.

Keywords:Pell split quaternions, Pell-Lucas split quaternions, Polar representation Mathematics Subject Classification: 11B37, 11R52, 53B30

1 Introduction

W. R. Hamilton [1] initiated the study of the quaternions an extension to the complex number in 1843. They have many applications in mathematics, physics, computer graphics, and engineering. Hamilton first introduced quaternions in 1843 as a way to represent rotations in three-dimensional space, and since then, they have become an important tool in many fields. One advantage of quaternions over other methods of representing rota tions is their ability to avoid the problem of gimbal lock, which can occur when using Euler angles. Quaternions also have a relatively simple algebraic structure and can be efficiently implemented in computer algorithms. In recent years, quaternions have been used in the development of virtual reality systems and computer games, where they are used to represent orientations of objects in three-dimensional space. They have also been applied in robotics, control theory, and signal processing. Overall, quaternions have become a valuable tool in many areas of mathematics and engineering, and their use continues to expand [2,3]. Sangwine and [4] introduced a quaternion polar representation that draws inspiration from the Cayley-Dickson form. In their formulation, they express quaternions using a complex modulus and argu ment. The Cayley-Dickson construction is a mathematical procedure that extends the concept of complex numbers to higher dimensions, paving the way for the development of quaternions. On the other hand, the complex argument represents the direction or orientation of the quaternion in a manner analogous to the argument of a complex number. This approach provides a concise and insightful way to represent quaternions, offering a geometric interpretation that aligns with the principles of complex analysis.

In 1849, Cockle [5] defined split quaternions by utilizing real quaternions. The multiplication rules for split quaternions differ in the vector part due to their association with Minkowski space, as opposed to the Euclidean 3-space associated with real quaternions. This distinction makes split quaternions particularly relevant in the context of theories involving spacetime, such as special relativity. It is known that split quaternions are effective for rotating vectors in [6–8], whereas real quaternions are efficient for rotating vectors in [13].

A split quaternion [9–11] is defined as the following quadruple

$$\gamma = \gamma_0 + \gamma_1 i_1 + \gamma_2 i_2 + \gamma_3 i_3$$

with $\gamma_{0-3} \in \mathbb{R}$ and quatenionic units i_1, i_2, i_3 satisfy

$$i_1^2 = -i_2^2 = -i_3^2 = -1,\ i_1 i_2 = -i_2 i_1 = i_3 \tag{1}$$

and can be shown as $\gamma = S_\gamma + V_\gamma$, γ with $S_\gamma = 0$ is a pure split quaternion. Let γ and δ be split quaternions, then addition and multiplication are

$$\gamma + \delta = (S_\gamma + S_\delta) + (V_\gamma + V_\delta)$$

$$\gamma\delta = S_\gamma S_\delta + \langle V_\gamma, V_\delta \rangle + S_\gamma V_\gamma + S_\delta V_\delta + V_\gamma \times V_\delta$$

respectively, where $\langle\ ,\ \rangle$ and $\times$ are inner and cross products in Minkowsky space $\mathbb{E}_1^3$. The conjugate and norm of $\gamma = \gamma_0 + \gamma_1 i_1 + \gamma_2 i_2 + \gamma_3 i_3$ are respectively as

$$\gamma^* = \gamma_0 - \gamma_1 i_1 - \gamma_2 i_2 - \gamma_3 i_3$$

and

$$|\gamma| = \sqrt{|\gamma\gamma^*|} = \sqrt{|\gamma_0^2 + \gamma_1^2 - \gamma_2^2 - \gamma_3^2|} = \sqrt{|\mathcal{I}_\gamma|}. \tag{2}$$

