Conditions of Safe Dominating Set in Some Graph Families

Original Article

Abstract

For a nontrivial connected graph G and nonempty subset $S \subseteq V(G)$, S is a safe dominating set of G if and only if S is a dominating set of G and every component X of $G[S]$ and every component Y of $G[V(G) \setminus S]$ adjacent to X, $|X| \geq |Y|$. Moreover, S is called a minimum safe dominating set if S is a safe dominating set of the smallest size in a given graph. The cardinality of the minimum safe dominating set of G is the safe domination number of G, denoted by $\gamma_s(G)$. In this paper, we characterized the safe dominating set and determine its corresponding safe domination number in some special classes of graphs.

Keywords: domination, safe domination, minimum safe dominating set, safe domination number, safe dominating set.

2024 Mathematics Subject Classification: 05C35

1 Introduction

Safe set is recently introduced parameter within the field of graph theory. The intention of this study is to help in terms of Facility Location Problem or (FLP) which refers to the placement and management of a facility in order to obtain the maximum goal with minimizing costs. Fujita et al., [4] studied the FLP and introduced the concept of safe set and connected safe set. They derived their concepts from a class of facility location problems, aiming to identify a "safe" subset of nodes within a network where facilities can be strategically positioned.

This paper extends the study of safe sets in some common graphs by combining the domination in safe sets to form a new parameter called safe dominating set. This study will investigate the safe dominating set and safe domination number in some graph families. Also, this paper aims to provide conditions of safe dominating set in some classes of graphs.

2 Preliminary Notes

Some definitions of the concepts covered in this study are included below. You may refer on the remaining terms and definitions in [5], [6], [8], [9], [11], [12].

Definition 2.1. [1] Let G be a simple graph. A set $S \subseteq V(G)$ is a **dominating set** of G, if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The **domination number** $\gamma(G)$ is the minimum cardinality of dominating set.

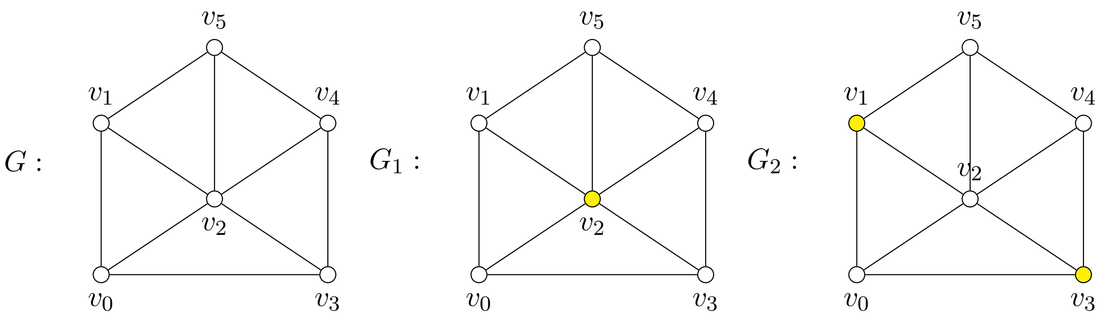

Figure 1: A graph G and its dominating sets

Definition 2.2. [3] The subgraph of a graph G induced by $S \subseteq V(G)$ is denoted by $\langle S \rangle_G$. A **component** of G is a connected induced subgraph of G with an inclusionwise maximal vertex set. A non-empty set $S \subseteq V(G)$ of vertices is a **safe set** if, for every component A of $\langle S \rangle_G$ and every component B of $\langle V(G) \setminus S \rangle_G$ adjacent to A, it holds that $|A| \geq |B|$. The **safe number** denoted by $s(G)$ of G is the minimum cardinality of a safe set of G.

