DISCRETE ERLANG MIXED DISTRIBUTIONS AND THEIR PROPERTIES

Abstract: The proposed research is on discrete Erlang mixtures. Properties of the mixed distributions analyzed include raw and central moments, which have been derived in terms of moments of the mixing distributions. Cumulants obtained from the cumulant generating function were also used in deriving the moments. The posterior distribution and posterior moments were also among properties presented. Bayesian estimation was used in parameter estimation of the mixed Erlang distributions. Some methods and special functions used in the study are the exponential series, logarithmic series, geometric series, Modified Bessel function of the first kind, and the Touchard polynomials. The discrete mixing distributions used are the geometric, Poisson and logarithmic.

Key Words: Discrete Erlang mixtures, moments, cumulant, cumulant generating function, posterior distribution, Poisson, geometric, logarithmic

1. Introduction

The Elang distribution is used in modeling the waiting time for an event in a Poisson process. It reduces to the exponential distribution when the shape parameter is equal to one. Its relation to both the Poisson and exponential distributions has contributed to its vast applications.

Mixed distributions are obtained by combining two or more distributions. They have a wider applicability compared to the basic distributions. They are used to model data that the basic distributions may fail to, and therefore are integral in situations that the basic distributions fail to address. They are devised by modifying the basic distributions using mixing weights to form finite mixtures, and by varying their shape parameters to form discrete mixtures and their rate/scale parameters to create continuous mixtures. Mixed Erlang distributions have been studied extensively over time. Zakerzadeh and Dolati [17], Shanker and Mishra [13], Merovci [10], Rashid et al. [12], Abouammoh et al. [1], Ghitany et al. [4], and Nadarajah et al. [11] are among people who derived finite Erlang mixtures, while McNolty [9], Jordanova and Stehlík [5], Jordanova et al. [6], and Kang [7] worked on continuous Erlang mixtures.

The focus of this work is on discrete Erlang mixed distributions, which are obtained by mixing the Erlang distribution with discrete mixing distributions. Tijms [14] showed that the Erlang mixture can be used in the approximation of any non-negative continuous distribution. Landriault et al. [8] evinced that the order statistics of independent mixed Erlang random variables belong to the same distribution class of Erlang mixtures. Cossette et al. [3] used mixtures of the Erlang distribution in moment based approximation. They conducted numerical experiments on the mixed Erlang approximation method, where the model was seen to provide an overall good fit. Willmot and Woo [15] demonstrated that a large number of distributions are of the discrete mixed Erlang type. They showed that the Laplace transform of the Erlang mixture can be expressed in terms of the probability generating function of the mixing distribution. They also discussed special cases of the Erlang mixture, which include the exponential distribution, the Erlang distribution and the non-central chi-square distribution.

Willmot and Woo [16] derived distributional properties of a class of multivariate mixed Erlang distributions with different scale parameters. Cossette et al. [2] presented the equilibrium function, among other properties, of the mixed Erlang distribution.
The outline of the paper is as follows: The mathematical formulation of the Erlang mixed distribution and its properties have been defined in section 2, and particular cases of the mixed distributions have been obtained in sections 3, 4 and 5 using the geometric, Poisson and logarithmic mixing distributions respectively. Section 6 contains the conclusion in brief.

2. Definitions, notations and Terminologies

- The probability density function of the conditional (Erlang) distribution is;

$$f(t|n) = \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1}, \quad t > 0; \lambda > 0, n = 1, 2, 3, .. \tag{2.1}$$

where n is the shape parameter and λ is the rate parameter.

- The mixed Erlang distribution is thus;

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} P_n \\ &= \lambda e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} P_n \\ &= \lambda e^{-\lambda t} E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right) \end{aligned} \tag{2.2}$$

where P_n is a discrete mixing distribution.

- The r^{th} moment of the Erlang mixture is given by;

$$\begin{aligned} E(T^r) &= EE(T^r|n) \\ &= E\int_0^{\infty} t^r f(t|n) dt \\ &= E\int_0^{\infty} t^r \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} dt \\ &= E\left(\frac{\lambda^n}{\Gamma n}\int_0^{\infty} t^{n+r-1} e^{-\lambda t} dt\right) \\ &= E\left(\frac{\lambda^n}{\Gamma n}\frac{\Gamma(n+r)}{\lambda^{n+r}}\right) \\ E(T^r) &= \frac{1}{\lambda^r} E\left(\frac{\Gamma(n+r)}{\Gamma n}\right) \end{aligned} \tag{2.3}$$