If $|\gamma| = 1$, then γ is a unit split quaternion, for any split quaternion γ with $|\gamma| \neq 0$, $\frac{\gamma}{\|\gamma\|}$ is a unit split quaternion. The split quaternion γ is space-like, time-like and light-like if $\mathcal{I}_\gamma < 0$, $\mathcal{I}_\gamma > 0$ and $\mathcal{I}_\gamma = 0$ respectively and

$$\mathcal{I}_\gamma = \gamma_0^2 + \gamma_1^2 - \gamma_2^2 - \gamma_3^2 = \mathcal{I}_{S_\gamma} + \mathcal{I}_{V_\gamma} \tag{3}$$

where $\mathcal{I}_{S_\gamma} = \gamma_0^2$ and $\mathcal{I}_{V_\gamma} = \gamma_1^2 - \gamma_2^2 - \gamma_3^2$. The multiplicative inverse of γ is $\gamma^{-1} = \frac{\gamma^*}{|\gamma|^2}$ and there is no inverse for light-like split quaternion. The Cayley-Dickson's form of a split quaternion γ is

$$\gamma = (\gamma_0 + \gamma_1 i_1) + (\gamma_2 + \gamma_3 i_1) i_2$$

which is based on two complex numbers.

Classify the polar representation according to the given the split quaternion γ and vector part,

respectively.
1. The polar representation for spacelike quaternion can be written in the form

$$\gamma = |\gamma| \left(\sinh\phi + \mu\cosh\phi\right) \tag{4}$$

where $\sinh\phi = \frac{\gamma_0}{|\gamma|}$, $\cosh\phi = \frac{\sqrt{-\gamma_1^2+\gamma_2^2+\gamma_3^2}}{|\gamma|}$ and $\mu = \frac{\gamma_1 i_1+\gamma_2 i_2+\gamma_3 i_3}{\sqrt{-\gamma_1^2+\gamma_2^2+\gamma_3^2}}$ is a spacelike unit vector.
2. The polar representation for timelike γ with spacelike vector part ($I_{V_\gamma} < 0$ for $V_\gamma = \gamma_1 i_1 + \gamma_2 i_2 + \gamma_3 i_3$ vector part of γ) can be written in the form

$$\gamma = |\gamma| \left(\cosh\phi + \mu\sinh\phi\right) \tag{5}$$

where $\sinh\phi = \frac{\sqrt{-\gamma_1^2+\gamma_2^2+\gamma_3^2}}{|\gamma|}$, $\cosh\phi = \frac{\gamma_0}{N(\gamma)}$ and $\mu = \frac{\gamma_1 i_1+\gamma_2 i_2+\gamma_3 i_3}{\sqrt{-\gamma_1^2+\gamma_2^2+\gamma_3^2}}$ is a spacelike unit vector.
3. The polar representation for timelike γ with timelike vector part can be written as

$$\gamma = |\gamma| \left(\cos\phi + \mu\sin\phi\right)$$

where $\sin\phi = \frac{\sqrt{\gamma_1^2-\gamma_2^2-\gamma_3^2}}{|\gamma|}$, $\cos\phi = \frac{\gamma_0}{|\gamma|}$ and $\mu = \frac{\gamma_1 i_1+\gamma_2 i_2+\gamma_3 i_3}{\sqrt{\gamma_1^2-\gamma_2^2-\gamma_3^2}}$ is a unit pure split quaternion. Indeed, a unit timelike quaternion γ with a timelike vector part represents a rotation of a three-dimensional non-lightlike Lorentzian vector by an angle 2ϕ about the axis defined by γ. This rotation operation is similar to the one described earlier with the unit timelike quaternion γ having a spacelike vector part. However, in this case, the axis of rotation is determined by the direction of the timelike vector part of γ.
4. If γ is a lightlike split quaternion, then

$$\gamma = 1 + \mu.$$

Here, γ is a lightlike (null) split vector. [12, 13]. Any Split quaternion can be written in the form

$$\gamma = Ae^{Bi_2}$$

where $A = A_0 + A_1 i_1$ and $B = A_2 + A_3 i_1$ are complex numbers, [4, 9]. For detailed information about the matrix notation of rotation and its relationship with quaternionic representations, we refer the reader to reference [6], as well as references [7] and [8]. These sources provide comprehensive explanations and discussions regarding the mathematical foundations and practical applications of quaternionic rotations in Lorentzian geometry.
In the literature, sequences of integers have an important place. The most famous of these sequences have been demonstrated in several areas of mathematics. The Pell P_n and Pell-Lucas Q_n number sequences are defined by

$$P_n = 2P_{n-1} + P_{n-2}, \qquad P_0 = 0, P_1 = 1, n \geq 2$$
$$Q_n = 2Q_{n-1} + Q_{n-2}, \qquad Q_0 = Q_1 = 2, n \geq 2$$

The characteristic equation of these number sequences is $x^2 - 2x - 2 = 0$, with roots $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.The Binet's forms of these sequences are

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } Q_n = \alpha^n + \beta^n \tag{6}$$

For more details and relations about these number sequences see [14–21].