Example 2.1. *Consider the graph in figure 2. We have $S = v_4, v_5, v_6$, then we have an induced subgraph of S in G which is A_1. Then we have $V(G) \setminus S$ with 2 components, B_1 and B_2. Clearly, A_1 is adjacent to B_1 and B_2 and $|A_1| = |B_1| > |B_2|$. Thus, S is a safe set.*

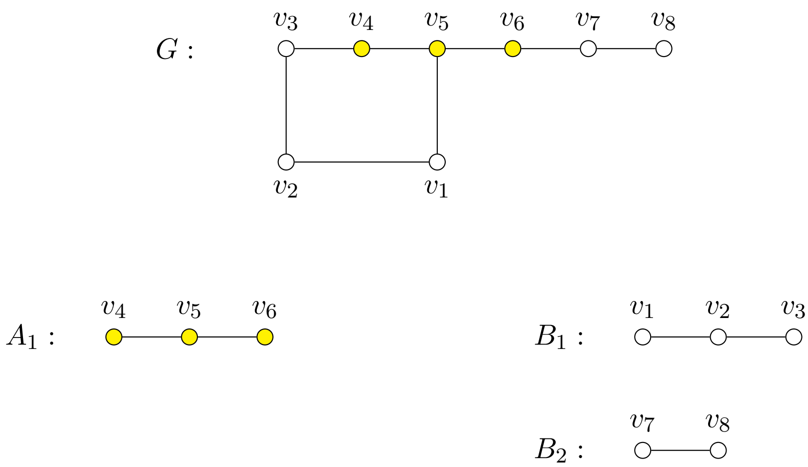

Figure 2: A graph G and its safe sets

Definition 2.3. [2] A nonempty subset $S \subseteq V(G)$ is a **safe dominating set** if and only if S is a dominating set of G and every component A of $G[S]$ and every component B of $G[V(G) \setminus S]$ adjacent to A, $|A| \geq |B|$. Moreover, S is called a **minimum safe dominating set** denoted by $\gamma_s - set$, if S is a safe dominating set of smallest size in a given graph. The cardinality of minimum safe dominating set of G is the **safe domination number** of G, denoted by $\gamma_s(G)$.

Example 2.2. *Consider the graph in figure 3. We have $S = v_4, v_5, v_6$, then we have an induced subgraph of S in G which is A_1. Then we have $V(G) \setminus S$ with 4 components, B_1 , B_2 , B_3 and B_4. Clearly, A_1 is adjacent to B_1 and B_2 , B_3 and B_4 and $|A_1| = |B_1| > |B_2| = |B_3| = B_4$. Thus, S is a safe dominating set.*

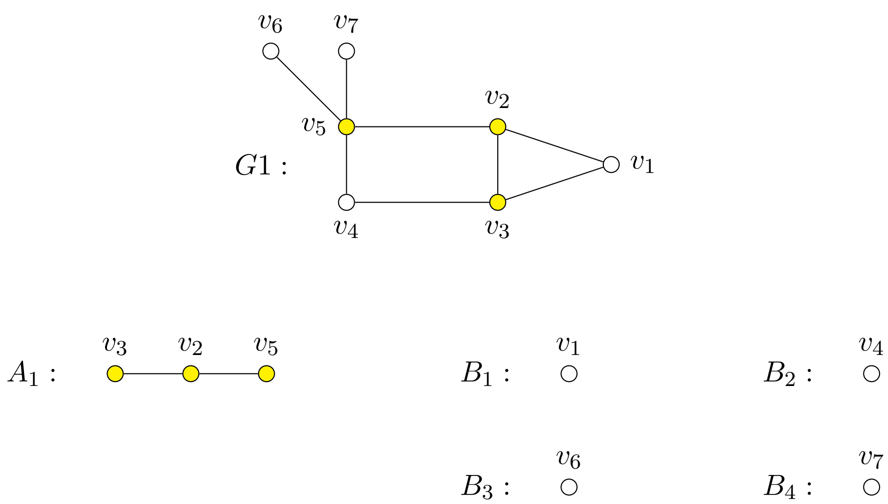

Figure 3: A graph G and its safe dominating set

3 Main Results

In this section, the characteristics of minimum rings dominating set in the total graph of some graph families are presented. We also determine the rings domination number for each of the graphs being classified in this paper.

