

# Conditions of Safe Dominating Set in Some Graph Families

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## Abstract

For a nontrivial connected graph  $G$  and nonempty subset  $S \subseteq V(G)$ ,  $S$  is a safe dominating set of  $G$  if and only if  $S$  is a dominating set of  $G$  and every component  $X$  of  $G[S]$  and every component  $Y$  of  $G[V(G) \setminus S]$  adjacent to  $X$ ,  $|X| \geq |Y|$ . Moreover,  $S$  is called a minimum safe dominating set if  $S$  is a safe dominating set of the smallest size in a given graph. The cardinality of the minimum safe dominating set of  $G$  is the safe domination number of  $G$ , denoted by  $\gamma_s(G)$ . In this paper, we determine the safe dominating set, minimum safe dominating set, and minimum safe domination number of some graph families.

*Keywords:* Domination, Safe domination, Minimum safe dominating set, Safe domination number, Safe dominating set.

2024 Mathematics Subject Classification: 05C35

## 1 Introduction

Safe set is recently introduced parameter within the field of graph theory. The intention of this study is to help in terms of Facility Location Problem or (FLP) which refers to the placement and management of a facility in order to obtain the maximum goal with minimizing costs. Fujita et al., [5] studied the FLP and introduced the concept of safe set and connected safe set. They derived their concepts from a class of facility location problems, aiming to identify a "safe" subset of nodes within a network where facilities can be strategically positioned.

This paper extends the study of safe sets in some common graphs by combining the domination in safe sets to form a new parameter called safe dominating set. This study will investigate the safe dominating set and safe domination number in some graph families. Also, this paper aims to provide conditions of safe dominating set in some classes of graphs.

## 2 Preliminary Notes

Some definitions of the concepts covered in this study are included below. You may refer on the remaining terms and definitions in [6], [7], [8], [9], [11], [12].

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**Definition 2.1.** [2] Let  $G$  be a simple graph. A set  $S \subseteq V(G)$  is a **dominating set** of  $G$ , if every vertex in  $V(G) \setminus S$  is adjacent to at least one vertex in  $S$ . The **domination number**  $\gamma(G)$  is the minimum cardinality of dominating set.

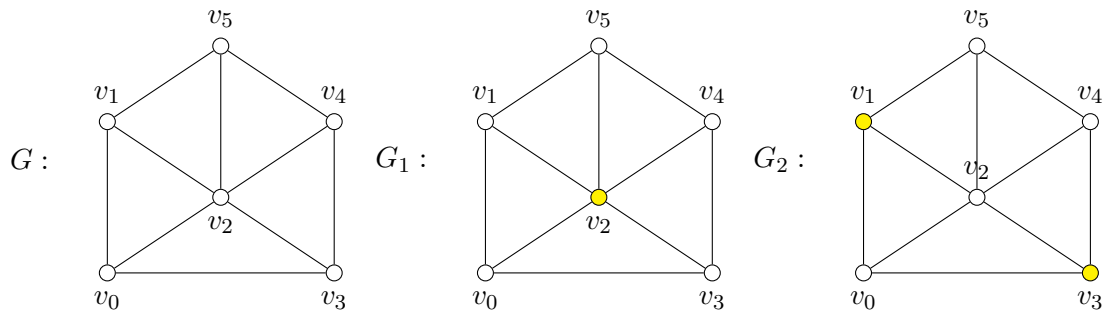


Figure 1: A graph  $G$  and its dominating sets

**Definition 2.2.** [4] The subgraph of a graph  $G$  induced by  $S \subseteq V(G)$  is denoted by  $\langle S \rangle_G$ . A **component** of  $G$  is a connected induced subgraph of  $G$  with an inclusionwise maximal vertex set. A non-empty set  $S \subseteq V(G)$  of vertices is a **safe set** if, for every component  $A$  of  $\langle S \rangle_G$  and every component  $B$  of  $\langle V(G) \setminus S \rangle_G$  adjacent to  $A$ , it holds that  $|A| \geq |B|$ . The **safe number** denoted by  $s(G)$  of  $G$  is the minimum cardinality of a safe set of  $G$ .

**Example 2.1.** Consider the graph in figure 2. We have  $S = v_4, v_5, v_6$ , then we have an induced subgraph of  $S$  in  $G$  which is  $A_1$ . Then we have  $V(G) \setminus S$  with 2 components,  $B_1$  and  $B_2$ . Clearly,  $A_1$  is adjacent to  $B_1$  and  $B_2$  and  $|A_1| = |B_1| > |B_2|$ . Thus,  $S$  is a safe set.

