On Unitary Quasi-Equivalence and w-Hyponormal Operators In Hilbert Spaces

Original Research Article

Abstract

In this paper, the properties of unitary quasi-equivalence on the class of w-hyponormal operators are presented using the Aluthge transform and polar decomposition property. We show that for any two unitary quasi-equivalent operators, F and G, if one is w-hyponormal, then the other operator is also w-hyponormal. This result also holds for p-hyponormal and log-hyponormal operators.

Keywords: Hilbert space, Unitary quasi-equivalence, w-hyponormal, p-hyponormal, log-hyponormal.

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1 Introduction

The Hilbert space, represented by H, is an inner product space that is complete [1]. The set of bounded linear operators in Hilbert spaces is denoted as $B(H)$. A mapping $F \in B(H)$ is an operator if it preserves the underlying structural properties of the space. An operator $F \in B(H)$ is hyponormal if $F^*F \geq FF^*$. An operator $F \in B(H)$ is said to have a polar decomposition if $F = U|F|$ where U is unitary. Two operators F and $G \in B(H)$ are said to be unitary quasi-equivalent if there exists a unitary operator U such that $F^*F = UG^*GU^*$ and $FF^* = UGG^*U^*$ where F^* denotes the adjoint of an operator F. This concept of unitary quasi-equivalence was introduced by Kutkut [3] under the idea of nearly equivalent operators by Othman [4]. Since then, various researchers have explored its properties on different operators in Hilbert space. Nzimbi and Wanyonyi [5] noted that unitary quasi-equivalence preserves the normality, hyponormality, and posinormality of operators. Lilian et al.[11] further established that unitary quasi-equivalence preserves the isometry, co-isometry, and partial isometry of an operator. Luketero and Khalagai [6] discovered that the operator is an equivalent relation. Luketero and Khalagai [6] also related unitary quasi-equivalence with other classes of equivalence relation, particularly unitary equivalence, and demonstrated that unitary quasi-equivalence implies unitary equivalence but the vice versa is not always true unless the operators are similar normal. Thereafter, Kikete et al. [7] introduced the concept of (n, m)-hyponormal and (n, m)-quasi-equivalence and proved it to be an equivalence relation. Kikete et al. [7] also proved that (n, m)-hyponormal preserves the (n, m)-quasi-equivalence of an operator. That is, if two operators are (n, m)-unitary quasi-equivalence and one of them is (n, m)-hyponormal then the other must be (n, m)-hyponormal. However, similar results on the class of w-hyponormal, p-hyponormal, and log-hyponormal operators have not been shown. Thus, this study focused on determining the properties of unitary quasi-equivalence on log-hyponormal, p-hyponormal, and w-hyponormal operators.

2 Definitions

Definition 2.1: hyponormal Operator [2]. An operator $F \in B(H)$ is hyponormal if $F^*F \geq FF^*$.

Definition 2.2: w-hyponormal operator [10]. An operator F is said to be *w-hyponormal* if it satisfies the condition $|\widetilde{F}| \geq |F| \geq |\widetilde{F}^*|$.

Definition 2.3: log-hyponormal [13], [8]. An invertible operator F is called *log-hyponormal* if $\log(F^*F) \geq \log(FF^*)$.

Definition 2.4: p-hyponormal [9], [12]. An operator $F \in B(H)$ is said to be p-hyponormal if $(F^*F)^p - (FF^*)^p \geq 0$ for $0 < p \leq 1$.

Definition 2.5: Aluthge Transform [12]. The first Aluthge transform of F is defined as $\widetilde{F} = |F|^{1/2}U|F|^{1/2}$ where $F = U|F|$, while the second transform of F is defined as $\widetilde{\widetilde{F}} = |\widetilde{F}|^{1/2}\widetilde{U}|\widetilde{F}|^{1/2}$.