Theorem 3.1. *Let $\varnothing \neq S \subseteq V(Cr_{n,n})$ be a safe dominating set of crown graph. Then S is a minimum safe dominating in $Cr_{n,n}$ if and only if $S = A$ or $S = B$.*

Proof. Let $\varnothing \neq S \subseteq V(Cr_{n,n})$ be a minimum safe dominating in $Cr_{n,n}$. Suppose, $S \neq A$ or $S \neq B$. If $|S| > |A|$, then S is not a minimum safe dominating set of $Cr_{n,n}$ since A is a safe dominating set. A contradiction to the assumption that S is a minimum safe dominating set in $Cr_{n,n}$. Suppose further, $|S| < |A|$. Then either S is not a dominating set or there exist a component Y of $Cr_{n,n}[V(Cr_{n,n}) \setminus S]$ such that $|Y| > |X|$, where X is a component of $Cr_{n,n}[S]$. In either case S is not a safe dominating set. A contradiction to the assumption that S is a safe dominating set. Thus, $S = A$. Similarly for $S = B$.

Conversely, suppose $S = A$ or $S = B$. Without loss of generality, let $S = A$. Clearly, S is a safe dominating set. Now, suppose S is not a minimum safe dominating set. Then there exists S_o such that $|S_o| < |S|$ a safe dominating set in $Cr_{n,n}$. This is not possible since S will not be a dominating set. Thus, $S = A$ is the minimum safe dominating set of $Cr_{n,n}$. Similarly for $S = B$. □

Corollary 3.2. *For a crown graph* $Cr_{n,n}$,

$$\gamma_s(Cr_{n,n}) = n$$

Proof: This immediately follows from Theorem 3.1.

Example 3.3. *Refer to Figure 4. Consider the crown graph* $Cr_{6,6}$ *with a vertex set of* $V(Cr_{6,6}) = \{u_1, u_2, u_3, u_4, u_5, u_6, v_1, v_2, v_3, v_4, v_5, v_6\}$. *The set* $S = \{\{u_1\}, \{u_2\}, \{u_3\}, \{u_4\}, \{u_5\}, \{u_6\}\}$ *is a dominating set and every component* X *in* $Cr_{6,6}[S]$, $|X| = 1$ *and every component* Y *in* $Cr_{6,6}[V(Cr_{6,6}) \setminus S]$, $|Y| = 1$. *Since* $|X| = |Y|$, *thus* S *is a safe dominating set. By Theorem 3.1, the minimum safe dominating set of* $Cr_{n,n}$ *equal is* A *or* B *where* A *and* B *are the* $2n$ *graph of* $Cr_{n,n}$. *Hence,* $\gamma_s(Cr_{6,6}) = |A| = |B| = 6$

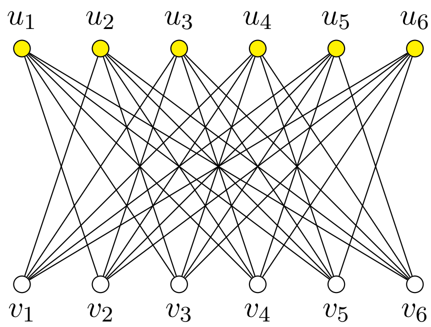

Figure 4: A graph $Cr_{6,6}$ with $|\gamma_s(Cr_{6,6})| = 6$

Theorem 3.4. *Let* G *be a barbell graph,* $B_{n,n}$. *Then* $\varnothing \neq S \subseteq V(G)$ *is a safe dominating set of* G *if and only if* $u_1, v_1 \in S$ *and* $|G[S]| \geq |G[V(G) \setminus S]|$.

Proof. Let $\varnothing \neq S \subseteq V(G)$ be a safe dominating set in $B_{n,n}$ Clearly, $u_1, v_1 \in S$ since $G[S]$ is connected, $|G[S]| \geq |G[V(G) \setminus S]|$ so that every component Y in $G[V(G) \setminus S]$, $|Y| \leq |G[S]|$.