2 Pell and Pell-Lucas split quaternions

In this section, we will give Pell and Pell-Lucas split quaternions and their fundemental identities..

Definition 1. *The Pell and Pell-Lucas split quaternions are defined as*

$$\mathcal{P}_n = P_n + P_{n+1}i_1 + P_{n+2}i_2 + P_{n+3}i_3 \tag{7}$$

$$\mathcal{Q}_n = Q_n + Q_{n+1}i_1 + Q_{n+2}i_2 + Q_{n+3}i_3 \tag{8}$$

respectively, where P_n and Q_n are Pell and Pell-Lucas numbers and i_1, i_2, i_3 follow the rules in (1).

From definition, the following recurrence relation can be prove easily

$$\mathcal{P}_n = 2\mathcal{P}_{n-1} + \mathcal{P}_{n-2},\ n \geq 2$$

and

$$\mathcal{Q}_n = 2\mathcal{Q}_{n-1} + \mathcal{Q}_{n-2},\ n \geq 2$$

Theorem 2. *The Binet's formulas for Pell and Pell-Lucas split quaternions are*

$$\mathcal{P}_n = \frac{\alpha^n \underline{\alpha} - \beta^n \underline{\beta}}{\alpha - \beta} \quad and \quad \mathcal{Q}_n = \alpha^n \underline{\alpha} + \beta^n \underline{\beta} \tag{9}$$

respectively, where

$$\underline{\alpha} = 1 + \alpha i_1 + \alpha^2 i_2 + \alpha^3 i_3 \quad and \quad \underline{\beta} = 1 + \beta i_1 + \beta^2 i_2 + \beta^3 i_3 \tag{10}$$

Proof. The proof can be done directly by using definition and Binet's formulas for Pell and Pell-Lucas sequences. □

Theorem 3 (Vajda's identities)**.** *For positive integers m, n and r with $n > m \geq r \geq 1$, we have*

$$\mathcal{P}_m\mathcal{P}_n - \mathcal{P}_{m-r}\mathcal{P}_{n+r} = \frac{P_r}{\sqrt{8}}(\alpha^n\beta^{m-r}\underline{\beta}\underline{\alpha} - \alpha^{m-r}\beta^n\underline{\alpha}\underline{\beta})$$
$$\mathcal{Q}_m\mathcal{Q}_n - \mathcal{Q}_{m-r}\mathcal{Q}_{n+r} = \sqrt{8}P_r(\alpha^{m-r}\beta^n\underline{\alpha}\underline{\beta} - \alpha^n\beta^{m-r}\underline{\beta}\underline{\alpha})$$

where P_n is n^{th} Pell number and

$$\underline{\alpha}\underline{\beta} = \mathcal{Q}_0 + 2\sqrt{2}(i_1 - 2i_2 + i_3) \quad and \quad \underline{\beta}\underline{\alpha} = \mathcal{Q}_0 - 2\sqrt{2}(i_1 - 2i_2 + i_3) \tag{11}$$