Definition 2.6: Unitary Quasi-equivalence Operator [11]. F and G $\in B(H)$ are said to be unitary quasi-equivalent if there exists a unitary operator U such that:

$$F^*F = UG^*GU^* \quad \text{and} \quad FF^* = UGG^*U^*$$

From the definition above, the clas of w-hyponormal p-hyponormal and log-hyponormal operators relates as follows;

Hyponormal $\subset$ Log-hyponormal $\subset$ w-Hyponormal $\subset$ Paranormal $\subset$ K-paranormal

Hyponormal $\subset$ p-Hyponormal $(0 < p < 1)$ $\subset$ w-Hyponormal $\subset$ Paranormal $\subset$ K-paranormal

[14]

3 Main Result

Lemma 4.1 [5] Let $F, G \in B(H)$ be unitarily quasi-equivalent. Then F is hyponormal if and only if G is hyponormal.

Lemma 4.2 [6] Let F be a hyponormal operator and G be another operator such that:

1. $F = UGU^*$ where U is an isometry, or

2. $F = U^*GU$ where U is a co-isometry.

Then G is also hyponormal.

Remark: From preceding lemmas it is evident that hyponormal operators are invariant under unitary quasi-equivalence, isometric and co-isometric transformations. To extend these result, the study aimed to establish the following results.

Theorem 4.3: If $F, G \in B(H)$ are unitary quasi-equivalent operators and F is a log-Hyponormal operator, then G is also log-Hyponormal.

Proof. Since F is unitary quasi-equivalent to G, then

$$F^*F = UG^*GU^* \tag{i}$$

$$FF^* = UGG^*U^* \tag{ii}$$

But F is log-Hyponormal thus by definition

$$\log(F^*F) \geq \log(FF^*) \tag{iii}$$

Substituting equations (i) and (ii) into (iii)

$$\log(UG^*GU^*) \geq \log(UGG^*U^*) \tag{iv}$$

Pre-multiplying and post-multiplying UG^*GU^* and UGG^*U^* in equation (iv) with U^* and U on the left and on the right, results to;

$$\log(U^*UG^*GU^*U) \geq \log(U^*UGG^*U^*U) \tag{v}$$

But

$$U^*U = UU^* = I$$

Thus (v) becomes

$$\log(IG^*GI) \geq \log(IGG^*I)$$

$$\log(G^*G) \geq \log(GG^*) \tag{vi}$$

Thus by definition G is a log-hyponormal operator. □

Theorem 4.4: Suppose $F, G \in B(H)$ are unitary quasi-equivalent operators. If F is a p-hyponormal operator, then G is also a p-hyponormal operator.

Proof. Since $F \overset{u.q.e}{\approx} G$ then we have that;

$$F^*F = UG^*GU^* \tag{i}$$

$$FF^* = UGG^*U^* \tag{ii}$$

However, F is p-hyponormal thus by definition:

$$(F^*F)^p \geq (FF^*)^p \tag{iii}$$

Replacing equations (i) and (ii) into equation (iii):

$$(UG^*GU^*)^p \geq (UGG^*U^*)^p \tag{iv}$$

By the polar decomposition property:

$$G = U|G| \text{ where } U \text{ is a unitary operator.} \tag{v}$$

$$G^* = |G^*|U^* \tag{vi}$$

Substituting equations (v) and (vi) into equation (iv):

$$(U|G^*|U^*U|G|U^*)^p \geq (UU|G||G^*|U^*U^*)^p \tag{vii}$$

From definition, equation (vii) becomes:

$$U((|G^*|U^*U|G|)^p)U^* \geq U((U|G||G^*|U^*)^p)U^* \tag{viii}$$

Post-multiplying and pre-multiplying both sides of equation (viii) with U^* on the left and U on the right;

$$U^*U((|G^*|U^*U|G|)^p)U^*U \geq U^*U((U|G||G^*|U^*)^p)U^*U \tag{ix}$$

But $UU^* = U^*U = I$. Replacing the identity in equation (ix):

$$I((|G^*|U^*U|G|)^p)I \geq I((U|G||G^*|U^*)^p)I$$

$$((|G^*|U^*U|G|)^p) \geq ((U|G||G^*|U^*)^p) \tag{x}$$

But from equations (v) and (vi), (x) simplifies to:

$$(G^*G)^p \geq (GG^*)^p \tag{xi}$$

And hence from (xi) G is a p-hyponormal operator. □

Theorem 4.5: If F and G are projection and unitary equivalent operators, and F is a w-hyponormal operator then G is also w hyponormal.