Conversely, suppose $u_1, v_1 \in S$ and $|G[S]| \geq |G[V(G) \setminus S]$. Then $G[S]$ has only one component which is itself and every component Y in $G[V(G) \setminus S]$, $|Y| \leq |G[S]|$. Since $u_1, v_1 \in S$ is a dominating set, thus S is a safe dominating set. □

Example 3.5. *Consider the graph in Figure 5. The set* $S = \{u_1, u_2, v_1, v_2\}$. *Observe that* u_1 *is in* S *and* u_1 *dominates all vertices in* U *and* v_1 *is in* S *and* v_1 *dominates all vertices in* V. *Thus,* S *is a dominating set. Observe further that* S *is a connected set, thus it has only one component and* $B_{n,n}[V(B_{n,n}) \setminus S]$ *is disconnected and it has two components* Y_1 *and* Y_2 *where* $Y_1 = \{u_3, u_4, u_5, u_6\}$ *and* $Y_2 = \{v_3, v_4, v_5, v_6\}$. *Clearly,* $|S| = |Y_1| = |Y_2|$. *Hence,* S *is a safe dominating set.*

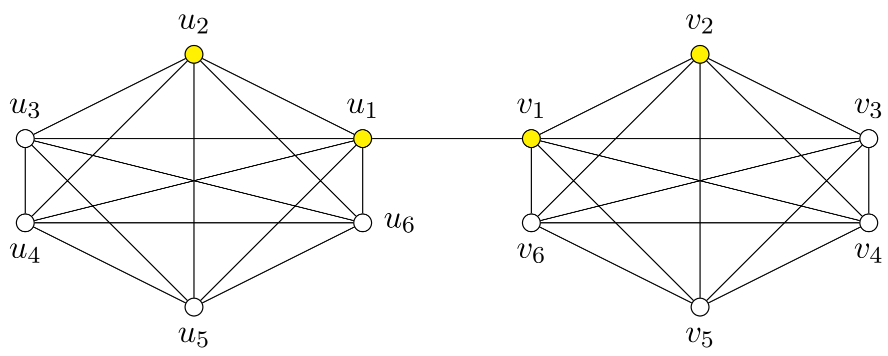

Figure 5: A barbell graph $B_{6,6}$ with $|\gamma_s(B_{6,6})| = 4$

Corollary 3.6. *For a barbell graph $B_{n,n}$,*

$$\gamma_s(B_{n,n}) = \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0 (\text{mod } 3) \\ \frac{2n+1}{3}, & \text{if } n \equiv 1 (\text{mod } 3) \\ \frac{2n+2}{3}, & \text{if } n \equiv 2 (\text{mod } 3) \end{cases}$$

Proof: Let $\varnothing \neq S \subseteq V(G)$ be a safe dominating set in G. Consider the following cases:

Case 1: $n \equiv 0\ (mod\ 3)$

Choose $S = \{u_1, v_1\} \cup S_1 \cup S_2$ where $S_1 \subseteq V(K_n)$ and $S_2 \subseteq V(K_m)$ where $m = n$ such that $|S_1| = \frac{n-3}{3}$ and $|S_2| = \frac{n-3}{3}$. Clearly, S is a dominating set. Now, $|S| = |\{u_1, v_1\}| + |S_1| + |S_2| = 2 + \frac{n-3}{3} + \frac{n-3}{3} = \frac{2n}{3}$. On the other hand, the components Y_1 and Y_2 of $G[V(G) \setminus S]$ where $Y_1[V(K_n) \setminus S_1]$ and $Y_2[V(K_m) \setminus S_2]$ has order $\frac{2n}{3} = |Y_1| = |Y_2|$. Thus $|S| = \frac{2n}{3} \geq \frac{2n}{3} = |Y_1| = |Y_2|$. Hence, S is a a safe set and it follows that S is a safe dominating set. We are left to show that S is the minimum safe dominating set. Suppose S is not a minimum safe dominating set. Then there exists $S_o \subseteq V(G)$ such that $|S_o| < |S|$. Since S_o is a domininating set, then u_1 and v_1 is in S_o. Hence, it is immediate to assume that $|S_o| \leq \frac{2n}{3} - 1$. This imply that there exists a component in $G[V(G) \setminus S_o]$, say Y_o, such that $|Y_o| \geq \frac{2n}{3} + 1$. Thus, $|S_o| \leq \frac{2n}{3} - 1 < \frac{2n}{3} + 1 \leq |Y_o|$, implying that $|S_o| < |Y_o|$. A contradiction, since S_o is not a safe set in G. Therefore, no such S_o exist with $|S_o| \leq \frac{2n}{3} - 1$. Thus, S is the minimum safe dominating set. Therefore, $\gamma_s(G) = |S| = \frac{2n}{3}$.