Proof. Using (9), (10) and Binet's formula, we have

$$\begin{aligned}\mathcal{P}_m\mathcal{P}_n - \mathcal{P}_{m-r}\mathcal{P}_{n+r} &= \frac{1}{8}\left(\alpha^m\underline{\alpha} - \beta^m\underline{\beta}\right)\left(\alpha^n\underline{\alpha} - \beta^n\underline{\beta}\right) \\ &- \frac{1}{8}\left(\alpha^{m-r}\underline{\alpha} - \beta^{m-r}\underline{\beta}\right)\left(\alpha^{n+r}\underline{\alpha} - \beta^{n+r}\underline{\beta}\right) \\ &= \frac{1}{8}\left(-\underline{\alpha}\underline{\beta}\,\alpha^m\beta^n - \underline{\beta}\underline{\alpha}\,\beta^m\alpha^n + \underline{\alpha}\underline{\beta}\,\alpha^{m-r}\beta^{n+r} + \underline{\beta}\underline{\alpha}\,\beta^{m-r}\alpha^{n+r}\right) \\ &= \frac{1}{\sqrt{8}}\left(\alpha^n\beta^m\underline{\beta}\underline{\alpha}\,\frac{\alpha^r - \beta^r}{\sqrt{8}\,\beta^r} - \underline{\alpha}\underline{\beta}\,\alpha^m\beta^n\frac{\alpha^r - \beta^r}{\sqrt{8}\,\beta^r}\right)\end{aligned}$$

By using (9), (10) and (11), the proof for Pell-Lucas can be done similarly. □

Corollary 4. *For $m = n$ in theorem 3 we get the Catalan's identities*

$$\mathcal{P}_{n-r}\mathcal{P}_{n+r} - \mathcal{P}_n^2 = \frac{P_r}{\sqrt{8}}(-1)^n(\alpha\ \underline{\beta}\underline{\alpha} - \beta\ \underline{\alpha}\underline{\beta})$$
$$\mathcal{Q}_{n-r}\mathcal{Q}_{n+r} - \mathcal{Q}_n^2 = \sqrt{8}P_r(-1)^n(\beta\ \underline{\alpha}\underline{\beta} - \alpha\ \underline{\beta}\underline{\alpha})$$

For $m = n$ and $r = 1$ we get the Cassini's identities

$$\mathcal{P}_{n-1}\mathcal{P}_{n+1} - \mathcal{P}_n^2 = \frac{(-1)^n}{\sqrt{8}}(\alpha\ \underline{\beta}\underline{\alpha} - \beta\ \underline{\alpha}\underline{\beta})$$
$$\mathcal{Q}_{n-1}\mathcal{Q}_{n+1} - \mathcal{Q}_n^2 = \sqrt{8}(-1)^n(\beta\ \underline{\alpha}\underline{\beta} - \alpha\ \underline{\beta}\underline{\alpha})$$

For $m = n$, $m = n+1$ and $r = 1$, we get the d'Ocagne's identities

$$\mathcal{P}_{n+1}\mathcal{P}_m - \mathcal{P}_n\mathcal{P}_{m+1} = \frac{1}{\sqrt{8}}(\alpha^m\beta^n\ \underline{\beta}\underline{\alpha} - \alpha^n\beta^m\ \underline{\alpha}\underline{\beta})$$
$$\mathcal{Q}_{n+1}\mathcal{Q}_m - \mathcal{Q}_n\mathcal{Q}_{m+1} = \sqrt{8}(\alpha^n\beta^m\ \underline{\alpha}\underline{\beta} - \alpha^m\beta^n\underline{\beta}\underline{\alpha})$$

3 The Different Polar Representation of Pell and Pell-Lucas split quaternions

In this section, we aim to demonstrate that every Pell and Pell-Lucas split quaternion possesses a polar form, in addition to the classical form mentioned. This polar form provides an alternative representation of Pell and Pell-Lucas split quaternions, offering insights into their geometric and algebraic properties. Indeed, the concept of constructing a polar form for Pell and Pell-Lucas split quaternions using two complex numbers, and then applying a specific operation (multiplying the second complex number by i_2) bears resemblance to the Cayley-Dickson construction. However, there are notable distinctions between this approach and the traditional Cayley-Dickson method.

Proposition 5. *The norm of Pell and Pell-Lucas split quaternions is*

$$N(\mathcal{P}_n) = \sqrt{2Q_{2n+3}}$$
$$N(\mathcal{Q}_n) = 4\sqrt{Q_{2n+3}}$$

where Q_n is n^{th} Pell-Lucas number.

