UNDER PEER REVIEW

On Unitary Quasi-Equivalence and w-Hyponormal Operators In Hilbert Spaces

Abstract

The concept of unitary quasi-equivalence was introduced in 1996 under the idea of nearly equivalent operators. Since then, various researchers have explored its properties on different operators in Hilbert space. For instance, it has been shown to preserve the binormality, normality, isometry, co-isometry, partial isometry, and hyponormality properties of an operator. It has also been shown to be symmetric, transitive, and reflexive, thus forming an equivalence relation. However, the properties of unitary quasi-equivalence on the class of w-hyponormal operators have not been shown. This study, therefore, focused on determining the properties of unitary quasi-equivalence on log-hyponormal, p-hyponormal, and w-hyponormal operators.

Keywords: Hilbert space, Unitary quasi-equivalence, w-hyponormal, p-hyponormal, log-hyponormal.

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1 Introduction

A Hilbert space, represented by H, is an inner product space that is complete. [1]. A set of bounded linear operators in Hilbert spaces is denoted as $B(H)$. A mapping $F \in B(H)$ is an operator if it preserves the underlying structural properties of a map. An operator $F \in B(H)$ is hyponormal if $F^*F \geq FF^*$. This concept was introduced by Stampfil [2]. Berbarian [3], later showed that every continuous hyponormal operator is also a normal operator. Aluthge [4], looked at the properties of p-hyponormal and the relationship that exists with self-adjoint, unitary, and isometry. Otieno [5], introduced the class of w-hyponormal operators and proved the Putnam-Fuglede commutative theorem on this operator. Halmos [6], introduced the concept of unitary quasi-equivalence. Two operators F and $G \in B(H)$ are said to be unitary quasi-equivalent if there exists a unitary operator U such that

$$F^*F = UG^*GU^* \quad \text{and} \quad FF^* = UGG^*U^*$$

where F^* denotes the adjoint of an operator F. Nzimbi and Wanyonyi [7], noted that unitary quasi-equivalence preserves the normality, hyponormality, and posinormality of operators. Luketero and Khalagai [8], noted that the operator is an equivalence relation and that unitary quasi-equivalence implies unitary equivalence, although the vice versa is not always true. Kikete et al. [9], introduced the concept of (n, m)-hyponormal and (n, m)-quasi-equivalence and proved it to be an equivalence relation. Kikete et al. [9], also proved that (n, m)-hyponormal preserves the (n, m)-quasi-equivalence of an operator. However, a similar result has not been established in the class of w-hyponormal, p-hyponormal, and log-hyponormal. This study therefore aimed to establish this gap.

2 Definitions

Definition 2.1: Hilbert Space [1]. Is an inner product space that is complete.
Definition 2.2: hyponormal Operator [1]. An operator $F \in B(H)$ is hyponormal if $F^*F \geq FF^*$.
Definition 2.3: w-hyponormal operator [12]. An operator F is said to be *w-hyponormal* if it satisfies the condition $|\widetilde{F}| \geq |F| \geq |\widetilde{F}^*|$.
Definition 2.4: log-hyponormal [10]. An invertible operator F is called *log-hyponormal* if $\log(F^*F) \geq \log(F^*FF^*)$.
Definition 2.5: p-hyponormal [11]. An operator $F \in B(H)$ is said to be p-hyponormal if $(F^*F)^p - (FF^*)^p \geq 0$ for $0 < p \leq 1$.
Definition 2.6: Aluthge Transform [5]. The first Aluthge transform of F is defined as $\widetilde{F} = |F|^{1/2}U|F|^{1/2}$ where $F = U|F|$, while the second transform of F is defined as $\widetilde{\widetilde{F}} = |\widetilde{F}|^{1/2}\widetilde{U}|\widetilde{F}|^{1/2}$.
Definition 2.7: Unitary Quasi-equivalence Operator [13]. F and G $\in B(H)$ are said to be unitary quasi-equivalent if there exists a unitary operator U such that:

$$F^*F = UG^*GU^* \quad \text{and} \quad FF^* = UGG^*U^*$$

From the definition above, the clas of w-hyponormal p-hyponormal and log-hyponormal operators relates as follows;