Case 2: $n \equiv 1\ (mod\ 3)$

Choose $S = \{u_1, v_1\} \cup S_1 \cup S_2$ where $S_1 \subseteq V(K_n)$ and $S_2 \subseteq V(K_m)$ where $m = n$ such that $|S_1| = \frac{n-1}{3}$ and $|S_2| = \frac{n-4}{3}$. Clearly, S is a dominating set since $u_1, v_1 \in S$. Now, $|S| = |\{u_1, v_1\}| + |S_1| + |S_2| = 2 + \frac{n-1}{3} + \frac{n-4}{3} = \frac{2n+1}{3}$. On the other hand, $G[V(G) \setminus S] = K_n[V(K_n) \setminus S_1] \cup K_m[V(K_m) \setminus S_2]$ where $|V(K_n) \setminus S_1| = \frac{2n+1}{3} - 1 = \frac{2n-2}{3}$ and $|V(K_m) \setminus S_2| = \frac{2n+1}{3} = \frac{2n+1}{3}$. Thus, $|S| = \frac{2n+1}{3} \geq \frac{2n-2}{3} = |V(K_n) \setminus S_1|$ and $|S| = \frac{2n+1}{3} \geq \frac{2n+1}{3} = |V(K_m) \setminus S_2|$. Thus, S is a safe set and it follows that S is a safe dominating set. We are left to show that S is the minimum safe dominating set. Suppose S is not a minimum safe dominating set. Then there exists $S_o \subseteq V(G)$ such that $|S_o| < |S|$. Since S_o is a dominating set, clearly $u_1, v_1 \in S_o$. Hence, it is immediate to assume that $|S_o| \leq \frac{2n-2}{3}$. Thus, $|S_o| \leq \frac{2n-2}{3} < \frac{2n+1}{3} = |V(K_m) \setminus S_2|$. Thus, S_o is not a safe set and it follows that S_o is not a safe dominating set. Hence, S is the minimum safe dominating set. Therefore, $\gamma_s(G) = |S| = \frac{2n+1}{3}$.

Case 3: $n \equiv 2\ (mod\ 3)$

Choose $S = \{u_1, v_1\} \cup S_1 \cup S_2$ where $S_1 \subseteq V(K_n)$ and $S_1 \subseteq V(K_m)$ where $m = n$ such that $|S_1| = \frac{n-2}{3}$ and $|S_2| = \frac{n-2}{3}$. Clearly, S is a dominating set. Now, $|S| = |\{u_1, v_1\}| + |S_1| + |S_2| = 2 + \frac{n-2}{3} + \frac{n-2}{3} = \frac{2n+2}{3}$. On the other hand, $G[V(G) \setminus S] = K_n[V(K_n) \setminus S_1] \cup K_m[V(K_m) \setminus S_2]$ where $|V(K_n) \setminus S_1| = |V(K_m) \setminus S_2| = \frac{2n-1}{3}$. Thus $|S| = \frac{2n+2}{3} > \frac{2n-1}{3} = |Y_1| = |Y_2|$. Thus, S is a a safe set and it follows that S is a safe dominating set. We are left to show that S is the minimum safe dominating set. Suppose S is not a minimum safe dominating set. Then there exists $S_o \subseteq V(G)$ such that $|S_o| < |S|$. Since S_o is a dominating set, then $u_1, v_1 \in S_o$. Hence, it is immediate to assume that $|S_o| \leq \frac{2n-1}{3} - 1 = \frac{2n-4}{3}$. This imply that there exists a component in $G[V(G) \setminus S_o]$, say Y_o, such that $|Y_o| \geq \frac{2n+5}{3}$. Thus, $|S_o| \leq \frac{2n-4}{3} < \frac{2n+5}{3} \leq |Y_o|$, implying that $|S_o| < |Y_o|$. A contradiction, since S_o is not a safe set in G. Therefore, no such S_o exist with $|S_o| \leq \frac{2n-4}{3}$. Thus, S is the minimum safe dominating set. Therefore, $\gamma_s(G) = |S| = \frac{2n+2}{3}$. □

For the next theorem, recall that for a helm graph H_n, for $n \geq 3$, there exists $S \subseteq V(H_n)$ such that $H_n[S]$ is a cycle graph.