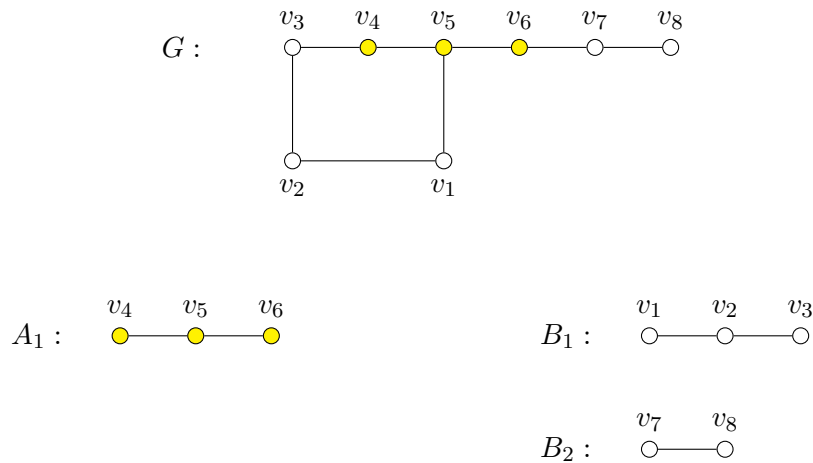


Figure 2: A graph  $G$  and its safe sets

**Definition 2.3.** [3] A nonempty subset  $S \subseteq V(G)$  is a **safe dominating set** if and only if  $S$  is a dominating set of  $G$  and every component  $A$  of  $G[S]$  and every component  $B$  of  $G[V(G) \setminus S]$  adjacent to  $A$ ,  $|A| \geq |B|$ . Moreover,  $S$  is called a **minimum safe dominating set** denoted by  $\gamma_s$ -set, if  $S$  is a safe dominating set of smallest size in a given graph. The cardinality of minimum safe dominating set of  $G$  is the **safe domination number** of  $G$ , denoted by  $\gamma_s(G)$ .

**Example 2.2.** Consider the graph in figure 3. We have  $S = v_4, v_5, v_6$ , then we have an induced subgraph of  $S$  in  $G$  which is  $A_1$ . Then we have  $V(G) \setminus S$  with 4 components,  $B_1$ ,  $B_2$ ,  $B_3$  and  $B_4$ . Clearly,  $A_1$  is adjacent to  $B_1$  and  $B_2$ ,  $B_3$  and  $B_4$  and  $|A_1| = |B_1| > |B_2| = |B_3| = |B_4|$ . Thus,  $S$  is a safe dominating set.

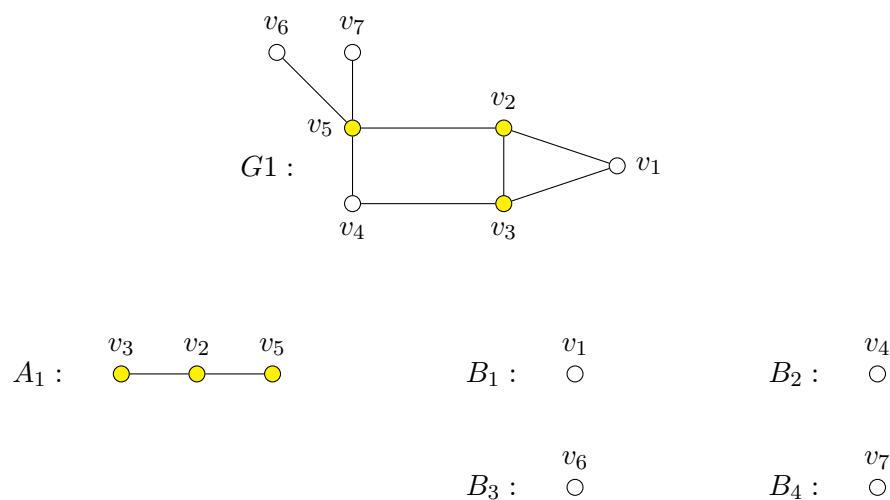


Figure 3: A graph  $G$  and its safe dominating set

### 3 Main Results

In this section, the characteristics of minimum rings dominating set in the total graph of some graph families are presented. We also determine the rings domination number for each of the graphs being classified in this paper.

#### 3.1 Results on Crown Graph

**Theorem 3.1.** Let  $\emptyset \neq S \subseteq V(Cr_{n,n})$  be a safe dominating set of crown graph. Then  $S$  is a minimum safe dominating in  $Cr_{n,n}$  if and only if  $S = A$  or  $S = B$ .

