EXTREME VALUES AND RELIABILITY MEASURES OF CONTINUOUS ERLANG MIXTURES

Abstract: Extreme value theory has been presented and the limiting distributions of continuous Erlang mixtures and their mixing distributions obtained. Erlang mixtures have also been grouped according to their extreme value types. The mean residual lifetime and equilibrium distributions of the mixed distributions have also been derived.

1. Introduction

Extreme value theory dates back to 1902 when Tippet, with the assistance of Fisher, derived three asymptotic limiting distributions which he used to model the strength of a cotton thread.

Extreme value theory has since been studied and applied in various fields.

Escalante-Sandoval (2007) compared the General Extreme Value distribution to the mixed Gumbel and mixed General Extreme Value distributions when he applied them in estimating floods in regions with heterogeneous data. He showed that there was a lower standard error of fit when using the former.

Ahn et al. (2012) compared two tail estimation methods; the extreme value theory (EVT) and class of log phase-type (LogPH) distributions. He noted that LogPH was more preferable to the EVT because unlike the EVT, the LogPH fits the whole data range and not only the tail part. The LogPH could also fit heavy-tailed loss data without separate modeling to the tail side due to its denseness.

Cramer (2014) studied extreme value theory of progressively Type-II censored order statistics and their connection and application to limit laws for upper, lower, central and intermediate progressively Type-IIcensored order statistics.

Reynkens (2017) noted that maximum domains of attraction (MDAs) could be described in two ways; one is through convergence to the generalized Pareto distribution and the other way is based on the regular variation of the tail quantile function. An application of extreme value theory to finance and insurance was made and in particular a recent financial crisis was investigated on being a Black Swan event.
Gwak et al. (2016) studied extreme value distributions of finite mixed distributions. They noted that slow convergence rate of extreme values to a General Extreme Value distribution could cause huge bias in quantile estimation. They also observed that the use of the nonparametric quantile estimator of the extreme value in order to reduce bias problem in quantile estimation leads to a high variance problem, thus the need to combine both methods.
Kang and Serfozo (1999) argued that the limiting distribution of finite mixed distributions or mixtures of convolutions of independent identically distributed Erlang random variables is determined by one or more distributions in the mixtures or convolutions and are of the Gumbel extreme-value type. They also showed that the limiting distribution of independent phase-type distributed random variables is the Gumbel extreme-value type. An application of these results was made to completion times of jobs with parallel tasks and serial subtasks with Erlang or normal distributions and also to completion times of jobs with parallel tasks that are Markovian PERT networks or task-graphs.
Mladenović (1999) presented extreme value distributions of finite mixtures of normal distributions, cauchy distributions and uniform and truncated exponential distributions. He noted that the extreme value distributions of component distributions were similar to those of the mixed distributions.
Kang and Serfozo (1997), using exponential and Erlang mixtures, showed that the asymptotic distribution of a finite mixture of continuous distributions was similar to that of the dominant component distribution in the mixture and the norming constants were also related to the component distribution's.
MacDonald et al. (2011) proposed an extreme value mixture model that combined a non-parametric kernel density estimator with a generalized Pareto distribution tail model. They noted that the model did not need a pre-specification of a parametric form due to the flexibility of the non-parametric component. They also observed that computation was simplified because the model had just one extra parameter.
Extreme value theory can thus be applied in many fields, such as, finance in modeling extreme profits or losses of valuable assets, in as-

sessing and quantifying the risk of extreme natural calamities such as floods, hurricanes and droughts, in engineering to construct structures that will not be affected by extreme conditions such as bridges, dams and aircrafts, in insurance in forecasting extreme events that could result in large insurance claims such as earthquakes and major accidents, and in predicting the probability of rare events such as the COVID 19 and extreme natural disasters.

The focus of this work is on limiting distributions of continuous Erlang mixtures. Kang (2003) studied Extreme value distributions of the Erlang uniform and Erlang exponential mixtures, and illustrated that Extreme Value distributions of continuous Erlang mixtures depended on their mixing distributions, and were either Type I (Gumbel) or Type II (Fretchet). He also noted that extreme value distributions of continuous Erlang mixtures were not always as those for the mixing distributions, unlike in the case of finite mixtures. The extreme value distribution of the Erlang-uniform distribution was shown to be Type I (Gumbel) when the minimum value parameter $l = 0$, and to be Type II (Fretchét) when the minimum value parameter $l > 0$. The extreme value distribution of the Erlang-exponential distribution was shown to be Type I (Gumbel).

