

# Block multi-derivative linear multistep methods for solving first, second and third order ordinary differential equations

## Abstract

This paper presents block multi-derivative linear multi-step method of single block to solve first and second and third-order initial value problems of ordinary differential equations. The derivation of the methods is achieved by apply the technique of interpolation and collocation to a power series polynomial which is considered as an approximate solution to the problems. Higher derivative terms are introduced to improve the order of accuracy of the method and also gives room to modified the method for solving second and third order initial value problems (IVPs) of ordinary differential equations (ODEs). Details conformation of the block method are presented which shows that the method are zero stable, consistence and convergent. The method will be applied in a block to solve real life problem of first second and third-order initial value problems (IVPs) of ordinary differential equations.

### *Keywords:*

Multi-derivative, Multistep, Higher derivative, Initial Value Problems (IVPs), Ordinary Differential Equations (ODEs)

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## 1. Introduction

Consider the initial value problem of first-order ordinary differential equations (ODEs) of the form

$$y' = f(x, y), \quad y(x_0) = y_0, \quad (1)$$

on a given interval  $[x_0, x_N]$ , where conditions about the existence of a unique solution are assumed. There are a lot of numerical methods available in the literature for solving the problem in (1), the Runge-Kutta and multistep methods are well-known schemes that have being used largely for this purpose. Other approach recently used for solving the above problem consists in the block formulation. Block methods were proposed firstly by Milne (1953) according to Olabode & Yusuph (2009) first proposed block method for implementing the implicit as a result of its nature called self-starting. This method was proposed in order to by-pass the burden of developing of separate predictors for the correctors which has been reported to be more costly since the subroutines for incorporating the stating values lead to lengthy computational time, Jator (2007). This setback was addressed by Ramos (2017), this was considered in the same light by Sarafyan (1965), Jator (2008), Mohammed and Adeniyi (2015), Kayode et al., (2014), Olabode and Momoh (2016), Akinfenwa

and Jator (2013), among others who independently proposed block methods for solving differential equation which do not require the development of separate predictor, but simultaneously generate approximations at different grid point within the interval of integration. This method of solution was studied and reported by some prominent authors such as Ogunfeyitimi & Ikhile (2020), Ramos (2019), Abdelrahim and Omole (2016), Ukpebor & Omole (2020), Nwachukwu & Okor (2018), Kashkari and Syam (2019). This article considers a block formulation of a one-step method that are obtained through the higher derivative .

## 2. Derivation of the Method

Power series polynomial of the form

$$y(t) = \sum_{j=0}^k a_j t^j \quad (2)$$

is allowed to approximate  $y(t)$  in (1) for the purpose of deriving a numerical method for solving the equation. The coefficients  $a_j$ 's are to be determined,  $t$  is continuous and differentiable within the interval of integration  $[a, b]$  and  $k$  represents the summation of the numbers of interpolation and collocation points. First, second and third derivatives of partial sums of (2) for  $k = 3$  gives the following:

$$y'(t) = \sum_{j=1}^3 j a_j t^{j-1} \quad (3)$$

$$y''(t) = \sum_{j=2}^3 j(j-1) a_j t^{j-2} \quad (4)$$

and

$$y'''(t) = \sum_{j=3}^3 j(j-1)(j-2) a_j t^{j-3} \quad (5)$$

interpolating (2) at  $x_n$  and collocating (3)-(5) at  $t = t_{n+i}$ ,  $i = 1$  where  $t_{n+i} = t_n + ih$  i.e  $t_{n+1} = t_{n+h}$  and setting  $t_n = 0$ , yields

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2h & 3h^2 \\ 0 & 0 & 2 & 6h \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_n \\ f_{n+1} \\ g_{n+1} \\ q_{n+1} \end{bmatrix} \quad (6)$$

