SOME SECOND ORDER ROTATABLE DESIGNS CONSTRUCTED USING TRIGONOMETRIC FUNCTIONS

Abstract

In this study, two new second order rotatable designs of thirty-five and forty-five points are constructed respectively using transformations of some trigonometric functions. The designs constructed permit response surfaces to be fitted easily and provides spherical information contours besides the economic use of scarce resources in relevant production processes.

Key words: Second Order, Rotatable Designs, Response Surface.

1.0 Introduction

In any experimental work, it is important to choose the best design in a class of existing designs. The choice is solely dependent of the interest of the experimenter and the adequacy of an experimental design that can be determined from the information matrix, Box [10]. The technique of fitting a response surface is one widely used to aid in the statistical analysis of experimental work in which the yield of a product depends in some unknown fashion on one or more controllable variables, Bose and Draper [1]. Before the details of such an analysis can be carried out, experiments must be performed at predetermined levels of controllable factors, i.e. an experimental design must be performed prior to experimentation. Box and Hunter [9] suggested designs of a certain type which they called rotatable as being suitable for such experimentation. They developed second order rotatable designs through geometrical configurations. Box and Wilson [11] pointed out that the technique of fitting a response surface is one widely used to aid in the statistical analysis of experimental work in which the response of the product depends in some unknown fashion, on one or more controllable variables. Mutiso [5] constructed specific optimal second order rotatable designs in three, four and five dimensions. Koske *eta*l. [6,7,8] constructed optimal second order rotatable designs and gave practical hypothetical examples. Cornelious [2,3,4] constructed optimal sequential third order rotatable designs in three four and five dimensions, and a second order rotatable design of 33 and 39 points respectively in three dimensions giving practical hypothetical examples. The current study gives new 35- points, and 45-Points second order rotatable designs in three dimensions constructed using transformations of Trigonometric functions.

2.0 Methods

2.1 Moment conditions for second order rotatability

Suppose a second order response surface is to be fitted, then the following model is ideal;

$$y_u = \beta_0 + \sum_{u=1}^{N} \beta_i x_{iu} + \sum_{u=1}^{N} \beta_{ii} x_{iu}^2 + \sum_{u=1}^{N} \beta_{ij} x_{iu} x_{ju} + e_u \qquad (1)$$

Where x_i denotes the level of the i^{th}factor (i=1,2,3…k) in the u^{th} run, (u=1,2,3…, N) of the experiment and e_u is the un correlated random error with mean zero and variance σ^2.

According to Box and Hunter [9], A second order response surface is achieved if the design points satisfy the following moment conditions.

i. $\sum_{u=1}^{N} x_{iu}^2 = N\lambda_2$ (2)
ii. $\sum_{u=1}^{N} x_{iu}^4 = 3N\lambda_4$ (3)
iii. $\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = N\lambda_4$ for (i$\neq j = 1,2,\ldots,k$) and $3, \lambda_2, \lambda_4$, are constants.
iv. $\sum_{u=1}^{N} x_{iu}^4 - 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 0$ (4)
v. All the other odd order moments are zero.

If all the above conditions are satisfied, then the set of points is said to form a rotatable arrangement.

2.2 Non-Singularity Conditions for Second Order Rotatability

For the second order rotatable design to exist, the following non-singularity conditions must also be satisfied.

$$\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2} \tag{5}$$

Where k is the number of factors which in this case will be 3, and, λ_2, and λ_4 are obtained from the moment conditions given in (1),(2) and (3).

2.3 3s- POINTS

Trigonometric functions were first introduced by Bose and Draper [7] in the construction of second order rotatable designs. They introduced transformations of the form,

$$T_1 = \begin{bmatrix} Cos\ \alpha, -Sin \propto, & 0 \\ sin\ \propto, \ cos \propto, 0 \\ 0 \quad\quad 0 & -1 \end{bmatrix} \tag{6}$$

and

$$T_2 = \begin{bmatrix} Cos\ \frac{\propto}{2}, & Sin\frac{\propto}{2}, & 0 \\ sin\ \frac{\propto}{2}, & cos\frac{\propto}{2}, & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{7}$$

Where $\propto = \frac{2\pi}{s}$

These transformations are applied in the current study to construct second order rotatable designs. The transformations given in (6) and (7) are applied to the point sets of the form G(r, 0, b), i.e., points on the plane y = 0, and to all other points obtained from repeated applications of T_1 and T_2.

The permutation group (I, W,W^2) generated by,

$$W=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad (8)$$

shall also be applied to G(r ,o ,b) to give T_z(r,o,b), T_x (b,r,o) and T_y (o,b,r).

Consider a set T(r, 0, b). Assuming b = 0, then T(r, 0, b) becomes T(r, 0, 0). Applying (6) and (8) respectively on T(r,0,0) gives the following set of coordinates,

$$(r\cos t\alpha,\ r\sin t\alpha,\ 0),(r\sin t\propto,\ 0, r\cos t\propto),\ (0, r\cos t\propto, r\sin t\propto) \quad (12)$$

Where; t = 0, 1, 2… (s-1) and s≥5.