2. Extreme value theory

The idea of extreme value distribution is borrowed from the central limit theory for sums of n random variables,

$$S_n = X_1 + X_2 + ... + X_n \tag{2.1}$$

which states that appropriately normalized sums $(S_n - a_n)/b_n$ converge to the standard normal distribution as n goes to infinity, that is,

$$\lim_{n\to\infty} P\left(\frac{S_n - a_n}{b_n} \leq x\right) = \Phi(x) \tag{2.2}$$

where $a_n = nE(X)$ and $b_n = \sqrt{Var(X)}$ are normalizing constants. (Embrechts et al. (2011))
Extreme value theory (EVT) deals with convergence of maxima (highest order statistic), $M_n = max(X_1, X_2, ..., X_n)$, of a given distribution, that is,

$$\lim_{n\to\infty} P\left(\frac{M_n - d_n}{C_n} \leq x\right) = \lim F^n(C_n x + d_n) = H(x), \quad x \in \mathbb{R} \tag{2.3}$$

where $H(x)$ is a non-degenerate distribution function (a limiting distribution that is not concentrated on a single point), and $F(x)$, the cumulative distribution function of x, is said to be in the maximum domain of attraction of the extreme value distribution of H, written as $F \in MDA(H)$ where $C_n > 0$, $d_n \in \mathbb{R}$ are normalizing constants. (Fisher and Tippett (1928); Gnedenko (1943); Embrechts et al. (2013))
The equation (2.3) above holds $\leftrightarrow$

$$\lim_{n\to\infty} n(1 - F(C_n x + d_n)) \to -logH(x), \quad x \in \mathbb{R} \tag{2.4}$$

for each x such that $H(x) > 0$. When $H(x) = 0$ the limit is interpreted as ∞.
The limiting distribution $H(x)$ is of three extreme value types, namely; Type I(Gumbel), Type II(Fretchét) and Type III(Weibull), their difference being their right endpoints and norming constants, among other factors.

3. Extreme Value Distributions

According to Fisher and Tippett (1928) and Gnedenko (1943), extreme value distributions fall under three categories;

$$TypeI(Gumbel) : \Lambda(x) = exp(-exp[-x]), \quad -\infty < x < \infty \tag{3.1}$$

$$TypeII(Fretch\acute{e}t) : \Phi_\alpha(x) = \begin{cases} 0, & if\ x \leq 0 \\ exp(-x^{-\alpha}), & if\ x > 0, for\ some\ \alpha > 0 \end{cases} \tag{3.2}$$

$$TypeIII(Weibull) : \Psi_\alpha(x) = \begin{cases} exp(-(-x)^\alpha), & if\ x \leq 0, for\ some\ \alpha > 0 \\ 1, & if\ x > 0 \end{cases} \tag{3.3}$$

1. Fretchet distribution $\Phi_\alpha(x)$

   A distribution function F is said to be in the maximum domain of attraction of the Fretchét distribution $\Phi_\alpha(x) \leftrightarrow$ its tail function is of the form $\bar{F}(x) = x^{-\alpha}L(x)$, where $\bar{F}(x) = 1 - F(x)$ is the tail function of the distribution, $L(x)$ is a positive, Lebesgue-measurable function on $(0, \infty)$ which is slowly varying at ∞, and $\alpha > 0$ is the tail index of the distribution.
   All $F \in MDA(\Phi_\alpha)$ have an infinite right endpoint $x_F = \infty$, and the normalizing constants are $C_n = F^{\leftarrow}(1 - n^{-1}) = n^{\frac{1}{\alpha}}L_1(n)$ and $d_n = 0$ where $L_1(n)$ is a slowly varying positive, Lebesgue-measurable function.

2. Weibull distribution $\Psi_\alpha(x)$
   A distribution function F is in the maximum domain of attraction of the Weibull distribution $\Psi_\alpha(x) \leftrightarrow$ its tail function is of the form $\bar{F}(x_F - x^{-1}) = x^{-\alpha}L(x)$, where $\bar{F}(x) = 1 - F(x)$ is the tail function of the distribution, $L(x)$ is a slowly varying positive, Lebesgue-measurable function and x_F is the right endpoint.
   All $F \in MDA(\Psi_\alpha)$ have a finite right endpoint $x_F < \infty$ and the norming constants are $C_n = x_F - F^{\leftarrow}(1 - n^{-1})$ and $d_n = x_F$.
   Remark: $\Psi_\alpha(-x^{-1}) = \Phi_\alpha(x), \quad x > 0.$
3. Gumbel distribution $\Lambda(x)$
   A distribution function F with right endpoint $x_F \leq \infty$ belongs to the maximum domain of the Gumbel distribution $\Lambda(x) \leftrightarrow$ it is tail equivalent to a Von Mises function, that is, $\bar{F}(x) =$