The determinant of the matrix  $X \neq 0$ , hence, matrix  $X$  is non-singular. The parameters  $a_j$ 's are obtained by using Gaussian elimination method are presented as follows;

$$a_0 = y_n \quad (7)$$

$$a_1 = \frac{1}{2} (2f_{n+1} - 2hg_{n+1} + h^2q_{n+1}) \quad (8)$$

$$a_2 = \frac{1}{2} (g_{n+1} - hq_{n+1}) \quad (9)$$

$$a_3 = \frac{1}{6} q_{n+1} \quad (10)$$

Substituting (7)-(10) into (2) with  $t_{n+i} = t_n + th$  gives a continuous formula of the form

$$y(t) = \alpha_0(t)y_n + h(\beta_1(t)f_{n+1}) + h^2(\gamma_1(t)g_{n+1}) + h^3(\mu_1(t)q_{n+1}) \quad (11)$$

with the following coefficients;

$$\alpha_0(t) = 1, \quad (12)$$

$$\beta_1(t) = h(t), \quad (13)$$

$$\gamma_1(t) = h^2 \left( \frac{t^2}{2} - t \right), \quad (14)$$

$$\mu_1(t) = h^3 \left( \frac{t}{2} - \frac{t^2}{2} + \frac{t^3}{6} \right). \quad (15)$$

Evaluating (11) at  $t = 1$ , gives the discrete scheme

$$y_{n+1} = y_n + hf_{n+1} - \frac{1}{2}h^2g_{n+1} + \frac{1}{6}h^3q_{n+1} \quad (16)$$

Equation (16) are the formulas that constitute the proposed method for solving first order ODEs. One of the novelty of this work is the adoption of the derived method for the directly solving second and third order ODEs. To adopt Equation (16) for solution of second and third order ODEs two additional methods are obtained by evaluating the first, and second derivative of equation (11) at  $t_n$  gives

$$f_n = f_{n+1} - hg_n + \frac{1}{2}h^2q_{n+1} \quad (17)$$

$$g_n = g_{n+1} - hq_{n+1} \quad (18)$$

### 3. Basic Properties of the Method

we shall consider in this section, the analysis of the basic properties of this method such as order, error constant, zero stability and consistency is investigated

#### 3.1. Order and Error Constant

Equation (16) derived is a discrete scheme belonging to the class of LMM of the form

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f_{n+j} + h^2 \sum_{j=0}^k \gamma_j g_{n+j} \quad (19)$$

According to Fatunla (1988) and Lambert (1991), we define the local truncation error associated with equation (16) by the difference operator

$$L[y(x); h] = \sum_{j=0}^k \alpha_j y(x_n + jh) - h^2 \beta_j y''(x_n + j) + h^3 \gamma_j y'''(x_n + j) \quad (20)$$

where  $y(x)$  is an arbitrary function, continuously differentiable on  $[a, b]$ . Expanding (20) in Taylor series about the point  $x$ , we obtain the expression

$$L[y(x); h] = c_0 y(x) + c_1 h y^1(x) + c_2 h^2 y^2(x) + c_3 h^3 y^3(x) + \dots + c_p h^p y^p(x) \quad (21)$$

where the  $c_0, c_1, c_2, \dots, c_{p+1}$  are obtain as

$$c_0 = \sum_{j=0}^k \alpha_j, c_1 = \sum_{j=1}^k j\alpha_j, c_2 = \frac{1}{2!} \sum_{j=1}^k j^2\alpha_j, c_q = \frac{1}{q!} \left[ \sum_{j=1}^k j^2\alpha_j - q(q-1) \sum_{j=1}^k \beta_j j^{q-2} \right]$$

according to Lambert (1991), for first-order methods, equation (16) is of order p if  $c_0 = c_1 = c_2 = \dots = c_p = 0$  and  $c_{p+r} \neq 0$  for  $r = 1$ . The  $c_{p+r} \neq 0$  is called the error constant and  $c_{p+r}h^{p+r}y^{p+r}(x_n)$  is the principal local truncation error at the point  $x_n$ .