Theorem 3.7. *Let H_n be helm graph. Then $\varnothing \neq S \subseteq V(H_n)$ is a minimum safe dominating set in H_n if and only if $S = V(C_n) \subseteq V(H_n)$.*

Proof. Let $\varnothing \neq S \subseteq V(H_n)$ be the minimum safe dominating set of H_n. Since S is a dominating set, the only dominating set of H_n are $V(H_n)$, $S = V(C_n)$, and $S = \{x\} \cup V_p$ where $\deg_{H_n}(x) = \Delta(H_n)$ and V_p is the set of the pendant vertices of H_n. Clearly, $S \neq V(H_n)$. If $S = \{x\} \cup V_p$, then $H_n[V(H_n) \setminus S] = C_n$, implying that it has only one component and $H_n[S]$ is an empty graph. Thus, for each component X in $H_n[S]$, $|X| < |V(C_n)|$. Thus, $S \neq \{x\} \cup V_p$. Now, if $S = V(C_n)$, then $H_n[S] = C_n$ and $H_n[V(H_n) \setminus S]$ is an empty graph. Thus, $|V(C_n)| > |Y|$, for every trivial graph component Y in $H_n[V(H_n) \setminus S]$. Therefore, $S = V(C_n)$.

Conversely, suppose $S = V(C_n)$. Then by the argument above, S is a safe dominating set. Since there can be no safe dominating set S_o such that $|S_o| < |S|$. Hence, S must be the minimum safe dominating set of H_n. □

Example 3.8. *Refer to Figure 6. Consider H_{12} with $|S| = V(C_{12}) = 12$. Observe that every component Y in $H_{12}[V(C_{12}) \setminus S]$, $|Y| = 1$. Since $|S| = 12 > 1 = |Y|$. Thus, S is a safe dominating set. By Theorem 3.3.1, the minimum safe dominating set of H_n is equal to $V(C_n)$ in H_n. Thus, S is the minimum safe dominating set.*

Corollary 3.9. *For a helm graph H_n,*

$$\gamma_s(H_n) = n$$

Proof: This immediately follows from Theorem 3.7.

For the next theorem, recall that for a caterpillar graph G, there exists $S \subseteq V(G)$ such that $G[S]$ is a path graph.

Theorem 3.10. *Let G be a caterpillar graph. Then $S \subseteq V(G)$ is a minimum safe dominating set if and only if $S = V(P_n) \subseteq V(G)$.*

Proof. Let $\varnothing \neq S \subseteq V(G)$ be the minimum safe dominating set of G. Since S is a dominating set, the only dominating set of G are $V(G)$, $S = V(P_n)$, and $S = V_p$ where V_p is the set of the pendant vertices of G. Clearly, $S \neq V(G)$. If $S = V_p$, then $G[V(G) \setminus S] = P_n$, implying that it has only one component and $G[S]$ is an empty graph. Thus, for each component X in $G[S]$, $|X| < |V(P_n)|$.

Hence, $S \neq V_p$. Now, if $S = V(P_n)$, then $G[S] = P_n$ and $G[V(G) \setminus S]$ is an empty graph. Thus, $|V(P_n)| > |Y|$, for every trivial graph component Y in $G[V(G) \setminus S]$. Therefore, $S = V(P_n)$.