*Proof.* Let  $\emptyset \neq S \subseteq V(Cr_{n,n})$  be a minimum safe dominating in  $Cr_{n,n}$ . Suppose,  $S \neq A$  or  $S \neq B$ . If  $|S| > |A|$ , then  $S$  is not a minimum safe dominating set of  $Cr_{n,n}$  since  $A$  is a safe dominating set. A contradiction to the assumption that  $S$  is a minimum safe dominating set in  $Cr_{n,n}$ . Suppose further,  $|S| < |A|$ . Then either  $S$  is not a dominating set or there exist a component  $Y$  of  $Cr_{n,n}[V(Cr_{n,n}) \setminus S]$  such that  $|Y| > |X|$ , where  $X$  is a component of  $Cr_{n,n}[S]$ . In either case  $S$  is not a safe dominating set. A contradiction to the assumption that  $S$  is a safe dominating set. Thus,  $S = A$ . Similarly for  $S = B$ .

Conversely, suppose  $S = A$  or  $S = B$ . Without loss of generality, let  $S = A$ . Clearly,  $S$  is a safe dominating set. Now, suppose  $S$  is not a minimum safe dominating set. Then there exists  $S_o$  such that  $|S_o| < |S|$  a safe dominating set in  $Cr_{n,n}$ . This is not possible since  $S$  will not be a dominating set. Thus,  $S = A$  is the minimum safe dominating set of  $Cr_{n,n}$ . Similarly for  $S = B$ .  $\square$

**Corollary 3.2.** For a crown graph  $Cr_{n,n}$ ,

$$\gamma_s(Cr_{n,n}) = n$$

*Proof:* This immediately follows from Theorem 3.1.

**Example 3.3.** Refer to Figure 4. Consider the crown graph  $Cr_{6,6}$  with a vertex set of  $V(Cr_{6,6}) = \{u_1, u_2, u_3, u_4, u_5, u_6, v_1, v_2, v_3, v_4, v_5, v_6\}$ . The set  $S = \{\{u_1\}, \{u_2\}, \{u_3\}, \{u_4\}, \{u_5\}, \{u_6\}\}$  is a dominating set and every component  $X$  in  $Cr_{6,6}[S]$ ,  $|X| = 1$  and every component  $Y$  in  $Cr_{6,6}[V(Cr_{6,6}) \setminus S]$ ,  $|Y| = 1$ . Since  $|X| = |Y|$ , thus  $S$  is a safe dominating set. By Theorem 3.1, the minimum safe dominating set of  $Cr_{n,n}$  equal is  $A$  or  $B$  where  $A$  and  $B$  are the  $2n$  graph of  $Cr_{n,n}$ . Hence,  $\gamma_s(Cr_{6,6}) = |A| = |B| = 6$

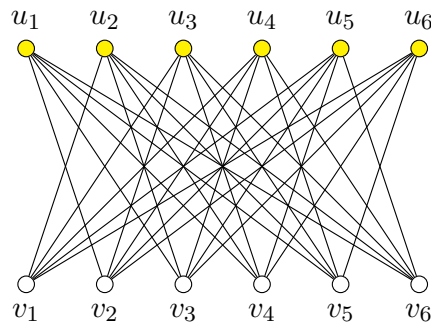


Figure 4: A graph  $Cr_{6,6}$  with  $|\gamma_s(Cr_{6,6})| = 6$

### 3.2 Results on Barbell Graph

**Theorem 3.4.** Let  $G$  be a barbell graph,  $B_{n,n}$ . Then  $\emptyset \neq S \subseteq V(G)$  is a safe dominating set of  $G$  if and only if  $u_1, v_1 \in S$  and  $|G[S]| \geq |G[V(G) \setminus S]|$ .

*Proof.* Let  $\emptyset \neq S \subseteq V(G)$  be a safe dominating set in  $B_{n,n}$ . Clearly,  $u_1, v_1 \in S$  since  $G[S]$  is connected,  $|G[S]| \geq |G[V(G) \setminus S]|$  so that every component  $Y$  in  $G[V(G) \setminus S]$ ,  $|Y| \leq |G[S]|$ .

Conversely, suppose  $u_1, v_1 \in S$  and  $|G[S]| \geq |G[V(G) \setminus S]|$ . Then  $G[S]$  has only one component which is itself and every component  $Y$  in  $G[V(G) \setminus S]$ ,  $|Y| \leq |G[S]|$ . Since  $u_1, v_1 \in S$  is a dominating set, thus  $S$  is a safe dominating set.  $\square$

**Example 3.5.** Consider the graph in Figure 5. The set  $S = \{u_1, u_2, v_1, v_2\}$ . Observe that  $u_1$  is in  $S$  and  $u_1$  dominates all vertices in  $U$  and  $v_1$  is in  $S$  and  $v_1$  dominates all vertices in  $V$ . Thus,  $S$  is a dominating set. Observe further that  $S$  is a connected set, thus it has only one component and  $B_{n,n}[V(B_{n,n}) \setminus S]$  is disconnected and it has two components  $Y_1$  and  $Y_2$  where  $Y_1 = \{u_3, u_4, u_5, u_6\}$  and  $Y_2 = \{v_3, v_4, v_5, v_6\}$ . Clearly,  $|S| = |Y_1| = |Y_2|$ . Hence,  $S$  is a safe dominating set.

