Original Research Article

On Sums of Squares Involving Integer Sequence: $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2)$

Abstract

The study of diophantine equations, which are polynomial equations seeking integer solutions, has been a very interesting journey in the field of number theory. Historically, these equations have attracted the attention of many mathematicians due to their intrinsic challenges and their significance in understanding the properties of integers. In this current study, we examine a diophantine equation relating the sum of squared integers from specific sequences to a variable d. In particular, the diophantine equation $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2)$ is introduced and partially characterized. The objective is to determine the conditions under which integer solutions for (w_r, d) exist within this diophantine equation. The methodology of solving this problem entails computational analysis, decomposing polynomials, factorizing polynomials, and exploring the solution set of the given equation.

Keywords: Diophantine Equation; Sums of Squares, Integer Sequence

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1 Introduction

Diophantine equations, tracing their roots back to the era of the ancient Greek mathematician Diophantus, continue to be a captivating challenge within number theory. These equations, seeking integer solutions, hold significant importance due to their real-life applications. Despite the extensive exploration of various Diophantine equations, including renowned challenges like Fermat's Last Theorem, Ramanujan Nagell equation, and Lebesgue Nagell equation, as well as studies focusing on polynomials of degree less than 5, the specific examination of the diophantine equation $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2)$ remains largely uncharted. Recent research has delved into the intricacies of polynomials with degrees less than 5, as referenced in [1, 2,3,10,11,12]. For a comprehensive understanding of studies related to Fermat's Last Theorem and Ramanujan Nagell equations, readers are encouraged to explore[4,5,6,7,8,9,13,14,15,16].Within the existing body of work, the literature concerning the diophantine equation $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2)$ remains largely unexplored. This study aims to contribute to this knowledge gap by introducing and developing the formula $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2)$, seeking to enhance our comprehension of this specific diophantine equation within the broader landscape of mathematical exploration.

2 Main Results

In the following sections, we begin by articulating our observations as a conjecture, and subsequently, we proceed to obtain solutions for particular instances of the aforementioned diophantine equation.

Conjecture 2.1. *For any integer n divisible by 3, the diophantine equation*

$$\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2) \cdots (1)$$

admits solutions in integers if $w_n - w_{n-1} = w_{n-1} - w_{n-2} = \cdots = w_2 - w_1 = d$

In the subsequent sections, the focus of this investigation revolves around identifying the values of the variables $(n, w_1, w_2, \cdots, w_n, d)$ that fulfill the conditions of equation (1). Consequently, distinct cases have been established.

Theorem 2.2. *Consider equation (1) satisfying the condition $(n, w_1, w_2, w_3, d) = (3, w_1, w_2, w_3, d)$ Then, the diophantine equation*

$$w_1^2 + w_2^2 + w_3^2 + d^2 = 3(d^2 + w_2^2)$$

has solution in integers if $w_3 - w_2 = w_2 - w_1 = d$.

Proof. Assume that $w_2 = w_1 + d, w_3 = w_1 + 2d, w_3 = w_1 + 2d$ and Consider the equation $w_1^2 + w_2^2 + w_3^2 + d^2 = 3(d^2 + w_2^2) \cdots (2.1)$. The, left hand side of equation (2.1) expressed as

$$w_1^2 + w_2^2 + w_3^2 + d^2 = w_1^2 + (w_1 + d)^2 + (w_1 + 2d)^2 + d^2$$

simplifies to

$$3w_1^2 + 6w_1 d + 6d^2 = 3(w_1^2 + 2w_1 d + 2d^2) \cdots (2.1.1).$$

Decomposing equation $(2.1.1)$ into thrice sums of two squares, we obtain,

$$3w_1^2 + 6w_1d + 6d^2 = 3(w_1^2 + 2w_1d + 2d^2)$$
$$= 3((w_1^2 + 2w_1d + d^2) + d^2) = 3(d^2 + (w_1 + d)^2) = 3(d^2 + w_2^2).$$

This complete the proof.

□

Theorem 2.3. *Consider equation* (1) *satisfying the condition* $(n, w_1, w_2, \cdots, w_6, 2d) = (6, w_1, w_2, \cdots, w_6, 2d)$. *Then, the diophantine equation*

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 + 2d^2 = 3(2d^2 + w_2^2 + w_5^2)$$

has solution in integers if $w_6 - w_5 = w_5 - w_4 = w_4 - w_3 = w_3 - w_2 = w_2 - w_1 = d.$

Proof. Let $w_2 = w_1 + d, w_3 = w_1 + 2d, w_3 = w_1 + 2d, w_4 = w_1 + 3d, w_5 = w_1 + 4d, w_6 = w_1 + 5d$ and Consider the equation $w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 + 2d^2 = 3(2d^2 + w_2^2 + w_5^2) \cdots (2.2)$. The, left hand side written as

$$w_1^2 + (w_1 + d)^2 + (w_1 + 2d)^2 + (w_1 + 3d)^2 + (w_1 + 4d)^2 + (w_1 + 5d)^2 + (w_1 + 6d)^2 + 2d^2$$

reduces to

$$6w_1^2 + 30w_1d + 57d^2 = 3(w_1^2 + 10w_1d + 19d^2) = 3(2d^2 + w_1^2 + 10w_1d + 17d^2) \cdots (2.2.1).$$

Breaking equation $(2.2.1)$ into thrice sums of sums of four squares, we get,

$$3(2d^2 + w_1^2 + 10w_1d + 17d^2) = 3(2d^2 + (w_1^2 + 2w_1d + d^2) + (w_1^2 + 8w_1d + 16d^2))$$
$$= 3(2d^2 + (w_1 + d)^2 + (w_1 + 4d)^2) = 3(2d^2 + w_2^2 + w_5^2)$$

This concludes the proof.