Conversely, suppose $S = V(P_n)$. Then by the argument above, S is a safe dominating set. Since there can be no safe dominating set S_o such that $|S_o| < |S|$. Hence, S must be the minimum safe dominating set of G. □

Example 3.11. *Refer to Figure 7. Consider the caterpillar graph $C_4(m_1+1, m_2+1, m_3+1, m_4+1)$ with a set $S = \{u_1, u_2, u_3, u_4\}$. Observe that S is a dominating set and has only one component X of $C_4(m_1+1, m_2+1, m_3+1, m_4+1)[S]$ such that $|X| = 4$ and every component Y in $C_4(m_1+1, m_2+1, m_3+1, m_4+1)[V(C_4(m_1+1, m_2+1, m_3+1, m_4+1)) \setminus S]$, $|Y| = 1$. Since $X > Y$, thus S is a safe dominating set. By Theorem 3.4.1, the minimum safe dominating set of $C_n(m_1+1, ..., m_n+1)$ is equal $V(P_n)$ where $V(P_n) \subseteq V(C_4(m_1+1, m_2+1, m_3+1, m_4+1))$ and $S = V(P_n)$. Thus, $\gamma_s(C_4(m_1+1, m_2+1, m_3+1, m_4+1)) = 4$.*

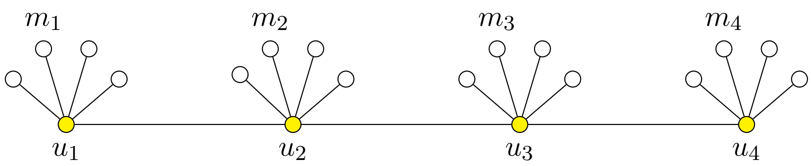

Figure 6: A caterpillar graph $C_4(m_1+1, m_2+1, m_3+1, m_4+1)$ with $\gamma_s(C_4(m_1+1, m_2+1, m_4+1, m_4+1) = 4$

Corollary 3.12. *For a caterpillar graph G,*

$$\gamma_s(G) = n$$

Proof: This immediately follows from Theorem 3.10.

Theorem 3.13. *Let $B_{n,k}$ be a Banana Tree graph. Then $\emptyset \neq S \subseteq V(B_{n,k})$ is a safe dominating set in $B_{n,k}$ if and only if S is a dominating set and $B_{n,k}[V \setminus S]$ is an empty graph.*

Proof. Suppose $S \subseteq V(B_{n,k})$ is a safe dominating set in $B_{n,k}$. Clearly, S is a dominating set in $B_{n,k}$. Now, suppose $B_{n,k}[V(B_{n,k}) \setminus S]$ is not an empty graph. Then there exist edge e_1 joining two vertices $v, u \in B_{n,k}[V(B_{n,k}) \setminus S]$. Hence, this edge e_1 forms a component B_1 in $B_{n,k}[V(B_{n,k}) \setminus S]$. Suppose further $deg(u) = 1$ and $deg(v) = k-1$. Then v must be in S_1 so that $u \in N[S] = V(B_{n,k})$. A contradiction. Suppose $deg(u) = 2$ and $deg(v) = n$. Then $u \in N[w]$ such that $deg(y) = 2$ and $y \in S$. Since the remaining vertices in $B_{n,k}[V(B_{n,k}) \setminus S]$ are isolated, every component X in $B_{n,k}[S]$ has $|S| = 2$ or $|X_0| = 1$ where X_0 is the trivial component containing w. Another contradiction. Thus, $B_{n,k}[V(B_{n,k}) \setminus S]$ is an empty graph.

Conversely, suppose S is a dominating set and $B_{n,k}[V(B_{n,k}) \setminus S]$ is an empty graph. Thus, every component Y of $B_{n,k}[V(B_{n,k}) \setminus S]$, $|Y| = 1$. Since S is a dominating set and every component X of $B_{n,k}[S]$, $|X| = 1$. Thus, $|Y| \leq |X|$. Hence, S is a safe dominating set.

□

Example 3.14. *Refer to Figure 8. Observe that every component X in $B_{5,5}[S]$, $|X| = 1$ and every component Y in $B_{5,5}[V(B_{5,5}) \setminus S]$, $|Y| = 1$. Since $|X| = |Y|$. Thus, S is a safe dominating set in $B_{5,5}$.*