Proof. From definition of norm for split quaternion, we have

$$N(\mathcal{P}_n) = \sqrt{|P_n^2 + P_{n+1}^2 - P_{n+2}^2 - P_{n+3}^2|}$$

and by using Binet's formula

$$P_n^2 + P_{n+1}^2 - P_{n+2}^2 - P_{n+3}^2 = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 + \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right)^2 - \left(\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\right)^2 - \left(\frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta}\right)^2$$

from using $\alpha\beta = -1$ and $\alpha - \beta = 2\sqrt{2}$ we get

$$P_n^2 + P_{n+1}^2 - P_{n+2}^2 - P_{n+3}^2 = \frac{1}{8}\left(\alpha^{2n}(1\alpha^2 - \alpha^4 - \alpha^6) + \beta^{2n}(1 + \beta^2 - \beta^4 - \beta^6)\right)$$

Because α and β are roots of $x^2 - 2x - 1 = 0$, then

$$\begin{aligned} P_n^2 + P_{n+1}^2 - P_{n+2}^2 - P_{n+3}^2 &= \frac{1}{8}\left(\alpha^{2n}(-80\alpha - 32) + \beta^{2n}(-80\beta - 32)\right) \\ &= -10(\alpha^{2n+1} + \beta^{2n+1}) - 4(\alpha^{2n} + \beta^{2n}) \\ &= -10Q_{2n+1} - 4Q_{2n} \\ &= -2Q_{2n+3} \end{aligned}$$

Then the result is clear, the proof for Pell-Lucas is similarly. □

Corollary 6. *The Pell and Pell-Lucas split quaternions are spacelike split quaternions with spacelike vector part.*

Theorem 7. *The classical polar representation of Pell split quaternion is*

$$\mathcal{P}_n = \sqrt{2Q_{2n+3}}(\sinh\Phi + \mu\cosh\Phi)$$

where $\mu = \frac{P_{n+1}(i_1+2i_2+5i_3)+P_n(i_2+2i_3)}{\sqrt{Q_{2n+6}+2Q_{2n+3}-2(-1)^n}}$ *is a pure unit split quaternion and*

$$\Phi = \tanh^{-1}\left(\frac{\sqrt{8}P_n}{\sqrt{Q_{2n+6} + 2Q_{2n+3} - 2(-1)^n}}\right)$$

Proof. Since $\mathcal{P}_n$ is a spacelike split quaternion, then by using (4), the polar representation is

$$\mathcal{P}_n = N(\mathcal{P}_n)(\sinh\Phi + \mu\cosh\Phi)$$

where

$$\begin{aligned} \mu &= \frac{P_{n+1}i_1 + P_{n+2}i_2 + P_{n+3}i_3}{\sqrt{-P_{n+1}^2 + P_{n+2}^2 + P_{n+3}^2}} \\ \sinh\Phi &= \frac{P_n}{N(\mathcal{P}_n)} \\ \cosh\Phi &= \frac{\sqrt{|P_{n+1}^2 - P_{n+2}^2 - P_{n+3}^2|}}{N(\mathcal{P}_n)} \end{aligned}$$

therefore $\tanh\Phi = \frac{P_n}{\sqrt{-P_{n+1}^2+P_{n+2}^2+P_{n+3}^2}}$ and we get

$$\mu = \frac{P_{n+1}(i_1 + 2i_2 + 5i_3) + P_n(i_2 + 2i_3)}{\sqrt{Q_{2n+6} + 2Q_{2n+3} - 2(-1)^n}}$$

and

$$\Phi = \tanh^{-1}\left(\frac{\sqrt{8}P_n}{\sqrt{Q_{2n+6} + 2Q_{2n+3} - 2(-1)^n}}\right)$$

□

Corollary 8. *The classical polar representation of Pell-Lucas split quaternion is*

$$\mathcal{Q}_n = 4\sqrt{Q_{2n+3}}(\sinh\Phi + \mu\cosh\Phi)$$

where $\mu = \frac{Q_{n+1}(i_1+2i_2+5i_3)+Q_n(i_2+2i_3)}{\sqrt{Q_{2n+6}+2Q_{2n+3}+2(-1)^n}}$ *is a pure unit split quaternion and*

$$\Phi = \tanh^{-1}\left(\frac{Q_n}{\sqrt{Q_{2n+6}+2Q_{2n+3}+2(-1)^n}}\right)$$

Proof. The Proof is similar to Theorem (7). □

Proposition 9. *Let* $D = Ai_2 + Bi_3 = (A + Bi_1)i_2$ *be an arbitrary split quaternion, for spacelike* D*, the exponential form is*