$c(x)exp\left[-\int_z^x \frac{1}{a(t)}dt\right] \quad z < x < x_F$, where $\bar{F}(x) = 1 - F(x)$ is the tail function of the distribution, c is some positive constant and $a(x)$, the auxiliary function of F, is a positive and absolutely continuous function with respect to Lebesgue measure with density $a'(x)$ and $\lim_{x\to x_F} a'(x) = 0$.
F is a Von Mises function with auxiliary function $a(x) = \frac{\bar{F}(x)}{f(x)} \leftrightarrow \lim_{x\to x_F} \frac{\bar{F}(x)F''(x)}{f^2(x)} = -1$.
$F \in MDA(\Lambda)$ can have both infinite and finite right endpoints $x_F \leq \infty$ and the norming constants are $d_n = F^{\leftarrow}(1 - n^{-1})$ and $C_n = a(d_n)$, where $a(x)$ is the auxiliary function of the distribution.

Remark: Continuous mixed Erlang distributions will be of either the Type I(Gumbel) or Type II(Fretchét) extreme value types, since their right endpoints are infinite, that is, $x_F = \infty$.

4. Definition of terms

- **Regular and slow variation**

  i. A positive, Lebesgue-measurable function L on $(0, \infty)$ is slowly varying at ∞, denoted by $L \in R_0$, if

  $$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1, \quad t > 0 \tag{4.1}$$

  ii. A positive, Lebesgue-measurable function L on $(0, \infty)$ is regularly varying at ∞ of index $\alpha \in \mathbb{R}$, denoted by $L \in R_\alpha$, if

  $$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = t^\alpha, \quad t > 0 \tag{4.2}$$

  (see Resnick (1987))

- **Karamata's theorem**
  It states that, if $L \in R_0$ is locally bound in $[x_0, \infty)$ for some $x_0 \geq 0$, then

i. for $j > -1$,

$$\int_{x_0}^{x} t^j L(t)dt \sim (j+1)^{-1} x^{j+1} L(x), \quad x \to \infty \tag{4.3}$$

ii. for $j < -1$,

$$\int_{x}^{\infty} t^j L(t)dt \sim -(j+1)^{-1} x^{j+1} L(x), \quad x \to \infty \tag{4.4}$$

(see Embrechts et al. (2013))

- **L'Hopital's rule**
  It states that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \tag{4.5}$$

- **Reliability Measures**
  Survival analysis is a branch of Statistics that deals with time to an event. It deals with both uncensored or complete and censored or incomplete data. The time to an event, also called the survival time, is a random variable which can be denoted by $T \geq 0$. Thus the survival function $S(t)$ is given by

$$\begin{aligned} S(t) &= Prob(T > t) \\ &= 1 - Prob(T \leq t) \\ &= 1 - F(t) \end{aligned} \tag{4.6}$$

where $F(t)$ is the cumulative distribution function of T. $S(t)$ is a non-increasing function of time. $S(0) = 1$ and $\lim_{t \to \infty} S(t) = S(\infty) = 0$. Other functions of survival time are the probability density function $f(t)$ and the hazard function $h(t)$, which is the rate at which events occur given no previous events, and it is given by

$$h(x) = \frac{f(x)}{S(x)} \tag{4.7}$$

i. Mean Residual Lifetime (mean excess loss) of a random variable X, denoted by $m(x)$, is the average remaining lifetime of a process that has survived beyond time x. It is given by

$$\begin{aligned} m(x) &= E(T - x|T > x) \\ &= \int_x^\infty \frac{1 - F(t)}{1 - F(x)} dt, \quad t > x \end{aligned} \tag{4.8}$$

where $F(x)$ is the cdf of X.

ii. The equilibrium distribution of a random variable X, denoted by $f_e(x)$, is used to determine whether a lifetime process preceding X, at a given time T, equals the mean of the distribution. It is given by

$$f_e(x) = \frac{S(x)}{E(X)}, \quad x > 0 \tag{4.9}$$

where $S(x)$ and $E(X)$ are the survival function and arithmetic mean of X respectively.

5. Frechét limiting distribution

In this section the mixed distributions are Pareto-like and thus the tail functions are of the form $\bar{F}(x) \sim Kx^{-\alpha}, \quad x \to \infty$ for some K, $\alpha > 0$, and they are therefore $F \in MDA(\phi_\alpha)$ and the norming constants are of the form $C_n = (Kn)^{\frac{1}{\alpha}}$ and $d_n = 0$.
Of importance to note also is that the mixing distributions of the mixtures are $F \in MDA(\Lambda)$.