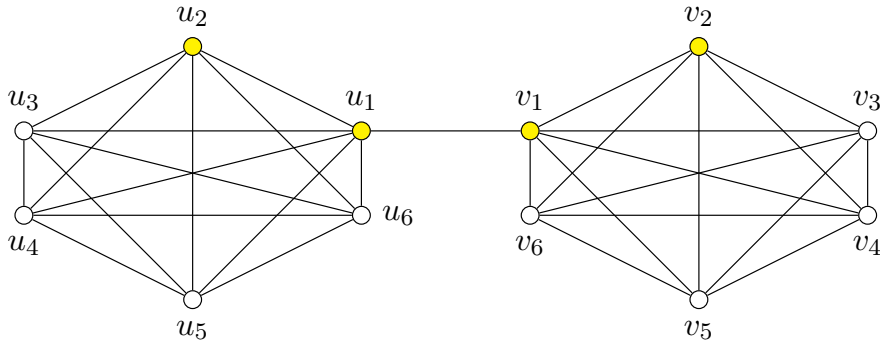


Figure 5: A barbell graph  $B_{6,6}$  with  $|\gamma_s(B_{6,6})| = 4$

**Corollary 3.6.** For a barbell graph  $B_{n,n}$ ,

$$\gamma_s(B_{n,n}) = \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0 \pmod{3} \\ \frac{2n+1}{3}, & \text{if } n \equiv 1 \pmod{3} \\ \frac{2n+2}{3}, & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

*Proof:* Let  $\emptyset \neq S \subseteq V(G)$  be a safe dominating set in  $G$ . Consider the following cases:

Case 1:  $n \equiv 0 \pmod{3}$

Choose  $S = \{u_1, v_1\} \cup S_1 \cup S_2$  where  $S_1 \subseteq V(K_n)$  and  $S_2 \subseteq V(K_m)$  where  $m = n$  such that  $|S_1| = \frac{n-3}{3}$  and  $|S_2| = \frac{n-3}{3}$ . Clearly,  $S$  is a dominating set. Now,  $|S| = |\{u_1, v_1\}| + |S_1| + |S_2| = 2 + \frac{n-3}{3} + \frac{n-3}{3} = \frac{2n}{3}$ . On the other hand, the components  $Y_1$  and  $Y_2$  of  $G[V(G) \setminus S]$  where  $Y_1[V(K_n) \setminus S_1]$  and  $Y_2[V(K_m) \setminus S_2]$  has order  $\frac{2n}{3} = |Y_1| = |Y_2|$ . Thus  $|S| = \frac{2n}{3} \geq \frac{2n}{3} = |Y_1| = |Y_2|$ . Hence,  $S$  is a safe set and it follows that  $S$  is a safe dominating set. We are left to show that  $S$  is the minimum safe dominating set. Suppose  $S$  is not a minimum safe dominating set. Then there exists  $S_o \subseteq V(G)$  such that  $|S_o| < |S|$ . Since  $S_o$  is a dominating set, then  $u_1$  and  $v_1$  is in  $S_o$ . Hence, it is immediate to assume that  $|S_o| \leq \frac{2n}{3} - 1$ . This imply that there exists a component in  $G[V(G) \setminus S_o]$ , say  $Y_o$ , such that  $|Y_o| \geq \frac{2n}{3} + 1$ . Thus,  $|S_o| \leq \frac{2n}{3} - 1 < \frac{2n}{3} + 1 \leq |Y_o|$ , implying that  $|S_o| < |Y_o|$ . A contradiction, since  $S_o$  is not a safe set in  $G$ . Therefore, no such  $S_o$  exist with  $|S_o| \leq \frac{2n}{3} - 1$ . Thus,  $S$  is the minimum safe dominating set. Therefore,  $\gamma_s(G) = |S| = \frac{2n}{3}$ .