Hyponormal $\subset$ Log-hyponormal $\subset$ w-Hyponormal $\subset$ Paranormal $\subset$ K-paranormal

Hyponormal $\subset$ p-Hyponormal $(0 < p < 1)$ $\subset$ w-Hyponormal $\subset$ Paranormal $\subset$ K-paranormal

3 Methodology

To achieve this objective, definitions of w-hyponormal, log-hyponormal and p-hyponormal operators were important. Additionally, the polar decomposition properties of an operator and the properties of unitary quasi-equivalence were essential. In particular, this study aimed to extend the following result.
Lemma 3.1: Let $F, G \in B(H)$ be unitarily quasi-equivalent. Then F is hyponormal if and only if G is hyponormal.

4 Main Result

This study was able to establish the following results.
Theorem 4.1: If $F, G \in B(H)$ are unitary quasi-equivalent operators and F is a Log-Hyponormal operator, then G is also Log-Hyponormal.

Proof. Since F is unitary quasi-equivalent to G, then

$$F^*F = UG^*GU^* \quad \text{(i)}$$

$$FF^* = UGG^*U^* \quad \text{(ii)}$$

But F is Log-Hyponormal thus by definition

$$\log(F^*F) \geq \log(FF^*) \quad \text{(iii)}$$

Substituting equations (i) and (ii) into (iii)

$$\log(UG^*GU^*) \geq \log(UGG^*U^*) \quad \text{(iv)}$$

Pre-multiplying and post-multiplying both sides of equation (iv) with U^* and U on the left and the right respectively, implies

$$\log(U^*UG^*GU^*U) \geq \log(U^*UGG^*U^*U) \quad \text{(v)}$$

But

$$U^*U = UU^* = I$$

Thus (v) becomes

$$\log(IG^*GI) \geq \log(IGG^*I)$$

$$\log(G^*G) \geq \log(GG^*) \quad \text{(vi)}$$

Thus by definition G is a log-hyponormal operator. □

Theorem 4.2: Suppose $F, G \in B(H)$ are unitary quasi-equivalent operators. If F is a p-Hyponormal operator, then G is also a p-Hyponormal operator.

Proof. Since $F \overset{u.q.e}{\approx} G$ then we have that;

$$F^*F = UG^*GU^* \quad \text{(i)}$$

$$FF^* = UGG^*U^* \quad \text{(ii)}$$

However, F is p-Hyponormal thus by definition:

$$(F^*F)^p \geq (FF^*)^p \text{ for a positive number } p. \quad \text{(iii)}$$

Replacing equations (i) and (ii) into equation (iii):

$$(UG^*GU^*)^p \geq (UGG^*U^*)^p \quad \text{(iv)}$$

By the polar decomposition property:

$$G = U|G| \text{ where } U \text{ is a unitary operator.} \quad \text{(v)}$$

$$G^* = |G^*|U^* \quad \text{(vi)}$$

Substituting equations (v) and (vi) into equation (iv):

$$(U|G^*|U^*U|G|U^*)^p \geq (UU|G||G^*|U^*U^*)^p \quad \text{(vii)}$$

From definition, equation (vii) becomes:

$$U((|G^*|U^*U|G|)^p)U^* \geq U((U|G||G^*|U^*)^p)U^* \tag{viii}$$

Post-multiplying and pre-multiplying both sides of equation (viii) with U^* on the left and U on the right;

$$U^*U((|G^*|U^*U|G|)^p)U^*U \geq U^*U((U|G||G^*|U^*)^p)U^*U \tag{ix}$$

But $UU^* = U^*U = I$. Replacing the identity in equation (ix):

$$I((|G^*|U^*U|G|)^p)I \geq I((U|G||G^*|U^*)^p)I$$

$$((|G^*|U^*U|G|)^p) \geq ((U|G||G^*|U^*)^p) \tag{x}$$

But from equations (v) and (vi), (x) simplifies to:

$$(G^*G)^p \geq (GG^*)^p \tag{xi}$$

And hence from (xi) G is a p-Hyponormal operator. □

Theorem 4.3: If F and G are projection and unitary equivalent operators, and F is a w-hyponormal operator then G is also hyponormal.