Erlang-Type II Gamma distribution

1. mixing distribution
The type II Gamma distribution is

$$f(y) = \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{y}{\beta}} y^{\alpha-1}, \quad y > 0; \beta > 0, \alpha > 0 \tag{5.1}$$

Using L'Hopital's rule, the tail of the mixing distribution can be obtained as follows;

$$\lim_{y\to\infty}\frac{\bar{F}(y)}{f(y)} = \lim_{y\to\infty}\frac{1}{\frac{1}{\beta}-\frac{(\alpha-1)}{y}} = \beta \tag{5.2}$$

and therefore $$\bar{F}(y) \sim \frac{1}{\beta^{\alpha-1}\Gamma(\alpha)}y^{\alpha-1}exp\left\{-\int_0^y \frac{1}{\beta}dt\right\} \tag{5.3}$$

and thus $F \in MDA(\Lambda)$ which is a Von Mises function with auxiliary function $a(y) = \beta$.
The norming constants are obtained by solving the below equation

$$\bar{F}(d_n) = \frac{1}{\beta^{\alpha-1}\Gamma(\alpha)}e^{-\frac{d_n}{\beta}}d_n^{\alpha-1} = n^{-1} \tag{5.4}$$

to obtain $d_n = \beta(lnn - ln\Gamma(\alpha) + (\alpha-1)lnlnn)$ and $C_n = a(d_n) = \beta^{-1}$.

2. mixed distribution

The mixed distribution is the generalized Pareto distribution with parameters $n, \alpha, \frac{1}{\beta}$ and has pdf

$$\begin{aligned} f_n(t) &= \frac{\left(\frac{1}{\beta}\right)^{\alpha} t^{n-1}}{B(\alpha,n)(t+\frac{1}{\beta})^{\alpha+n}} \\ &= \frac{\left(\frac{1}{\beta}\right)^{\alpha}}{B(\alpha,n)}t^{-(\alpha+1)}\left(1+\frac{1}{\beta t}\right)^{-(\alpha+n)} \end{aligned} \tag{5.8}$$

(see Gathongo (2019))
By Karamata's theorem, the tail function is given by

$$\begin{aligned} \bar{F}(t) &\sim \frac{\left(\frac{1}{\beta}\right)^{\alpha}}{\alpha B(\alpha,n)}t^{-\alpha}\left(1+\frac{1}{\beta t}\right)^{-(n+\alpha)} \\ &\sim \frac{\left(\frac{1}{\beta}\right)^{\alpha}}{\alpha B(\alpha,n)}t^{-\alpha} \end{aligned} \tag{5.9}$$

which is of the form $t^{-\alpha}L(t)$ and is regularly varying since $\lim_{x\to\infty}(1+\frac{1}{\beta t})^{-(n+\alpha)} = 1$ and thus $F \in MDA(\Phi_\alpha)$. The norming constants are thus $C_n = \left(\frac{n}{\alpha\beta^\alpha B(\alpha,n)}\right)^{\frac{1}{\alpha}}$.

3. Reliability measures

The mean excess loss function of the mixture is

$$m(t) = \frac{t}{\alpha - 1} \tag{5.10}$$

and the equilibrium distribution is

$$f_e(t) = \frac{\left(\frac{1}{\beta}\right)^{\alpha-1}}{\alpha t^\alpha B(\alpha-1, n+1)} \tag{5.11}$$

Erlang-Type I Two-Parameter Lindley distribution

1. mixing distribution

The Type I Two-Parameter Lindley distribution is

$$f(y) = \frac{\theta^{\alpha+1}}{\theta+1}\frac{(\alpha+y)}{\Gamma(\alpha+1)}e^{-\theta y}y^{\alpha-1}, \quad y>0; \theta>0, \alpha>0 \tag{5.12}$$

Its tail function can be obtained using L'Hopital's rule as

$$\lim_{y\to\infty}\frac{\bar{F}(y)}{f(y)} = \lim_{y\to\infty}\frac{1}{\left(\theta - \frac{1}{\alpha+y} - \frac{(\alpha-1)}{y}\right)} = \theta^{-1} \tag{5.13}$$

and therefore $$\bar{F}(y) \sim \frac{\theta^\alpha}{\theta+1}\frac{(\alpha+y)}{\Gamma(\alpha+1)}y^{\alpha-1}exp\left\{-\int_0^y \frac{1}{\frac{1}{\theta}}dt\right\} \tag{5.14}$$

Thus $F \in MDA(\Lambda)$ and is a Von Mises function with auxiliary function $a(y) = \theta^{-1}$ and norming constants $d_n = \overleftarrow{F}(1-n^{-1}) = \theta^{-1}[lnn + (\alpha-1)(-ln\theta + lnlnn) + \alpha ln\theta - ln(\theta+1) - ln\Gamma(\alpha+1) + ln(\alpha + \theta^{-1}lnn)]$ and $C_n = a(d_n) = \theta^{-1}$.