Case 2:  $n \equiv 1 \pmod{3}$

Choose  $S = \{u_1, v_1\} \cup S_1 \cup S_2$  where  $S_1 \subseteq V(K_n)$  and  $S_2 \subseteq V(K_m)$  where  $m = n$  such that  $|S_1| = \frac{n-1}{3}$  and  $|S_2| = \frac{n-4}{3}$ . Clearly,  $S$  is a dominating set since  $u_1, v_1 \in S$ . Now,  $|S| = |\{u_1, v_1\}| + |S_1| + |S_2| = 2 + \frac{n-1}{3} + \frac{n-4}{3} = \frac{2n+1}{3}$ . On the other hand,  $G[V(G) \setminus S] = K_n[V(K_n) \setminus S_1] \cup K_m[V(K_m) \setminus S_2]$  where  $|V(K_n) \setminus S_1| = \frac{2n+1}{3} - 1 = \frac{2n-2}{3}$  and  $|V(K_m) \setminus S_2| = \frac{2n+1}{3} = \frac{2n+1}{3}$ . Thus,  $|S| = \frac{2n+1}{3} \geq \frac{2n-2}{3} = |V(K_n) \setminus S_1|$  and  $|S| = \frac{2n+1}{3} \geq \frac{2n+1}{3} = |V(K_m) \setminus S_2|$ . Thus,  $S$  is a safe set and it follows that  $S$  is a safe dominating set. We are left to show that  $S$  is

the minimum safe dominating set. Suppose  $S$  is not a minimum safe dominating set. Then there exists  $S_o \subseteq V(G)$  such that  $|S_o| < |S|$ . Since  $S_o$  is a dominating set, clearly  $u_1, v_1 \in S_o$ . Hence, it is immediate to assume that  $|S_o| \leq \frac{2n-2}{3}$ . Thus,  $|S_o| \leq \frac{2n-2}{3} < \frac{2n+1}{3} = |V(K_m) \setminus S_2|$ . Thus,  $S_o$  is not a safe set and it follows that  $S_o$  is not a safe dominating set. Hence,  $S$  is the minimum safe dominating set. Therefore,  $\gamma_s(G) = |S| = \frac{2n+1}{3}$ .

Case 3:  $n \equiv 2 \pmod{3}$

Choose  $S = \{u_1, v_1\} \cup S_1 \cup S_2$  where  $S_1 \subseteq V(K_n)$  and  $S_2 \subseteq V(K_m)$  where  $m = n$  such that  $|S_1| = \frac{n-2}{3}$  and  $|S_2| = \frac{n-2}{3}$ . Clearly,  $S$  is a dominating set. Now,  $|S| = |\{u_1, v_1\}| + |S_1| + |S_2| = 2 + \frac{n-2}{3} + \frac{n-2}{3} = \frac{2n+2}{3}$ . On the other hand,  $G[V(G) \setminus S] = K_n[V(K_n) \setminus S_1] \cup K_m[V(K_m) \setminus S_2]$  where  $|V(K_n) \setminus S_1| = |V(K_m) \setminus S_2| = \frac{2n-1}{3}$ . Thus  $|S| = \frac{2n+2}{3} > \frac{2n-1}{3} = |Y_1| = |Y_2|$ . Thus,  $S$  is a safe set and it follows that  $S$  is a safe dominating set. We are left to show that  $S$  is the minimum safe dominating set. Suppose  $S$  is not a minimum safe dominating set. Then there exists  $S_o \subseteq V(G)$  such that  $|S_o| < |S|$ . Since  $S_o$  is a dominating set, then  $u_1, v_1 \in S_o$ . Hence, it is immediate to assume that  $|S_o| \leq \frac{2n-1}{3} - 1 = \frac{2n-4}{3}$ . This imply that there exists a component in  $G[V(G) \setminus S_o]$ , say  $Y_o$ , such that  $|Y_o| \geq \frac{2n+5}{3}$ . Thus,  $|S_o| \leq \frac{2n-4}{3} < \frac{2n+5}{3} \leq |Y_o|$ , implying that  $|S_o| < |Y_o|$ . A contradiction, since  $S_o$  is not a safe set in  $G$ . Therefore, no such  $S_o$  exist with  $|S_o| \leq \frac{2n-4}{3}$ . Thus,  $S$  is the minimum safe dominating set. Therefore,  $\gamma_s(G) = |S| = \frac{2n+2}{3}$ .  $\square$

### 3.3 Results on Helm Graph

For the next theorem, recall that for a helm graph  $H_n$ , for  $n \geq 3$ , there exists  $S \subseteq V(H_n)$  such that  $H_n[S]$  is a cycle graph.