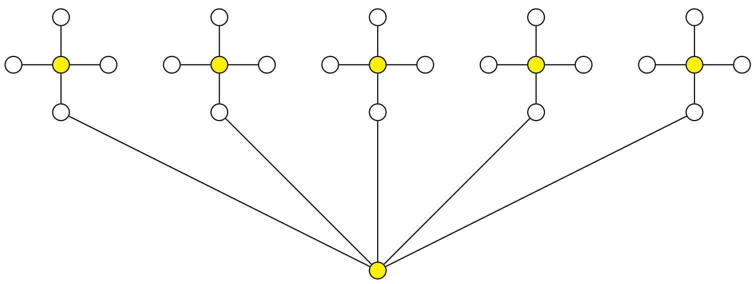

Figure 7: A banana tree graph $B_{5,5}$ with $\gamma_s(B_{5,5}) = 6$

Corollary 3.15. *For a banana graph $B_{n,k}$,*

$$\gamma_s(B_{n,k}) = n + 1$$

Proof. Choose $S \subseteq V(B_{n,k})$ to be $S = \{x \mid \deg(x) = n \text{ or } \deg(x) = k-1\}$. Then by Theorem 3.0.9, S is a safe dominating set and $|S| = n+1$. Now, suppose S is not the minimum safe dominating set. Then there exists S_o such that S_o is a safe dominating set in $B_{n,k}$ such that $|S| > |S_o|$, implying that $|S_o| \leq n$. Without loss of generality, suppose $|S_o| = n$. By Theorem 2.2.4 the domination number of $B_{n,k} = n + 1$. Hence S_o is not a dominating set. Thus, $S = n + 1$ is the minimum safe dominating set. □

Theorem 3.16. *Let $B_{n,n}$ be a bistar graph. Then $\varnothing \neq S \subsetneq V(B_{n,n})$ is a safe dominating set of $B_{n,n}$ if and only if u and $v \in S$. Consequently, $\gamma_s(B_{n,n}) = 2$.*

Proof. Let $\varnothing \neq S \subsetneq V(B_{n,n})$be a minimum safe dominating set in $B_{n,n}$. Clearly, $u, v \in S$ so that if X is a component in $B_{n,n}[S]$ and Y is a component in $B_{n,n}[V(B_{n,n}) \setminus S]$, $|X| > |Y|$.

Conversely, suppose $u, v \in S$. Then every component in $B_{n,n}[V(B_{n,n}) \setminus S]$ is empty, that is if Y is a component of $B_{n,n}[V(B_{n,n}) \setminus S]$, $|Y| = 1$. Thus, if X is a component of $B_{n,n}[S]$, then $|Y| = 1 < |X| = 2$. □

Example 3.17. *Consider the graph in Figure 9. Observe that $S = \{u, v\}$ and u, v dominates all other vertices. Since u, v are apex vertices and the rest are pendant vertices thus $|V(B_{5,5} \setminus S)| = 1$. Clearly, $|S| > |V(B_{5,5} \setminus S)|$. Thus, S is a safe dominating set in $B_{5,5}$ and $\gamma_s(B_{5,5}) = 2$*

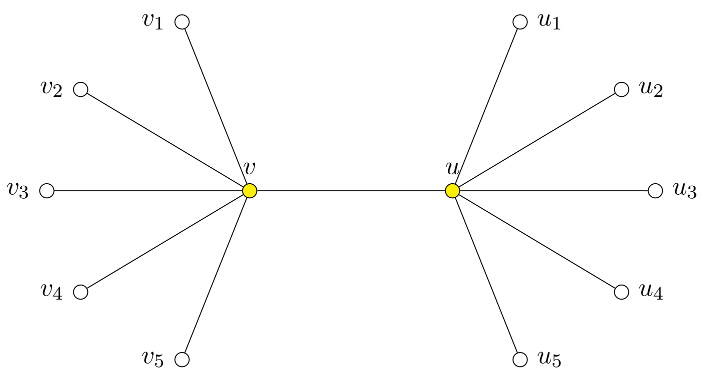

Figure 8: A bistar graph $B_{5,5}$ and its safe dominating set $\{v, u\}$

4 Conclusion

In this article, conditions of safe dominating sets in some graph families are studied. Further, the safe domination number is also determined. Lastly, we intend to examine the safe dominating set and safe domination number for few unstudied graph families in the future.

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Competing Interests

The authors declare that they have no competing interests.

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