Raw moments and central moments of the mixed Erlang distribution in terms of moments of the mixing distribution are;

- Raw moments

$$E(T) = \frac{1}{\lambda}E(n) \tag{2.4}$$

$$E(T^2) = \frac{1}{\lambda^2}E\left(\frac{\Gamma(n+2)}{\Gamma n}\right) = \frac{1}{\lambda^2}E[n(n+1)] = \frac{1}{\lambda^2}[E(n^2) + E(n)] \tag{2.5}$$

$$E(T^3) = \frac{1}{\lambda^3}E\left(\frac{\Gamma(n+3)}{\Gamma n}\right) = \frac{1}{\lambda^3}E[n(n+1)(n+2)] = \frac{1}{\lambda^3}[E(n^3) + 3E(n^2) + 2E(n)] \tag{2.6}$$

$$\begin{aligned} E(T^4) &= \frac{1}{\lambda^4}E\left(\frac{\Gamma(n+4)}{\Gamma n}\right) = \frac{1}{\lambda^4}E[n(n+1)(n+2)(n+3)] \\ &= \frac{1}{\lambda^4}[E(n^4) + 6E(n^3) + 11E(n^2) + 6E(n)] \end{aligned} \tag{2.7}$$

- Central moments

i. Variance

$$\begin{aligned} \mu_2 &= E\left[T - E(T)\right]^2 = E(T^2) - [E(T)]^2 \\ &= \frac{1}{\lambda^2}[E(n^2) + E(n)] - \frac{1}{\lambda^2}[E(n)]^2 \\ &= \frac{1}{\lambda^2}\left\{E(n^2) + E(n) - [E(n)]^2\right\} \\ &= \frac{1}{\lambda^2}\{Var(n) + E(n)\} \end{aligned} \tag{2.8}$$

ii. Third moment

$$\begin{aligned} \mu_3 &= E\left[T - E(T)\right]^3 = E(T^3) - 3E(T^2)E(T) + 2[E(T)]^3 \\ &= \frac{1}{\lambda^3}[E(n^3) + 3E(n^2) + 2E(n)] - \frac{3}{\lambda^3}[E(n^2) + E(n)]E(n) + \frac{2}{\lambda^3}[E(n)]^3 \\ &= \frac{1}{\lambda^3}\left\{E(n^3) + 3E(n^2) + 2E(n) - 3E(n^2)E(n) - 3[E(n)]^2 + 2[E(n)]^3\right\} \\ &= \frac{1}{\lambda^3}\left\{E[n - E(n)]^3 + 3Var(n) + 2E(n)\right\} \end{aligned} \tag{2.9}$$

iii. Fourth moment

$$\begin{aligned} \mu_4 &= E\left[T - E(T)\right]^4 = E(T^4) - 4E(T^3)E(T) + 6E(T^2)[E(T)]^2 - 3[E(T)]^4 \\ &= \frac{1}{\lambda^4}[E(n^4) + 6E(n^3) + 11E(n^2) + 6E(n)] - \frac{4}{\lambda^4}[E(n^3) + 3E(n^2) + \\ &\quad 2E(n)]E(n) + \frac{6}{\lambda^4}[E(n^2) + E(n)][E(n)]^2 - \frac{3}{\lambda^4}[E(n)]^4 \\ &= \frac{1}{\lambda^4}\{E(n^4) + 6E(n^3) + 11E(n^2) + 6E(n) - 4E(n^3)E(n) - 12E(n^2)E(n) - \\ &\quad 8[E(n)]^2 + 6E(n^2)[E(n)]^2 + 6[E(n)]^3 - 3[E(n)]^4\} \\ &= \frac{1}{\lambda^4}\{E[n - E(n)]^4 + 6E[n - E(n)]^3 + 6Var(n)E(n) + 11Var(n) + 3[E(n)]^2 + 6E(n)\} \\ &= \frac{1}{\lambda^4}\{E[n - E(n)]^4 + 6E[n - E(n)]^3 + Var(n)[6E(n) + 11] + 3[E(n)]^2 + 6E(n)\} \end{aligned} \tag{2.10}$$