**Theorem 3.7.** *Let  $H_n$  be helm graph. Then  $\emptyset \neq S \subseteq V(H_n)$  is a minimum safe dominating set in  $H_n$  if and only if  $S = V(C_n) \subseteq V(H_n)$ .*

*Proof.* Let  $\emptyset \neq S \subseteq V(H_n)$  be the minimum safe dominating set of  $H_n$ . Since  $S$  is a dominating set, the only dominating set of  $H_n$  are  $V(H_n)$ ,  $S = V(C_n)$ , and  $S = \{x\} \cup V_p$  where  $\deg_{H_n}(x) = \Delta(H_n)$  and  $V_p$  is the set of the pendant vertices of  $H_n$ . Clearly,  $S \neq V(H_n)$ . If  $S = \{x\} \cup V_p$ , then  $H_n[V(H_n) \setminus S] = C_n$ , implying that it has only one component and  $H_n[S]$  is an empty graph. Thus, for each component  $X$  in  $H_n[S]$ ,  $|X| < |V(C_n)|$ . Thus,  $S \neq \{x\} \cup V_p$ . Now, if  $S = V(C_n)$ , then  $H_n[S] = C_n$  and  $H_n[V(H_n) \setminus S]$  is an empty graph. Thus,  $|V(C_n)| > |Y|$ , for every trivial graph component  $Y$  in  $H_n[V(H_n) \setminus S]$ . Therefore,  $S = V(C_n)$ .

Conversely, suppose  $S = V(C_n)$ . Then by the argument above,  $S$  is a safe dominating set. Since there can be no safe dominating set  $S_o$  such that  $|S_o| < |S|$ . Hence,  $S$  must be the minimum safe dominating set of  $H_n$ .  $\square$

**Example 3.8.** *Refer to Figure 6. Consider  $H_{12}$  with  $|S| = V(C_{12}) = 12$ . Observe that every component  $Y$  in  $H_{12}[V(C_{12}) \setminus S]$ ,  $|Y| = 1$ . Since  $|S| = 12 > 1 = |Y|$ . Thus,  $S$  is a safe dominating set. By Theorem 3.3.1, the minimum safe dominating set of  $H_n$  is equal to  $V(C_n)$  in  $H_n$ . Thus,  $S$  is the minimum safe dominating set.*

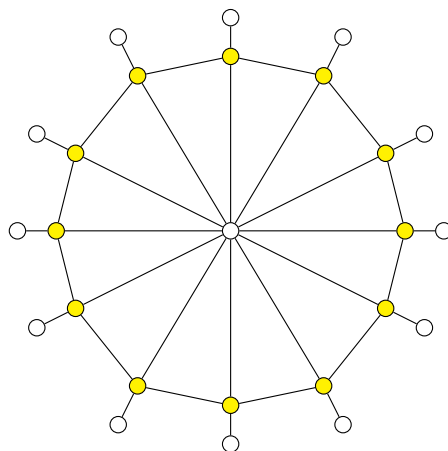


Figure 6: A helm graph  $H_{12}$  with  $|\gamma_s(H_{12})| = 12$

**Corollary 3.9.** For a helm graph  $H_n$ ,

$$\gamma_s(H_n) = n$$

*Proof:* This immediately follows from Theorem 3.7.

### 3.4 Results on Caterpillar Graph

For the next theorem, recall that for a caterpillar graph  $G$ , there exists  $S \subseteq V(G)$  such that  $G[S]$  is a path graph.

**Theorem 3.10.** Let  $G$  be a caterpillar graph. Then  $S \subseteq V(G)$  is a minimum safe dominating set if and only if  $S = V(P_n) \subseteq V(G)$ .

*Proof.* Let  $\emptyset \neq S \subseteq V(G)$  be the minimum safe dominating set of  $G$ . Since  $S$  is a dominating set, the only dominating set of  $G$  are  $V(G)$ ,  $S = V(P_n)$ , and  $S = V_p$  where  $V_p$  is the set of the pendant vertices of  $G$ . Clearly,  $S \neq V(G)$ . If  $S = V_p$ , then  $G[V(G) \setminus S] = P_n$ , implying that it has only one component and  $G[S]$  is an empty graph. Thus, for each component  $X$  in  $G[S]$ ,  $|X| < |V(P_n)|$ . Hence,  $S \neq V_p$ . Now, if  $S = V(P_n)$ , then  $G[S] = P_n$  and  $G[V(G) \setminus S]$  is an empty graph. Thus,  $|V(P_n)| > |Y|$ , for every trivial graph component  $Y$  in  $G[V(G) \setminus S]$ . Therefore,  $S = V(P_n)$ .