2. mixed distribution
The Erlang-Type I Two-Parameter Lindley distribution has pdf

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}\theta^{\alpha+1}}{B(n,\alpha+1)(t+\theta)^{n+\alpha+1}}\left(\frac{1+\alpha(t+\theta)(n+\alpha)^{-1}}{\theta+1}\right) \\ &= \frac{\alpha\theta^{\alpha+1}}{(\theta+1)}\frac{t^{-(\alpha+1)}}{(\alpha+n)B(n,\alpha+1)}\left(1+\frac{\theta}{t}\right)^{-(n+\alpha+1)}\left(\frac{\alpha(\theta+1)+n}{\alpha t}+1\right) \end{aligned} \tag{5.15}$$

(see Gathongo (2019))
Using Karamata's theorem, the tail function can be obtained as

$$\begin{aligned} \bar{F}(t) &\sim \frac{\theta^{\alpha+1}}{(\theta+1)}\frac{t^{-\alpha}}{(\alpha+n)B(n,\alpha+1)}\left(1+\frac{\theta}{t}\right)^{-(n+\alpha+1)}\left(\frac{\alpha(\theta+1)+n}{\alpha t}+1\right) \\ &\sim \frac{\theta^{\alpha+1}}{(\theta+1)}\frac{t^{-\alpha}}{(\alpha+n)B(n,\alpha+1)} \end{aligned} \tag{5.16}$$

which is of the form $t^{-\alpha}L(t)$ and is a regularly varying function since $\lim_{t\to\infty}\left(1+\frac{\theta}{t}\right)^{-(n+\alpha+1)}\left(\frac{\alpha(\theta+1)+n}{\alpha t}+1\right)=1$ and therefore $F\in MDA(\Phi_\alpha)$.
The norming constants are $C_n=\left(\frac{n\theta^{\alpha+1}}{(\theta+1)(\alpha+n)B(n,\alpha+1)}\right)^{\frac{1}{\alpha}}$.

3.Reliability measures
The mean excess loss function of the mixed distribution is

$$m(t)=\frac{t}{\alpha-1} \tag{5.17}$$

and the equilibrium distribution is

$$f_e(t)=\frac{\theta^{\alpha}}{t^{\alpha}B(\alpha-1,n+1)[\alpha(\theta+1)-1]} \tag{5.18}$$

6. Gumbel limiting distribution

Mixed distributions are in the form of Confluent Hypergeometric Functions and are $F \in MDA(\Lambda)$ and the mixing distributions are $F \in MDA(\Psi_\alpha)$, $F \in MDA(\Phi_\alpha)$ and $F \in MDA(\Lambda)$.

Confluent Hypergeometric Functions

1. **Kummer's Confluent Hypergeometric Function**
   The Kummer's confluent hypergeometric function (Kummer's series) is defined as

$$ {}_1F_1(a, c; x) = \sum_{n=0}^{\infty} \frac{a_{(n)}}{c_{(n)}} \frac{x^n}{n!} \tag{6.1}$$

   where $a_{(n)} = a(a+1)(a+2)...(a+n-1), c_{(n)} = c(c+1)(c+2)...(c+n-1), c \neq 0, -1, -2, ..., a_{(0)} = c_{(0)} = 1$ and its integral representation is given by

$$ {}_1F_1(a, c; x) = \frac{1}{B(a, c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{xt} dt \tag{6.2}$$

   where a and c are the parameters and x is the argument of the function.
   Properties:

$$\begin{aligned} {}_1F_1(a, c; x) &= e^x_1F_1(c-a, c; -x) \\ {}_1F_1(a, a; x) &= e^x \\ \frac{x^a}{a} {}_1F_1(a, a+1; -x) &= \int_0^t t^{a-1} e^{-t} dt = \gamma(a, x) \end{aligned}$$

   where $\gamma(a, x)$ is the lower incomplete gamma function.
   The n^{th} derivative of Kummer's Confluent Hypergeometric Func-

tion is given by

$$\frac{d^n}{dx^n}{}_1F_1(a,c;x) = \frac{a_{(n)}}{c_{(n)}}{}_1F_1(a+n,c+n;x) \tag{6.3}$$

$$\text{where} \quad x_{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} x(x+1)...(x+n-1), & \textit{if } n>1 \\ 1, & \textit{if } n=0, \end{cases}$$