Thus, the block (16) is of order p=3 and error constants  $c_{p+r} = [-\frac{1}{24}]$

### 3.2. Zero-stability of the method

The general form of block method is given as

$$A^0Y_m = A^rY_{m-1} + h^\rho[B^iF_m + B^0F_{m-1}] \tag{22}$$

A method is said to be zero stable, if the roots

$$\det[\lambda A^0 - A^r] = 0 \tag{23}$$

the first characteristic polynomial satisfies  $|\lambda| \leq 1$  and for the roots with  $|\lambda| = 1$ , the multiplicity must not exceed the order of the differential equation. Fatunla, (1988). This kind of stability concerned the behaviour of the numerical method. As  $h \rightarrow 0$ , the derived method given the system of equations that can be written as;

$$A^0Y_m = A^rY_{m-1} \tag{24}$$

where  $A^0$  is identity matrix Consider (16) for  $K = 1$  and setting  $h = 0$ , we have

$$\begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \end{bmatrix} - \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} y_n \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\rho(\lambda) = \lambda \begin{bmatrix} 1 \end{bmatrix} - \begin{bmatrix} 1 \end{bmatrix}$$

$$\lambda - 1 = 0$$

Since  $\lambda_1 = 1$ , the block methods are zero stable

### 3.3. Consistency

A method is said to be consistent, if it has order greater than one and zero-stable Henrici (1996). From the above analysis, it is show that our method is consistent.

### 3.4. Convergency

The necessary and sufficient condition for a LMM to be convergent is that it must be consistent and zero stable Henrici (1996), hence the methods are convergent since it satisfies all the conditions.

### 3.5. Region of Absolute Stability

The area in the complex  $z$ -plane in which the numerical method exhibits the behaviours of the real solution is known as the region of absolute stability, the behaviour is equivalent to that method for which  $h > 0$ . The behaviour for the method is studied here by considering the application of the method to any of the following equations

$$y' = -\lambda y \quad (25)$$

depending on the type of problems that the method is intended to solve. Applying the formulas (16) to the test problem (25) yields

$$BY_n = GY_{n-1} \quad (26)$$

where

$$Y_n = (y_{n+1})^T, \quad Y_{n-1} = (y_n)^T \text{ and } z = h\lambda$$

$$B = \left( \frac{1}{6}(z+3)z^2 + z + 1 \right) \quad (27)$$

$$G = \begin{pmatrix} 0 & -1 \end{pmatrix} \quad (28)$$

The amplification matrix

$$M(h) = B^{-1}G \quad (29)$$

$$M(h) = \left( \frac{1}{6}(z+3)z^2 + z + 1 \right)^{-1} \begin{pmatrix} 0 & -1 \end{pmatrix} \quad (30)$$

where the dominant eigenvalue  $\left( \frac{1}{6}(z+3)z^2 + z + 1 \right)$  is a function of  $z$

From the above figure, it is observed that the method exhibit a Runge-Kutta like region of absolute stability