Proof. From hypothesis $F \overset{u.q.e}{\approx} G$ implying that:

$$F^*F = UG^*GU^*$$

And

$$FF^* = UGG^*U^*$$

Also F is w-hyponormal which means that

$$|\widetilde{F}| \geq |F| \geq |\widetilde{F}^*| \tag{i}$$

Where:

$$F = U|F| \tag{ii}$$

$$|\widetilde{F}| = |F|^{1/2}U|F|^{1/2} \tag{iii}$$

$$|\widetilde{F}^*| = |F^*|^{1/2}U^*|F^*|^{1/2} \tag{iv}$$

Substituting equations (ii), (iii), and (iv) into equation (i) we get:

$$||F|^{1/2}U|F|^{1/2}| \geq |U|F|| \geq ||F^*|^{1/2}U^*|F^*|^{1/2}| \tag{v}$$

Since F is a projection,

$$F^* = UG^*U^*$$

And

$$F = UGU^*$$

Substituting F and F^* into equation (v) we get:

$$||UGU^*|^{1/2}U|UGU^*|^{1/2}| \geq |U|UGU^*|| \geq ||UG^*U^*|^{1/2}U^*|UG^*U^*|^{1/2}| \tag{vi}$$

By the polar decomposition property equation (iv) can be rewritten as:

$$U(|G|^{1/2})U^*UU(|G|^{1/2})U^* \geq UU|G|U^* \geq U(|G^*|^{1/2})U^*U^*U(|G^*|^{1/2})U^* \tag{vii}$$

Since U is unitary, then

$$U^*U = I \quad \text{where } I \text{ is the identity.}$$

Thus:

$$U|(|G|^{1/2})U(|G|^{1/2})|U^* \geq UU||G||U^* \geq U|(|G^*|^{1/2})U^*I(|G^*|^{1/2})|U^* \quad \text{(viii)}$$

Pre-multiplying both sides of equation (viii) by U^* on the left and post-multiplying U on the right side we get:

$$U^*U(|G|^{1/2}U|G|^{1/2})U^*U \geq U^*UU||G||U^*U \geq U^*U(|G^*|^{1/2}U^*I|G^*|^{1/2})U^*U$$

But

$$U^*U = I$$

$$I(|G|^{1/2}U|G|^{1/2})I \geq IU||G||I \geq I(|G^*|^{1/2}U^*I|G^*|^{1/2})I$$

Thus

$$(|G|^{1/2}U|G|^{1/2}) \geq U||G|| \geq (|G^*|^{1/2}U^*|G^*|^{1/2}) \quad \text{(ix)}$$

But

$$(|G|^{1/2}U|G|^{1/2}) = |\widetilde{G}|$$

$$U|G| = T$$

And

$$(|G^*|^{1/2}U^*|G^*|^{1/2}) = |\widetilde{G}^*|$$

Thus equation (ix) simplifies to

$$|\widetilde{G}| \geq |G| \geq |\widetilde{G}^*|$$

And hence by definition 2.3 G is also w-hyponormal.

□

5 Conclusion

Based on theorem 4.1, 4.2 and 4.3 results, it is now evident that unitary quasi-equivalence preserves the properties of w-hyponormal, log-hyponormal and p-hyponormal. This means that if any two operators are unitary quasi equivalent and one of operator is w-hyponormal, log-hyponormal or p-hyponormal then the other operators is also w-hyponrmal, log-hyponormal or p-hyponormal.

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