□

Theorem 2.4. *Consider equation* (1) *satisfying the condition*

$$(n, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, 3d) = (9, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, 3d).$$

Then, the diophantine equation

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 + w_7^2 + w_8^2 + w_9^2 + 3d^2 = 3(3d^2 + w_2^2 + w_5^2 + w_8^2)$$

has solution in integers if $w_9 - w_8 = w_8 - w_7 = w_7 - w_6 = w_6 - w_5 = w_5 - w_4 = w_4 - w_3 = w_3 - w_2 = w_2 - w_1 = d.$

Proof. Consider the equation $w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 + w_7^2 + w_8^2 + w_9^2 + 3d^2 = 3(3d^2 + w_2^2 + w_5^2 + w_8^2)$. Assume that $w_2 = w_1 + d, w_3 = w_1 + 2d, w_3 = w_1 + 2d, w_4 = w_1 + 3d, w_5 = w_1 + 4d, w_6 = w_1 + 5d, w_7 = w_1 + 6d, w_8 = w_1 + 7d, w_9 = w_1 + 8d$. The, left hand side expressed as,

$$w_1^2 + (w_1 + d)^2 + (w_1 + 2d)^2 + (w_1 + 3d)^2 + (w_1 + 4d)^2 + (w_1 + 5d)^2 + (w_1 + 6d)^2 + (w_1 + 7d)^2 + (w_1 + 8d)^2 + 3d^2$$

simplifies to

$$9w_1^2 + 72w_1d + 207d^2 = 3(3w_1^2 + 24w_1d + 69d^2) \cdots (2.3).$$

Decomposing equation (2.3) into triple sums of squares, we obtain,

$$= 3(3d^2 + (w_1^2 + 2w_1d + d^2) + (w_1^2 + 8w_1d + 16d^2) + (w_1^2 + 14w_1d + 49d^2))$$
$$= 3(3d^2 + (w_1 + d)^2 + (w_1 + 2d)^2) + (w_1 + 4d)^2 + (w_1 + 7d)^2 = 3(3d^2 + w_2^2 + w_5^2 + w_8^2)$$

This complete the proof.

□

Theorem 2.5. *Consider equation* (1) *satisfying the condition* $(n, w_1, w_2, \cdots, w_{12}, 4d) = (12, w_1, w_2, \cdots, w_{12}, 4d)$. *Then, the diophantine equation*

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 + w_7^2 + w_8^2 + w_9^2 + w_{10}^2 + w_{11}^2 + w_{12}^2 + 4d^2 = 3(4d^2 + w_2^2 + w_5^2 + w_8^2 + w_{11}^2)$$

has solution in integers if $w_{12} - w_{11} = w_{11} - w_{10} = w_{10} - w_9 = w_9 - w_8 = w_8 - w_7 = w_7 - w_6 = w_6 - w_5 = w_5 - w_4 = w_4 - w_3 = w_3 - w_2 = w_2 - w_1 = d.$

Proof. Consider the equation,

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 + w_7^2 + w_8^2 + w_9^2 + w_{10}^2 + w_{11}^2 + w_{12}^2 + 4d^2 = 3(4d^2 + w_2^2 + w_5^2 + w_8^2 + w_{11}^2)$$

and Suppose that $w_2 = w_1 + d, w_3 = w_1 + 2d, w_3 = w_1 + 2d, w_4 = w_1 + 3d, w_5 = w_1 + 4d, w_6 = w_1 + 5d, w_7 = w_1 + 6d, w_8 = w_1 + 7d, w_9 = w_1 + 8d, w_{10} = w_1 + 9d, w_{11} = w_1 + 10d, w_{12} = w_1 + 11d.$
The, left hand side expressed as
$w_1^2 + (w_1 + d)^2 + (w_1 + 2d)^2 + (w_1 + 3d)^2 + (w_1 + 4d)^2 + (w_1 + 5d)^2 + (w_1 + 6d)^2 + (w_1 + 7d)^2 + (w_1 + 8d)^2 + (w_1 + 10d)^2 + (w_1 + 11d)^2 + 4d^2.$
simplifies to

$$12w_1^2 + 132w_1d + 510d^2 = 3(4w_1^2 + 44w_1d + 170d^2) \cdots (2.4).$$

Splitting equation (2.4) into thrice sums of squares, we obtain,
$3(4d^2 + (w_1^2 + 2w_1d + d^2) + (w_1^2 + 8w_1d + 16d^2) + (w_1^2 + 14w_1d + 49d^2) + (w_1^2 + 18w_1d + 81d^2) + (w_1^2 + 20w_1d + 100d^2) + (w_1^2 + 22w_1d + 121d^2))$

$$= 3(4d^2 + (w_1 + d)^2 + (w_1 + 4d)^2 + (w_1 + 7d)^2 + (w_1 + 10d)^2)$$

$$= 3(4d^2 + w_2^2 + w_5^2 + w_8^2 + w_{11}^2).$$

This complete the proof.

□

3 Conclusion

In summary, the solution of the diophantine equation $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2)$ under the specified conditions of a common difference d between consecutive terms $w_n, w_{n-1}, \cdots,$ w_2, w_1 where $w_n - w_{n-1} = w_{n-1} - w_{n-2} = \cdots = w_2 - w_1 = d$ has been achieved for $n \leq 12$. This solution provides valuable insights into the relation among the sequence terms, enhancing our understanding of the inherent patterns and structures within the equation. For future investigations, it is recommended to explore extensions of this diophantine equation by proving conjecture (1).

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