$$\text{Hence} \quad \frac{d}{dx}{}_1F_1(a,c;x) = \frac{a}{c}{}_1F_1(a+1,c+1;x) \tag{6.4}$$

$$\text{and} \quad \frac{d^2}{dx^2}{}_1F_1(a,c;x) = \frac{a(a+1)}{c(c+1)}{}_1F_1(a+2,c+2;x) \tag{6.5}$$

and from Kummer's differential equation $x\frac{d^2\omega}{dx^2}+(c-x)\frac{d\omega}{dx}-a\omega=0$,

$$\begin{aligned}\frac{a}{c}{}_1F_1(a+1,c+1;x) &= \frac{a}{c-x}{}_1F_1(a,c;x) - \frac{xa(a+1)}{c(c+1)(c-x)}{}_1F_1(a+2,c+2;x) \\ &\sim \frac{a}{c-x}{}_1F_1(a,c;x) \end{aligned}\tag{6.6}$$

where $\omega = {}_1F_1(a,c;x)$.

2. **Tricomi Confluent Hypergeometric Function**

The Tricomi confluent hypergeometric function is defined as

$$\Psi(a,c;x) = \frac{1}{\Gamma a}\int_0^\infty t^{a-1}(1+t)^{c-a-1}e^{-xt}dt \tag{6.7}$$

where a and c are the parameters and x is the argument of the function.

The Tricomi function is defined for all values of a, c and x. It is generally complex for $x<0$ and singular at $x=0$.

Properties:

$$\begin{aligned}\Psi(a,c;x) &= x^{1-c}\Psi(a-c+1,2-c;x) \\ \Psi(a,a;x) &= e^x\int_x^\infty t^{-a}e^{-t}dt \\ \Psi(1-a,1-a;x) &= e^x\int_x^\infty t^{a-1}e^{-t}dt = e^x\Gamma(a,x) \\ \Psi(1,1+a;x) &= x^{-a}e^x\Gamma(a,x)\end{aligned}$$

where $\Gamma(a,x)$ is the upper incomplete gamma function.
The n^{th} derivative of Tricomi Confluent Hypergeometric Function is given by

$$\frac{d^n}{dx^n}\Psi(a,c;x) = (-1)^n a_{(n)}\Psi(a+n,c+n;x) \tag{6.8}$$

$$\text{where} \quad x_{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} x(x+1)...(x+n-1), & if\ n>1 \\ 1, & if\ n=0, \end{cases}$$

$$\text{Hence} \quad \frac{d}{dx}\Psi(a,c;x) = -a\Psi(a+1,c+1;x) \tag{6.9}$$

$$\text{and} \quad \frac{d^2}{dx^2}\Psi(a,c;x) = a(a+1)\Psi(a+2,c+2;x) \tag{6.10}$$

and from Kummer's differential equation $x\frac{d^2\omega}{dx^2}+(c-x)\frac{d\omega}{dx}-a\omega=0$,

$$\begin{aligned}-a\Psi(a+1,c+1;x) &= \frac{-a}{x-c}\Psi(a,c;x)+\frac{xa(a+1)}{x-c}\Psi(a+2,c+2;x) \\ &\sim \frac{-a}{x-c}\Psi(a,c;x) \end{aligned} \tag{6.11}$$

where $\omega = \Psi(a,c;x)$.
The Tricomi and Kummer's confluent hypergeometric functions are related by

$$\Psi(a,c;x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}{}_1F_1(a,c;x)+\frac{\Gamma(c-1)x^{1-c}}{\Gamma a}{}_1F_1(a-c+1,2-c;x)$$

$$c \neq 0,-1,-2,... \tag{6.12}$$

Erlang-Scaled Beta distribution

1. **mixing distribution**
   The mixing (Scaled Beta) distribution,

$$f(y) = \frac{(\frac{y}{\mu})^{\alpha-1}(1-\frac{y}{\mu})^{\beta-1}\frac{1}{\mu}}{B(\alpha,\beta)}, \quad 0<y<\mu; \alpha>0, \beta>0, \mu>0 \tag{6.13}$$

has a finite right endpoint $y_F = \mu$ and $\overline{F}(\mu - y^{-1})$ is of the form $y^{-\beta}L(y)$ and is a regularly varying function since $\lim_{y\to\infty}\left(1 - \frac{1}{\mu y}\right)^{\alpha-1} = 1$, that is, by Karamata's theorem;

$$\overline{F}(\mu - y^{-1}) \sim \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)}\frac{y^{-\beta}}{\mu^{\beta}}\left(1 - \frac{1}{\mu y}\right)^{\alpha-1} \tag{6.14}$$

Hence $F \in MDA(\psi_\beta)$ with norming constants $d_n = y_F = \mu$ and $C_n = \mu - \overleftarrow{F}(1-n^{-1}) = \mu\left(\frac{n\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)}\right)^{\frac{-1}{\beta}}$.