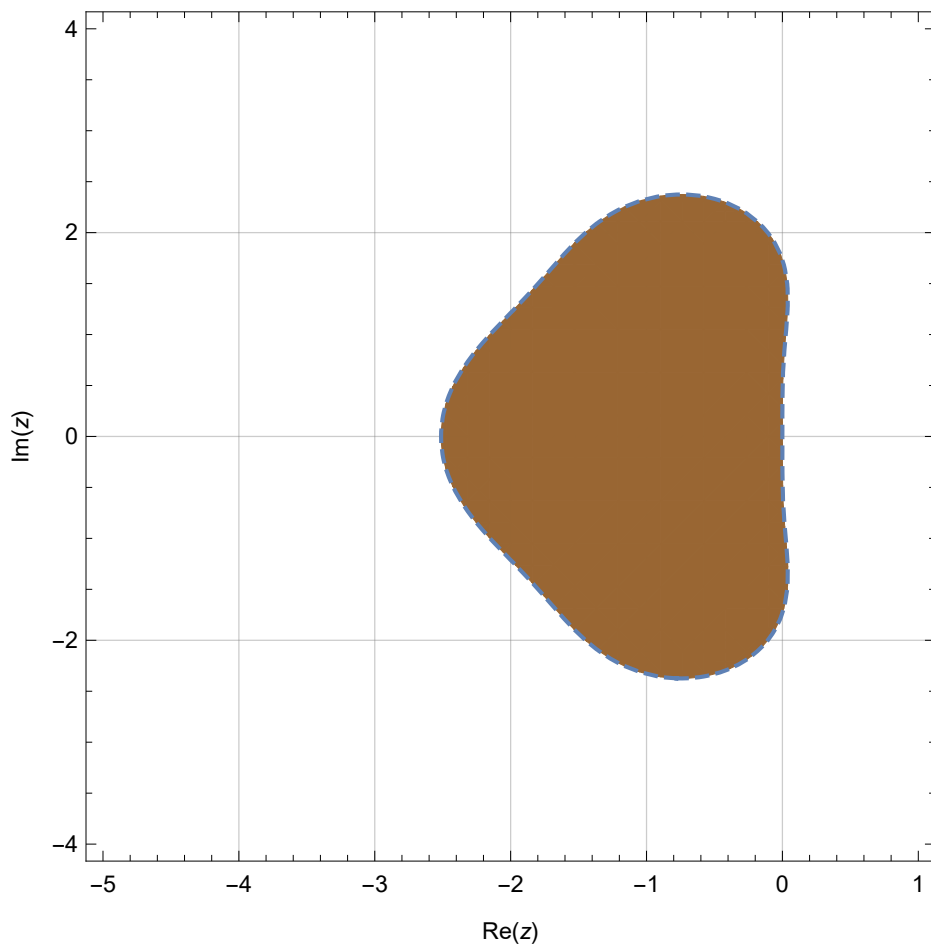


Figure 1: Region of absolute stability of the method

## 4. Numerical Examples

The performance of the methods is tested on four numerical examples

### *Problem 1*

Consider an AI system integration into a hardware setup, like a server or a robotic unit, where maintaining an optimal temperature is crucial for performance and longevity. The system has a cooling mechanism (like fans or liquid cooling), and the effectiveness of this cooling changes as the system's temperature approaches the operating temperature. Let  $T(t)$  represent the temperature of the AI system at time  $t$ . The rate of change of temperature could be influenced by the difference between the current temperature and the desired optimal temperature. A simple first-order differential equation could be:

$$\frac{dT}{dt} = -k \cdot (T(t) - T_{opt})$$

Where:

- .  $\frac{dT}{dt}$  is the rate of change of the system's temperature over time.
- .  $k$  is a positive constant representing cooling system's efficiency.
- .  $T(t)$  is the current temperature at time  $t$ .
- .  $T_{opt}$  is the optimal operating temperature for the AI system.

*Interpretation*

. where the system’s temperature  $T(t)$  is much higher than  $T_{opt}$ , the cooling system works effectively, leading to a faster decrease in temperature ( $\frac{dT}{dt}$ ).

. As  $T(t)$  approaches  $T_{opt}$ , the cooling system’s effectiveness reduces (as less cooling is needed), leading to a slower rate of temperature change.

.The model captures the balancing act of the cooling system: it actively cools the system when the temperature is far from optimal and reduces its cooling effect as the temperature nears the desired level, preventing over cooling.

The model

$$\frac{dT}{dt} = -k(T - T_{opt})$$

has exact solution

$$T(t) = T_o e^{-k(t)} - (e^{-kt} - 1)T_{opt}$$

where

$$T_o = T(t_o)$$

Given that

$T_{opt} = 24^{\circ}c$ ;  $T_o = T(o) = 18^{\circ}c$  and  $k = 0.8$  (i.e 80% cooling efficiency)