Conversely, suppose  $S = V(P_n)$ . Then by the argument above,  $S$  is a safe dominating set. Since there can be no safe dominating set  $S_o$  such that  $|S_o| < |S|$ . Hence,  $S$  must be the minimum safe dominating set of  $G$ .  $\square$

**Example 3.11.** Refer to Figure 7. Consider the caterpillar graph  $C_4(m_1+1, m_2+1, m_3+1, m_4+1)$  with a set  $S = \{u_1, u_2, u_3, u_4\}$ . Observe that  $S$  is a dominating set and has only one component  $X$  of  $C_4(m_1+1, m_2+1, m_3+1, m_4+1)[S]$  such that  $|X| = 4$  and every component  $Y$  in  $C_4(m_1+1, m_2+1, m_3+1, m_4+1)[V(C_4(m_1+1, m_2+1, m_3+1, m_4+1)) \setminus S]$ ,  $|Y| = 1$ . Since  $X > Y$ , thus  $S$  is a safe dominating set. By Theorem 3.4.1, the minimum safe dominating set of  $C_n(m_1+1, \dots, m_n+1)$  is equal  $V(P_n)$  where  $V(P_n) \subseteq V(C_4(m_1+1, m_2+1, m_3+1, m_4+1))$  and  $S = V(P_n)$ . Thus,  $\gamma_s(C_4(m_1+1, m_2+1, m_3+1, m_4+1)) = 4$ .

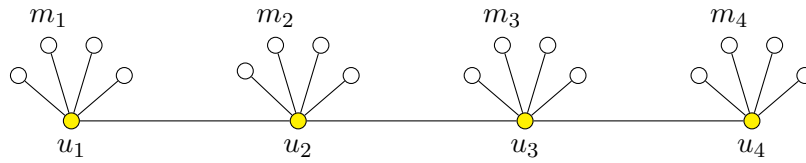


Figure 7: A caterpillar graph  $C_4(m_1 + 1, m_2 + 1, m_3 + 1, m_4 + 1)$  with  $\gamma_s(C_4(m_1 + 1, m_2 + 1, m_3 + 1, m_4 + 1)) = 4$

**Corollary 3.12.** For a caterpillar graph  $G$ ,

$$\gamma_s(G) = n$$

*Proof:* This immediately follows from Theorem 3.10.

### 3.5 Results on Banana Tree Graph

**Theorem 3.13.** Let  $B_{n,k}$  be a Banana Tree graph. Then  $\emptyset \neq S \subseteq V(B_{n,k})$  is a safe dominating set in  $B_{n,k}$  if and only if  $S$  is a dominating set and  $B_{n,k}[V \setminus S]$  is an empty graph.

*Proof.* Suppose  $S \subseteq V(B_{n,k})$  is a safe dominating set in  $B_{n,k}$ . Clearly,  $S$  is a dominating set in  $B_{n,k}$ . Now, suppose  $B_{n,k}[V(B_{n,k}) \setminus S]$  is not an empty graph. Then there exist edge  $e_1$  joining two vertices  $v, u \in B_{n,k}[V(B_{n,k}) \setminus S]$ . Hence, this edge  $e_1$  forms a component  $B_1$  in  $B_{n,k}[V(B_{n,k}) \setminus S]$ . Suppose further  $\deg(u) = 1$  and  $\deg(v) = k - 1$ . Then  $v$  must be in  $S_1$  so that  $u \in N[S] = V(B_{n,k})$ . A contradiction. Suppose  $\deg(u) = 2$  and  $\deg(v) = n$ . Then  $u \in N[w]$  such that  $\deg(y) = 2$  and  $y \in S$ . Since the remaining vertices in  $B_{n,k}[V(B_{n,k}) \setminus S]$  are isolated, every component  $X$  in  $B_{n,k}[S]$  has  $|S| = 2$  or  $|X_0| = 1$  where  $X_0$  is the trivial component containing  $w$ . Another contradiction. Thus,  $B_{n,k}[V(B_{n,k}) \setminus S]$  is an empty graph.

Conversely, suppose  $S$  is a dominating set and  $B_{n,k}[V(B_{n,k}) \setminus S]$  is an empty graph. Thus, every component  $Y$  of  $B_{n,k}[V(B_{n,k}) \setminus S]$ ,  $|Y| = 1$ . Since  $S$  is a dominating set and every component  $X$  of  $B_{n,k}[S]$ ,  $|X| = 1$ . Thus,  $|Y| \leq |X|$ . Hence,  $S$  is a safe dominating set. □

**Example 3.14.** Refer to Figure 8. Observe that every component  $X$  in  $B_{5,5}[S]$ ,  $|X| = 1$  and every component  $Y$  in  $B_{5,5}[V(B_{5,5}) \setminus S]$ ,  $|Y| = 1$ . Since  $|X| = |Y|$ . Thus,  $S$  is a safe dominating set in  $B_{5,5}$ .