Proof. From hypothesis $F \overset{u.q.e}{\approx} G$ implying that:

$$F^*F = UG^*GU^*$$

And

$$FF^* = UGG^*U^*$$

Also F is w-hyponormal which means that

$$|\widetilde{F}| \geq |F| \geq |\widetilde{F}^*| \tag{i}$$

Where:

$$F = U|F| \tag{ii}$$

$$|\widetilde{F}| = |F|^{1/2}U|F|^{1/2} \tag{iii}$$

$$|\widetilde{F}^*| = |F^*|^{1/2}U^*|F^*|^{1/2} \tag{iv}$$

Substituting equations (ii), (iii), and (iv) into equation (i) we get:

$$||F|^{1/2}U|F|^{1/2}| \geq |U|F|| \geq ||F^*|^{1/2}U^*|F^*|^{1/2}| \tag{v}$$

Since F is a projection,

$$F^* = UG^*U^*$$

And

$$F = UGU^*$$

Substituting F and F^* into equation (v) we get:

$$||UGU^*|^{1/2}U|UGU^*|^{1/2}| \geq |U|UGU^*|| \geq ||UG^*U^*|^{1/2}U^*|UG^*U^*|^{1/2}| \tag{vi}$$

By the polar decomposition property equation (iv) can be rewritten as:

$$U(|G|^{1/2})U^*UU(|G|^{1/2})U^* \geq UU|G|U^* \geq U(|G^*|^{1/2})U^*U^*U(|G^*|^{1/2})U^* \qquad \text{(vii)}$$

Since U is unitary, then

$$U^*U = I \quad \text{where } I \text{ is the identity.}$$

Thus:

$$U|(|G|^{1/2})U(|G|^{1/2})|U^* \geq UU||G||U^* \geq U|(|G^*|^{1/2})U^*I(|G^*|^{1/2})|U^* \qquad \text{(viii)}$$

Pre-multiplying both sides of equation (viii) by U^* on the left and post-multiplying U on the right side we get:

$$U^*U(|G|^{1/2}U|G|^{1/2})U^*U \geq U^*UU||G||U^*U \geq U^*U(|G^*|^{1/2}U^*I|G^*|^{1/2})U^*U$$

But

$$U^*U = I$$

$$I(|G|^{1/2}U|G|^{1/2})I \geq IU||G||I \geq I(|G^*|^{1/2}U^*I|G^*|^{1/2})I$$

Thus

$$(|G|^{1/2}U|G|^{1/2}) \geq U||G|| \geq (|G^*|^{1/2}U^*|G^*|^{1/2}) \qquad \text{(ix)}$$

But

$$(|G|^{1/2}U|G|^{1/2}) = |\widetilde{G}|$$

$$U|G| = T$$

And

$$(|G^*|^{1/2}U^*|G^*|^{1/2}) = |\widetilde{G}^*|$$

Thus equation (ix) simplifies to

$$|\widetilde{G}| \geq |G| \geq |\widetilde{G}^*|$$

And hence by definition 2.2 G is also w-hyponormal.

□

4 Conclusion

The preceding results has shown that unitary quasi-equivalence preserves the properties of w-hyponormal, log-hyponormal and p-hyponormal. That is for any two operators which are unitary quasi-equivalence and one of them being w-hyponormal, log-hyponormal or p-hyponormal then the other operators must also w-hyponormal, log-hyponormal or p-hyponormal. These findings will be valuable for those using these operators in quantum mechanics to develop theoretical formulations. Also, the results will be significant in applied algebra and the study of differential operators for the calculation and differentiation of wave functions.

Competing Interests

Authors have declared no competing interest.

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