$$e^D = \sinh|D| + \frac{A}{|D|}\cosh|D|i_2 + \frac{B}{|D|}\cosh|D|i_3 = \alpha_0 + \alpha_2 i_2 + \alpha_3 i_3$$

and for timelike D*, then*

$$e^D = \cosh|D| + \frac{A}{|D|}\sinh|D|i_2 + \frac{B}{|D|}\sinh|D|i_3 = \beta_0 + \beta_2 i_2 + \beta_3 i_3$$

That is, it is a split quaternions with no i_1*'s term.*

Proof. Suppose μ is a spacelike unit split quaternion, that is $N(\mu) = 1$, then from (4) we have

$$e^{\mu\theta} = \sinh\theta + \mu\cosh\theta$$

if we rewrite $D = |D|\frac{D}{|D|}$, then by taking $\mu = \frac{D}{|D|}$ and $\theta = |D|$ we get the result, we can prove similarly for timelike D by using 5. □

Now we give the new polar representations for Pell and Pell-Lucas split quaternions by using Cayley-Dikson's form.

Theorem 10. *Every Pell split quaterninon* $\mathcal{P}_n = P_n + P_{n+1}i_1 + P_{n+2}i_2 + P_{n+3}i_3$ *can be given in the form* $\mathcal{P}_n = Ae^{Bi_2}$*, where* A *and* B *are complex numbers, that is*

$$A = \frac{2(P_n + P_{n+1}i_1)}{\sqrt{Q_{2n+1}+Q_{2n}}}$$

$$B = \frac{Q_{2n+3}+Q_{2n+2}-8(-1)^n i_1}{\sqrt{(Q_{2n+1}+Q_{2n})(Q_{2n+5}+Q_{2n+4})}}\tanh^{-1}\left(\sqrt{\frac{Q_{2n+1}+Q_{2n}}{Q_{2n+5}+Q_{2n+4}}}\right)$$

Proof. Suppose that $A = a + bi_1$ and $e^{Bi_2} = \alpha_0 + \alpha_2 i_2 + \alpha_3 i_3$, then

$$\mathcal{P}_n = Ae^{Bi_2} = a\alpha_0 + b\alpha_0 i_1 + (a\alpha_2 - b\alpha_3)i_2 + (a\alpha_3 + b\alpha_2)i_3$$

if $\alpha_0 = 0$, then we can select $a = 1$ and $b = 0$, we will get $A = 1$. For $\alpha_0 \neq 0$, we construct a complex number $\gamma = a\alpha_0 + b\alpha_0 i_1 = P_n + P_{n+1}i_1$ and then $A = \frac{\gamma}{|\gamma|}$, by using (7) the explicit form of A is

$$A = \frac{2(P_n + P_{n+1}i_1)}{\sqrt{Q_{2n+1}+Q_{2n}}}|\mathcal{P}_n|$$

Since A is a unit complex number then $A^{-1} = \overline{A} = \frac{2(P_n - P_{n+1}i_1)}{\sqrt{Q_{2n+1}+Q_{2n}}}|\mathcal{P}_n|$, where $\overline{A}$ is conjugate of A and

$$e^{Bi_2} = \overline{A}\mathcal{P}_n = \frac{2}{\sqrt{Q_{2n+1}+Q_{2n}}}\left(P_n^2 + P_{n+1}^2 + (P_nP_{n+2} + P_{n+1}P_{n+3})i_2 + (P_nP_{n+3} - P_{n+1}P_{n+2})i_3\right)$$

and the norm of e^{Bi_2} is

$$\begin{aligned}|e^{Bi_2}| &= \frac{2}{\sqrt{Q_{2n+1}+Q_{2n}}}\sqrt{(P_n^2 + P_{n+1}^2)^2 - (P_nP_{n+2} + P_{n+1}P_{n+3})^2 - (P_nP_{n+3} - P_{n+1}P_{n+2})^2} \\ &= \sqrt{P_{n+2}^2 + P_{n+3}^2 - P_n^2 - P_{n+1}^2}\end{aligned}$$