2. **mixed distribution**
   The pdf of the mixture is given by

$$\begin{aligned} f_n(t) &= \frac{\mu^n t^{n-1}}{\Gamma(n)}\frac{B(n+\alpha, n+\alpha+\beta)}{B(\alpha,\beta)}{}_1F_1(n+\alpha, n+\alpha+\beta; -\mu t) \\ &= \frac{\mu^n e^{-\mu t} t^{n-1}}{\Gamma(n)}\frac{B(n+\alpha, n+\alpha+\beta)}{B(\alpha,\beta)}{}_1F_1(\beta, n+\alpha+\beta; \mu t) \end{aligned} \tag{6.15}$$

(see Gathongo (2019))
An application of L'Hopital's rule on $\overline{F}(t)/f(t)$ yields

$$\lim_{t\to\infty}\frac{\bar{F}(t)}{f(t)} = \lim_{t\to\infty}\frac{1}{\mu - \frac{(n-1)}{t} - \frac{\beta}{n+\alpha+\beta-\mu t}} = \mu^{-1} \tag{6.16}$$

and thus
$$\overline{F}(t) \sim \frac{(\mu t)^{n-1}}{\Gamma(n)}\frac{B(n+\alpha, n+\alpha+\beta)}{B(\alpha,\beta)}{}_1F_1(\beta, n+\alpha+\beta; \mu t)exp\left\{-\int_0^t \frac{1}{\frac{1}{\mu}}dy\right\} \tag{6.17}$$

Therefore $F \in MDA(\Lambda)$ and is a von Mises function with auxiliary function $a(t) = \mu^{-1}$ and norming constants $d_n = \overleftarrow{F}(1-n^{-1}) = \mu^{-1}[(n-1)lnlnn - ln\Gamma(n) + lnB(n+\alpha, n+\alpha+\beta) - lnB(\alpha,\beta) + ln{}_1F_1(\beta, n+\alpha+\beta; lnn)]$ and $C_n = a(d_n) = \mu^{-1}$.

3. **Reliability measures**
   The equilibrium distribution of the mixed distribution is

$$f_e(t) = \frac{\mu^n}{n} e^{-\mu t} t^{n-1} \frac{B(n+\alpha, n+\alpha+\beta)}{B(\alpha-1,\beta)} {}_1F_1(\beta, n+\alpha+\beta; \mu t)$$

Erlang-Pareto I distribution

1. **mixing distribution**
   The Pareto I distribution is

$$f(y) = \alpha\beta^\alpha y^{-(\alpha+1)}, \quad y > \beta; \beta > 0, \alpha > 0 \tag{6.18}$$

   By Karamata's theorem, the tail function can be obtained as

$$\bar{F}(y) \sim \beta^\alpha y^{-\alpha} \tag{6.19}$$

   and is of the form $y^{-\alpha}L(y)$ which is regularly varying since $lim_{y\to\infty}1 = 1$. Thus $F \in MDA(\phi_\alpha)$ with norming constants $C_n = \beta n^{\frac{1}{\alpha}}$.

2. **mixed distribution**
   The Erlang-Pareto I mixture has pdf

$$f_n(t) = \frac{\alpha\beta^n}{\Gamma(n)} e^{-\beta t} t^{n-1} \psi(1, n-\alpha+1; \beta t) \tag{6.20}$$

   (see Gathongo (2019))
   and a tail function, obtained using L'Hopital's rule, as

$$\lim_{t\to\infty} \frac{\bar{F}(t)}{f(t)} = \lim_{t\to\infty} \frac{1}{\beta - \frac{(n-1)}{t} + \frac{1}{\beta t-(n-\alpha+1)}} = \beta^{-1} \tag{6.21}$$

   and thus $$\bar{F}(t) \sim \frac{\alpha}{\Gamma(n)} (\beta t)^{n-1} \psi(1, n-\alpha+1; \beta t) exp\left\{ -\int_0^t \frac{1}{\frac{1}{\beta}} dy \right\} \tag{6.22}$$

   and hence $F \in MDA(\Lambda)$ and is a Von Mises function with auxiliary function $a(t) = \beta^{-1}$ and norming constants $d_n = \overleftarrow{F}(1-n^{-1}) = \beta^{-1}[lnn + (n-1)lnlnn + ln\alpha - ln\Gamma(n) + ln\psi(1, n-\alpha+1; lnn)]$ and $C_n = a(d_n) = \beta^{-1}$.