The sums and products of the set up to power four for the co-ordinates listed in (12) are given by;

$$\sum_{u=1}^{N} x_{iu}^2 = sr^2, \sum_{u=1}^{N} x_{iu}^4 = \frac{3}{4} sr^4, \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \frac{1}{8} sr^4, \quad (13)$$

The excess function for T(r, o, o) is given by;

$$\sum_{u=1}^{N} x_{iu}^4 - 3\sum_{u=1}^{N} x_{iu}^2\, x_{ju}^2 = \frac{3}{8}\, sr^4 \quad (14)$$

2.5 Eight Point Set

Similarly, the co-ordinates for the eight points set denoted G (a, a, a) are as listed below,

G (a, a, a), G (-a, a, a), G (a, -a, a), G (a, a, -a)

G (-a, - a, a), G (-a, a, -a), G (a, - a, -a), G (- a, -a, - a) (15)

The sums and products up to power four for G (a, a, a) is given by,

$$\sum_{u=1}^{N} x_{iu}^2 = 8a^2$$

$$\sum_{u=1}^{N} x_{iu}^4 = 8a^4 \quad (16)$$

$$\sum_{u=1}^{N} x_{iu}^2\, x_{ju}^2 = 8a^4$$

The excess for the above set of points is given by

$$\sum_{u=1}^{N} x_{iu}^4 - 3\sum_{u=1}^{N} x_{iu}^2\, x_{ju}^2 = -16a^4 \quad (17)$$

2.6 The Twelve Point Set

Consequently, the co-ordinates for the twelve-point set denoted by G (p, p, o) are listed as given below.

G (p, p, 0), G (-p, p, 0), G (p, -p, 0), G (-p, -p, 0)

G (0, p, p), G (0, -p, p), G (0, p, -p), G (0, -p,-p) (18)

G (p, 0, p), G (-p, 0, p), G (p, 0, -p), G (-p, 0, -p)

The sums and products up to the power four for G (p, p, o), are given by,

$\sum_{u=1}^{N} x_{iu}^{2}=8p^2$, $\sum_{u=1}^{N} x_{iu}^{4}=8p^4$, $\sum_{u=1}^{N} x_{iu}^{2}\, x_{ju}^{2}=4p^4$ (19)

The excess for the above set of points is given by

$\sum_{u=1}^{N} x_{iu}^{4}-3\sum_{u=1}^{N} x_{iu}^{2}\, x_{ju}^{2}=-4p^4$ (20)

2.7 The Twenty-Four Point Set

Finally, the co-ordinates for the twenty-four points set denoted by G (p, q, o) are listed as given below.

G (q, p, 0), G (-q, p, 0), G (q, -p, 0), G (-q, -p, 0)

G (p, q, 0), G (-p, q, 0), G (p, -q, 0), G (-p, -q, 0)

G (0, p, q), G (0, -p, q), G (0, p, -q), G (0, -p, -q)

G (0, q, p), G (0, -p, p), G (0, q, -p), G (0, -q, -p) (21)

G (p, 0, q), G (-p, 0, q), G (p, 0, -q), G (-p, 0,-p)

G (q, 0, p), G (-q, 0, p), G (q, 0, -p), G (-q, 0, -p)

The sums and products up to the power four for G (p, q, 0), are given by,

$\sum_{u=1}^{N} x_{iu}^{2}=8p^2+8q^2$, $\sum_{u=1}^{N} x_{iu}^{4}=8p^4+8q^4$, $\sum_{u=1}^{N} x_{iu}^{2}\, x_{ju}^{2}=8p^2q^2$ (22)

The excess for the above set of points is given by

$\sum_{u=1}^{N} x_{iu}^{4}-3\sum_{u=1}^{N} x_{iu}^{2}\, x_{ju}^{2}=8p^4+8q^4-24\, p^2q^2$ (23)

3.0 Results

3.1 Construction of Thirty-Five Points Second Order Rotatable Design in Three Dimensions

The thirty-five points is obtained by combining three sets i.e. the set denoted by s_1, s_2, s_3.

Where;

$s_1 = 3s$ for s=5 given in (12), s_2= 2 G (p, p, 0) given in (18), and s_3= s (a, a, a) given in (15).