Since $\frac{e^{Bi_2}}{|e^{Bi_2}|}$ is a unit spacelike split quaternion, then its classical polar form is

$$\frac{e^{Bi_2}}{|e^Bi_2|} = \sinh\theta + \mu\cosh\theta$$

then we can write

$$\begin{aligned}\sinh\theta &= \frac{\sqrt{P_n^2 + P_{n+1}^2}}{\sqrt{P_{n+2}^2 + P_{n+3}^2 - P_n^2 - P_{n+1}^2}} \\ \cosh\theta &= \frac{\sqrt{P_{n+2}^2 + P_{n+3}^2}}{\sqrt{P_{n+2}^2 + P_{n+3}^2 - P_n^2 - P_{n+1}^2}} \\ \mu &= \frac{(P_nP_{n+2} + P_{n+1}P_{n+3})i_2 + (P_nP_{n+3} - P_{n+1}P_{n+2})i_3}{\sqrt{(P_n^2 + P_{n+1}^2)(P_{n+2}^2 + P_{n+3}^2)}}\end{aligned}$$

which gives

$$\tanh\theta = \frac{\sqrt{P_n^2 + P_{n+1}^2}}{\sqrt{P_{n+2}^2 + P_{n+3}^2}}$$

From $Bi_2 = \mu\theta$ we get

$$Bi_2 = \frac{(P_nP_{n+2} + P_{n+1}P_{n+3} + (P_nP_{n+3} - P_{n+1}P_{n+2})i_1)i_2}{\sqrt{(P_n^2 + P_{n+1}^2)(P_{n+2}^2 + P_{n+3}^2)}}\tanh^{-1}\left(\frac{\sqrt{P_n^2 + P_{n+1}^2}}{\sqrt{P_{n+2}^2 + P_{n+3}^2}}\right)$$

The explicit form of B can be write easily. □

Corollary 11. *Every Pell-Lucas split quaterninon $\mathcal{Q}_n = Q_n + Q_{n+1}i_1 + Q_{n+2}i_2 + Q_{n+3}i_3$ can be given in the form $\mathcal{Q}_n = Ae^{Bi_2}$, where A and B are complex numbers, that is*

$$\begin{aligned}A &= \frac{Q_n + Q_{n+1}i_1}{\sqrt{Q_{2n+2}+Q_{2n}}} \\ B &= \frac{Q_{2n+4} + Q_{2n+2} + 16(-1)^n i_1}{\sqrt{(Q_{2n+2}+Q_{2n})(Q_{2n+6}+Q_{2n+4})}}\tanh^{-1}\left(\sqrt{\frac{Q_{2n+2}+Q_{2n}}{Q_{2n+6}+Q_{2n+4}}}\right)\end{aligned}$$

Proof. The proof can be done similar to Theorem (10). □

Example 12. *Find the new polar representation for* $\mathcal{P}_1 = 1 + 2i_1 + 5i_2 + 12i_3$ *and* $\mathcal{Q}_0 = 2 + 2i_1 + 6i_2 + 14i_3$.

We have $\mathcal{P}_1 = Ae^{Bi_2}$*, where*

$$A = \frac{2(P_n + P_{n+1}i_1)}{\sqrt{Q_{2n+1} + Q_{2n}}} = \frac{1 + 2i_1}{\sqrt{5}}$$

and

$$B = \frac{Q_{2n+3} + Q_{2n+2} - 8(-1)^n i_1}{\sqrt{(Q_{2n+1} + Q_{2n})(Q_{2n+5} + Q_{2n+4})}} \tanh^{-1}\left(\sqrt{\frac{Q_{2n+1} + Q_{2n}}{Q_{2n+5} + Q_{2n+4}}}\right) = \frac{Q_5 + Q_4 + 8i_1}{\sqrt{(Q_3 + Q_2)(Q_7 + Q_6)}} \tanh^{-1}\left(\sqrt{\frac{Q_3 + Q_2}{Q_7 + Q_6}}\right) = \frac{29 + 2i_1}{13\sqrt{5}} \tanh^{-1}\left(\frac{\sqrt{5}}{13}\right)$$

4 Conclusions

In this paper, we give the Pell and Pell-Lucas split quaternions and obtain some fundamental identities. After that we introduce a new class of quaternions known as Pell and Pell-Lucas split quaternions. Our work is mainly concerned with polar representations of Pell and Pell-Lucas split quaternions similar to the real quaternions. An arbitrary Pell and Pell-Lucas split quaternion has been used to compute the argument and modulus for this. In further the current paper, it would be valuable to replicate similar approaches in dual split quaternions with Pell and Pell-Lucas number sequences.