3. **Reliability measures**
The mean excess loss function of the mixed distribution is

$$m(t) = \left(\frac{1}{\beta} - 1\right) \frac{e^{\beta t}\Gamma(n, \beta t)}{\beta^n t^{n-1}}$$

and the equilibrium distribution is

$$f_e(t) = \frac{\alpha + 1}{n} \frac{\beta^n}{\Gamma(n)} e^{-\beta t} t^{n-1} \psi(1, n - \alpha + 1; \beta t)$$

Erlang-Shifted Gamma(Pearson Type III) distribution

1. **mixing distribution**
The Shifted Gamma(Pearson Type III) distribution,

$$f(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta(y-\mu)} (y - \mu)^{\alpha-1}, \quad y > \mu; \alpha > 0, \beta > 0, \mu > 0 \tag{6.23}$$

has a tail $$\bar{F}(y) \sim \frac{e^{\beta\mu}(\beta(y - \mu))^{\alpha-1}}{\Gamma(\alpha)} exp\left\{ - \int_0^y \frac{1}{\frac{1}{\beta}} dt \right\} \tag{6.24}$$

which is obtained by L'Hopital's rule as illustrated below.

$$\lim_{y\to\infty} \frac{\bar{F}(y)}{f(y)} = \lim_{y\to\infty} \frac{1}{\beta - \frac{(\alpha-1)}{y-\mu}} = \beta^{-1} \tag{6.25}$$

and thus $F \in MDA(\Lambda)$ and is a Von Mises function with auxiliary function $a(y) = \beta^{-1}$. The norming constants are $d_n = \overleftarrow{F}(1 - n^{-1}) = \beta^{-1}[lnn + \beta\mu + (\alpha - 1)ln(lnn - \beta\mu) - ln\Gamma(\alpha)]$ and $C_n = a(d_n) = \beta^{-1}$.

2. **mixed distribution**
The mixed distribution has pdf

$$f_n(t) = \frac{\mu^{n+\alpha}}{\Gamma(n)} t^{n-1} \beta^\alpha e^{-\mu t} \psi(\alpha, \alpha + n + 1; (t + \beta)\mu) \tag{6.26}$$

(see Gathongo (2019))
L'Hopital's rule can be applied in obtaining the tail of the distribution as shown below.

$$\lim_{t\to\infty}\frac{\bar{F}(t)}{f(t)} = \lim_{t\to\infty}\frac{1}{\mu - \frac{(n-1)}{t} + \frac{\alpha}{(t+\beta)\mu-(\alpha+n+1)}} = \mu^{-1} \tag{6.27}$$

hence $$\bar{F}(t) \sim \frac{(\beta\mu)^{\alpha}(\mu t)^{n-1}}{\Gamma(n)}\psi(\alpha,\alpha+n+1;(t+\beta)\mu)exp\left\{-\int_0^t \frac{1}{\frac{1}{\mu}}dy\right\} \tag{6.28}$$

Thus $F \in MDA(\Lambda)$ and is a Von Mises function with auxiliary function $a(t) = \mu^{-1}$ and norming constants $d_n = \overleftarrow{F}(1-n^{-1}) = \mu^{-1}[lnn+(n-1)lnlnn+\alpha ln(\beta\mu)-ln\Gamma(n)+ln\psi(\alpha,\alpha+n+1;(lnn+\beta\mu))]$ and $C_n = a(d_n) = \mu^{-1}$.

3. **Reliability measures**
The mean excess loss function of the mixed distribution is

$$m(t) = \left(\frac{1}{\mu\Gamma(\alpha)} - 1\right)\frac{\Gamma(n,\mu t)e^{\mu t}}{\mu^n t^{n-1}}$$

and the equilibrium distribution is

$$f_e(t) = \frac{1}{n}\frac{\mu^n}{\Gamma(n)}e^{-\mu t}t^{n-1}\frac{\Psi(\alpha,\alpha+n+1;(t+\beta)\mu)}{\Psi(\alpha,\alpha;\beta\mu)}$$

7. Conclusion

Extreme value distributions have been derived for continuous Erlang mixtures and their mixing distributions. It has also been determined that the limiting distributions of the Erlang mixtures are either Type I (Gumbel) or Type II (Fretchét) as their right endpoints are infinite, and the mixed distributions have been classified into the two extreme value types. It has been noted that all Erlang mixed distributions with the Fretchét limiting distribution have mixing distributions with the Gumbel extreme value type. Mixtures with the Gumbel have mixing distributions with either type of the three limiting distributions. Some reliability measures of the Erlang mixtures have also been presented which are the mean excess loss and equlibrium distributions.

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