The sums of squires for the combined sets (12) and (15) and (18) up to the power of four gives

$\sum_{u=1}^{35} x_{iu}^2 = Sr^2 + 8p^2 + 8a^2$

$\sum_{u=1}^{35} x_{iu}^4 = \frac{3}{4}Sr^4 + 8p^4 + 8a^4$ (24)

$\sum_{u=1}^{35} x_{iu}^2 x_{ju}^2 = \frac{1}{8}Sr^4 + 4p^4 + 8a^4$

And all other odd odder powers are zero

The set of thirty-five points denoted by R_3 is given by;

R_3 = s (p, p, 0) + s (a, a,) + 3s (25)

Substituting (24) to the moment conditions given in (2),(3),(4) gives;

$Sr^2 + 8p^2 + 8a^2 = N\lambda_2$, $\frac{3}{4}Sr^4 + 8p^4 + 8a^4 = 3N\lambda_4$,

$\frac{1}{8}Sr^4 + 4p^4 + 8a^4 = N\lambda_4$. (26)

The excess function for s_1 given in (14) is added to the excess function for s_2 and s_3 given in (17) and (20) respectively assuming the value of S to be 5 to give;

$\sum_{u=1}^{35} x_{iu}^4 = 3\sum_{u=1}^{35} x_{iu}^2 x_{ju}^2 = \frac{15}{8} sr^4 - 4p^4 - 16a^4 = 0$ (27)

Equation (27) is solved to obtain the values of p and a in terms of r as follows;

Let $p^2 = xr^2$ and $a^2 = yr^2$ (28)

Substituting (28) to (27) gives

$1.875r^4 - 4x^2r^4 - 16y^2r^4 = 0$ (29)

Making x the subject of the formula in (29) results to;

$x = \sqrt{0.1171875 - 0.25y^2}$ (30)

The values of y which together with the corresponding values of x makes real and positive values are;

$0 \leq y \geq 0.632455532$, (31)

Taking an arbitrary value of x to be 0.05, the corresponding value of y is o.341412507,

Substituting the values of x and y to (28) gives

$p^2 = 0.05r^2, a^2 = 0.341412507r^2$ (32)

Substituting (32) to (26) for the value of s = 5 gives,

$\lambda_2 = 0.232322858p^2$, $\lambda_4 = 0.044785714p^4$ (33)

Substituting (33) to the non-singularity conditions given in (2),(3) and (4) for k=3 factors proves that the condition in (5) is satisfied thus the 35 points forms a second order rotatable design in three dimensions.

3.2 Construction of Forty-Five Points Second Order Rotatable Design in Three Dimensions

The forty-five points is obtained by combining two sets i.e. the set denoted by s_1 and s_5 ,

Where;

s_1= 3 s for s=5 given in (12)

s_4= s (p, q, 0) given in (21)

The sums of squires for the combined sets (12) and (21) up to the power of four gives

$\sum_{u=1}^{45} x_{iu}^2 = Sr^2 + 8p^2 + 8q^2$

$\sum_{u=1}^{45} x_{iu}^4 = \frac{3}{4}Sr^4 + 8p^4 + 8q^4$ (34)

$\sum_{u=1}^{45} x_{iu}^2 x_{ju}^2 = \frac{1}{8}Sr^4 + 8p^2q^2$

And all other odd powers are zero

The set of forty-five points denoted by R_6 is given by;

R_6 = 3s + s (p, q, 0) (35)

Subjecting (34) to the moment conditions given in (2),(3), and (4) results to,

$Sr^2 + 8p^2 + 8q^2 = N\lambda_2$, $\frac{3}{4}Sr^4 + 8p^4 + 8q^4 = 3N\lambda_4$, $\frac{1}{8}Sr^4 + 8p^2q^2 = N\lambda_4$ (36)

The excess function for s_1 given in (14) is added to the excess function for s_5 given in (23) assuming the value of S=7 to give;

$\sum_{u=1}^{45} x_{iu}^4 = 3\sum_{u=1}^{45} x_{iu}^2 x_{ju}^2 = \frac{21}{8}r^4 + 8p^4 + 8q^4 - 24p^2q^2 = 0$ (37)

Equation (37) is solved to obtain the values of p and q in terms of r as follows;

Let $p^2 = xr^2$ and $q = yr^2$ (38)

Substituting (38) to (37) gives;

$$2.625r^4 + 8x^2 r^4 + 8 y^2 r^4 - 24x^2 y^2 r^4 = 0 \quad (39)$$

Making x the subject of the formula in (39) results to;

$$x = \frac{24y \pm \sqrt{320y^2 - 84}}{16} \quad (40)$$

The values of y which together with the corresponding values of x makes real and positive values are;

$$y \geq 0.0.5124, \quad (41)$$

Taking an arbitrary value of x to be 0.54, the corresponding value of y is 0.6192776888. Substituting the values of x and y to (38) gives

$$p^2 = 0.54r^2, q^2 = 0.6192776888r^2 \quad (42)$$

Substituting (42) to (36) for the value of s = 7 gives,

$$\lambda_2 = 0.3616493667r^2, \lambda_4 = 0.07889510257r^4 \quad (43)$$

Substituting (43) to the non-singularity conditions given in (2),(3),and (4) for k=3 factors proves that the condition in (5) is satisfied thus the 45 points forms a second order rotatable design in three dimensions.

4.0 Conclusions and Recommendations

The response surface methodology in this case is used to approximate the functional relationship between the performance characteristics and the design variables. After experimentation, the resulting response is used to construct response surface approximation model using least squares regression analysis. This study recommends the extension of this class of designs to more than three factors to cater for experiments where more than three factors are required.

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