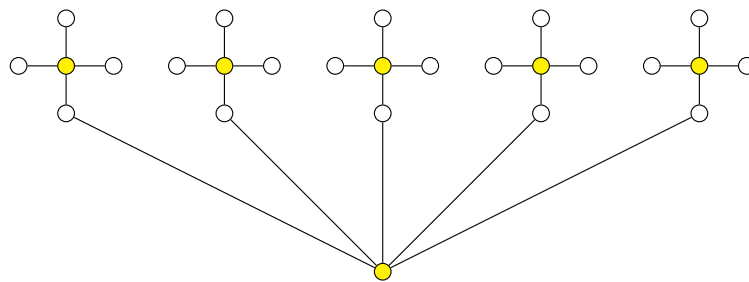


Figure 8: A banana tree graph  $B_{5,5}$  with  $\gamma_s(B_{5,5}) = 6$

**Corollary 3.15.** For a banana graph  $B_{n,k}$ ,

$$\gamma_s(B_{n,k}) = n + 1$$

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*Proof.* Choose  $S \subseteq V(B_{n,k})$  to be  $S = \{x \mid \deg(x) = n \text{ or } \deg(x) = k - 1\}$ . Then by Theorem 3.0.9,  $S$  is a safe dominating set and  $|S| = n + 1$ . Now, suppose  $S$  is not the minimum safe dominating set. Then there exists  $S_o$  such that  $S_o$  is a safe dominating set in  $B_{n,k}$  such that  $|S| > |S_o|$ , implying that  $|S_o| \leq n$ . Without loss of generality, suppose  $|S_o| = n$ . By Theorem 2.2.4 the domination number of  $B_{n,k} = n + 1$ . Hence  $S_o$  is not a dominating set. Thus,  $|S| = n + 1$  is the minimum safe dominating set.  $\square$

### 3.6 Results on Bistar Graph

**Theorem 3.16.** *Let  $B_{n,n}$  be a bistar graph. Then  $\emptyset \neq S \subsetneq V(B_{n,n})$  is a safe dominating set of  $B_{n,n}$  if and only if  $u$  and  $v \in S$ . Consequently,  $\gamma_s(B_{n,n}) = 2$ .*

*Proof.* Let  $\emptyset \neq S \subsetneq V(B_{n,n})$  be a minimum safe dominating set in  $B_{n,n}$ . Clearly,  $u, v \in S$  so that if  $X$  is a component in  $B_{n,n}[S]$  and  $Y$  is a component in  $B_{n,n}[V(B_{n,n}) \setminus S]$ ,  $|X| > |Y|$ .

Conversely, suppose  $u, v \in S$ . Then every component in  $B_{n,n}[V(B_{n,n}) \setminus S]$  is empty, that is if  $Y$  is a component of  $B_{n,n}[V(B_{n,n}) \setminus S]$ ,  $|Y| = 1$ . Thus, if  $X$  is a component of  $B_{n,n}[S]$ , then  $|Y| = 1 < |X| = 2$ .  $\square$

**Example 3.17.** *Consider the graph in Figure 9. Observe that  $S = \{u, v\}$  and  $u, v$  dominates all other vertices. Since  $u, v$  are apex vertices and the rest are pendant vertices thus  $|V(B_{5,5} \setminus S)| = 1$ . Clearly,  $|S| > |V(B_{5,5} \setminus S)|$ . Thus,  $S$  is a safe dominating set in  $B_{5,5}$  and  $\gamma_s(B_{5,5}) = 2$*

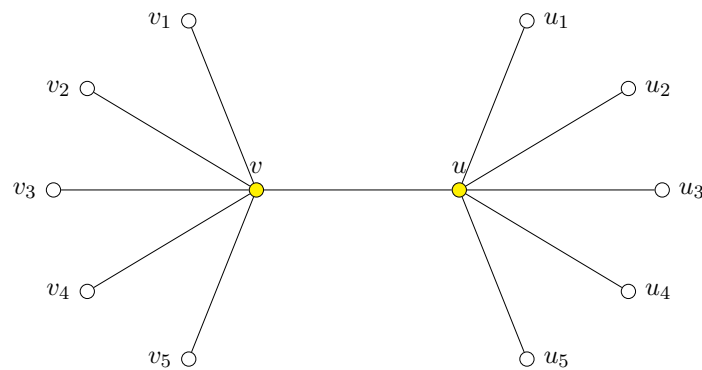


Figure 9: A bistar graph  $B_{5,5}$  and its safe dominating set  $\{v, u\}$

## 4 Conclusion

In this article, conditions of safe dominating sets in some graph families are studied. Further, the safe domination number is also determined. Lastly, we intend to examine the safe dominating set and safe domination number for few unstudied graph families in the future.

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