$$\frac{dT}{dt} = -0.8(T(t) - 24)$$

where

$$T_o = T(o) = 18$$

has exact solution

$$T(t) = 18e^{-0.8(t)} - (e^{-0.8t} - 1)24$$

Table 1: Solution of Problem 1 for k=1, p=3 and h=0.01

x-value	y - exact	y-computed	Error
0.1	18.461301921680	18.461301912288	9.392415734055 E-09
0.2	18.887137266203	18.887137248862	1.734058940883 E-08
0.3	19.280232833601	19.280232809590	2.401106513616 E-08
0.4	19.643105777558	19.643105748004	2.955334821309 E-08
0.5	19.978079723786	19.978079689685	3.410147542127 E-08
0.6	20.287299649163	20.287299611388	3.777554624662 E-08
0.7	20.572745616907	20.572745576224	4.068309777949 E-08
0.8	20.836245455742	20.836245412821	4.292027000474 E-08
0.9	21.079486464240	21.079486419667	4.457294977556 E-08
1.0	21.304026215297	21.304026169579	4.571779754770 E-08

*Problem 2*

Consider the following linear initial value problem

$$y'' = y'$$

$$y(x) = 1 - e^x, \quad y(0) = 0, \quad y'(0) = -1, \quad h = 0.1, 0.01$$

Table 2: Comparison of Error for Problem 2 (k=1, p=3 and h=0.01)

x-value	y - exact	y-computed	Error k=1, p=8	Ogunware & Ezekiel 2023
0.1	-0.010050167084	-0.010050167083	9.478690351 E-13	5.831942458 E-12
0.2	-0.020201340027	-0.020201340023	3.785581178 E-12	1.087659387 E-11
0.3	-0.030454533954	-0.030454533945	8.551027165 E-12	1.511754063 E-11
0.4	-0.040810774192	-0.040810774184	1.528266847 E-11	1.853811766 E-11
0.5	-0.051271096376	-0.051271096352	2.401954492 E-11	2.112140771 E-11
0.6	-0.061836546545	-0.061836546511	3.480128250 E-11	2.285023525 E-11
0.7	-0.072508181254	-0.072508181207	4.766810111 E-11	2.370716438 E-11
0.8	-0.083287067675	-0.083287067612	6.266082257 E-11	2.367449502 E-11
0.9	-0.094174283705	-0.094174283625	7.982087859 E-11	2.273425941 E-11
1.0	-0.105170918076	-0.105170917976	9.919031899 E-11	2.086821844 E-11

*Problem 3*

Consider the third-order IVP.

$$y''' - y'' + y' - y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1, \quad h = 0.01$$

with the exact solution

$$y(x) = \cos x$$

Table 3: Comparison of Error for Problem 3 (k=1, p=3 and h=0.01)

x-value	y -exact	y - computed	Error	Adeyefa 2021 k=1,p=7
0.1	0.999950000417	0.999950000445	2.831901E-11	1.746452E-07
0.2	0.999800006667	0.999800006892	2.252265E-10	4.159672E-07
0.3	0.999550033749	0.999550034509	7.598979E-10	1.402140E-06
0.4	0.999200106661	0.999200108464	1.803134E-09	3.291541E-06
0.5	0.998750260395	0.998750263922	3.527363E-09	6.348473E-06
0.6	0.998200539935	0.998200546042	6.106603E-09	1.082232E-05
0.7	0.997551000253	0.997551009970	9.716457E-09	1.694540E-05
0.8	0.996801706303	0.996801720837	1.453409E-08	2.493442E-05
0.9	0.995952733012	0.995952753750	2.073822E-08	3.498913E-05
1.0	0.995004165278	0.995004193787	2.850908E-08	1.052497E-04

## 5. Discussion

In this research work, we have applied the procedure of collocation and interpolation to develop block multi-derivative linear multistep methods for solving initial value problem of first, second and third order ordinary differential equation. Table 1 show the solution of Artificial Intelligent (IT) problem while table 2 and table 3 show that the our new method is more accurate than the method proposed by Ogunware & Ezekiel (2023), and Adeyefa (2021).

## 6. Conclusion

A more suitable numerical method for the solution of first, second and third-order initial value problems has been achieved. Numerical experiment perform using this method show that the method is consistent, efficient and accurate. Therefore, this article is recommended for direct integration of first, second and third-order ordinary differential equation

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