References

[1] W.R.Hamilton, "On quaternions, or on a new system of imagniaries in algebra," Edinb. Dublin Philosophical Magazine, 25(3), pp.489-495 (1844).

[2] J. B.Kuipers, Quaternions and rotation sequences: a primer with applications to orbits, aerospace, and virtual reality, Princeton Uni- versity Press (1999).

[3] S. L. Altmann, Rotations, Quaternions, and Double Groups, Oxford University Press, Oxford (1986).

[4] S.J.Sangwine, N.L. Bihan, Quaternion Polar Representation with a Complex Modulus and Complex Argument Inspired by the Cayley Dickson Form, Advances in Applied Clifford Algebras, 20, pp. 111–120 (2010).

[5] J. Cockle , On Systems of Algebra Involving More than One Imag- inary, and on Equations of the Fifth Degree, The London, Edin- burgh, and Dublin Philosophical Magazine and Journal of Science, 35(3), pp.434–437 (1849).

[6] M.Ozdemir, A.A.Ergin, Rotations with unit timelike quaternions in Minkowski 3-space, Journal of Geometry and Physics, 56(2), pp.322-336 (2006).

[7] L. Kula, Y. Yaylı, Split Quaternions and Rotations in Semi- Euclidean Space R24, Journal of the Korean Mathematical Society, 44(6), pp.1313–1327 (2007).

[8] L. Kula, Y. Yaylı, Dual split quaternions and screw motion in Minkowski 3-space, Iranian Journal of Science Technology, Trans- action A, 30(A3), pp.245–258 (2006).

[9] E. Ata, Y. Yıldırım, A different polar representation for generalized and generalized dual quaternions, Advances in Applied Clifford Al- gebras, 28, pp.1–20 (2018).

[10] A. Atasoy, E. Ata, Y. Yaylı, Y. Kemer, A New Polar Representation for Split and Dual Split Quaternions,“ Adv. Appl. Clifford Alg. 27, pp.2307–2319 (2017).

[11] Sangwine SJ, Bihan NL. Quaternion polar representation with a complex modulus and complex argument inspired by the Cayley-Dickson form. Adv. Appl. Clifford Alg. 2010;20(1):111–120.

[12] Hanson, A.J.: Visualizing Quaternion. Elsevier, Morgan-Kaufman (2005)

[13] Ozdemir, M., Ergin, A.A.: Rotations with unit timelike quaternions in Minkowski 3-space. J. Geom. Phys. 56, 322–336 (2006)

[14] A. F. Horadam, *Complex Fibonacci Numbers and Fibonacci Quaternions.* American Math. Monthly 70 (1963), 289-291.

[15] T. Koshy, *Fibonacci and Lucas Numbers with Applications.* A Wiley-Intersience Publication, USA, 2001.

[16] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section.* Ellis Horwood Limited Publ., England, 1989.

[17] E. Verner and Jr. Hoggatt, *Fibonacci and Lucas Numbers.* The Fibonacci Association, 1969.

[18] P. Koshy T., Pell and Pell-Lucas Numbers with Applications. London: Springer New York Heidelberg Dordrecht, (2014) 31-35.

[19] Bicknell N (1975) A primer on the Pell sequence and related sequence. Fibonacci Quart 13(4):345–349

[20] Catarino P (2013) A note involving two-by-two matrices of the k-Pell and k-Pell-Lucas sequences. Int Math Forum 8(32):1561–1568

[21] Catarino P, Vasco P (2013) Some basic properties and a two-by-two matrix involving the k-Pell numbers. Int J Math Anal 7(45):2209–2215