- The moment generating function of the Erlang mixture is given by

$$\begin{aligned} M_t(s) = E(e^{ts}) &= EE(e^{ts}|n) \\ &= E\int_0^{\infty} e^{ts} f(t|n)dt \\ &= E\int_0^{\infty} e^{ts}\frac{\lambda^n}{\Gamma n}e^{-\lambda t}t^{n-1}dt \\ &= E\left(\frac{\lambda^n}{\Gamma n}\int_0^{\infty} t^{n-1}e^{-(\lambda-s)t}dt\right) \\ &= E\left(\frac{\lambda^n}{\Gamma n}\frac{\Gamma n}{(\lambda-s)^n}\right) = E\left(\frac{\lambda}{\lambda-s}\right)^n \end{aligned} \tag{2.11}$$

and hence the cumulant generating function is

$$K_t(s) = \text{log}M_t(s) = \text{log}E\left(\frac{\lambda}{\lambda-s}\right)^n \tag{2.12}$$

The r^{th} cumulant of the mixed distribution, $K_r(t)$, is the r^{th} derivative of the cumulant generating function at $s=0$, and the first, second and third cumulants are the expected value, second and third central moments respectively.

$$K_t^{'}(s) = \frac{E\left[\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right]}{E\left(\frac{\lambda}{\lambda-s}\right)^n} \quad \text{and} \quad K_1(t) = K_t^{'}(0) = \frac{1}{\lambda}E(n) \tag{2.13}$$

$$K_t^{''}(s) = \frac{E\left(\frac{\lambda}{\lambda-s}\right)^n E\left[\frac{n(n+1)\lambda^n}{(\lambda-s)^{n+2}}\right] - \left\{E\left[\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right]\right\}^2}{\left[E\left(\frac{\lambda}{\lambda-s}\right)^n\right]^2} \quad \text{and}$$

$$K_2(t) = K_t^{''}(0) = \frac{1}{\lambda^2}\left\{E(n^2) + E(n) - [E(n)]^2\right\} \tag{2.14}$$

$$K_t^{'''}(s) = \frac{\left[E\left(\frac{\lambda}{\lambda-s}\right)^n\right]\left\{E\left(\frac{\lambda}{\lambda-s}\right)^n E\left[\frac{n(n+1)(n+2)\lambda^n}{(\lambda-s)^{n+3}}\right] - E\left[\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right]E\left[\frac{n(n+1)\lambda^n}{(\lambda-s)^{n+2}}\right]\right\} -}{\left[E\left(\frac{\lambda}{\lambda-s}\right)^n\right]^2}$$

$$\frac{2E\left[\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right]\left\{E\left(\frac{\lambda}{\lambda-s}\right)^n E\left[\frac{n(n+1)\lambda^n}{(\lambda-s)^{n+2}}\right] - \left[E\left(\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right)\right]^2\right\}}{} \quad \text{and}$$

$$K_3(t) = K_t^{'''}(0) = \frac{1}{\lambda^3}\left\{E(n^3) + 3E(n^2) + 2E(n) - 3E(n^2)E(n) - 3[E(n)]^2 + 2[E(n)]^3\right\} \tag{2.15}$$

- The Posterior distribution is given by

$$\begin{aligned} g(n|T) &= \frac{f(t|n)P_n}{f(t)} \\ &= \frac{\frac{\lambda^n}{\Gamma n}e^{-\lambda t}t^{n-1}P_n}{\lambda e^{-\lambda t}E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} = \frac{\frac{(\lambda t)^{n-1}}{(n-1)!}P_n}{E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} \end{aligned} \tag{2.16}$$

where $f(t|n)$ is the likelihood function, which is the Erlang distribution, and P_n is the prior distribution.

The posterior r^{th} moment is given by

$$E(n^r|T) = \sum_{n=1}^{\infty} n^r g(n|t)$$
$$= \frac{\sum_{n=1}^{\infty} n^r \frac{(\lambda t)^{n-1}}{(n-1)!} P_n}{E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} = \frac{E\left(\frac{n^r(\lambda t)^{n-1}}{(n-1)!}\right)}{E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} \tag{2.17}$$

and the posterior mean is

$$E(n|T) = \frac{E\left(\frac{n(\lambda t)^{n-1}}{(n-1)!}\right)}{E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} \tag{2.18}$$

- The posterior mean $E(n|T)$ is the Bayes estimator of the parameter n, assuming squared error loss function.

3. Erlang-Geometric distribution

The geometric mixing distribution is;

$$P_n = p(1-p)^{n-1}, \quad n = 1, 2, 3, ...; 0 < p < 1 \tag{3.1}$$

and, $$E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right) = \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} p(1-p)^{n-1}$$
$$= p\sum_{n=1}^{\infty} \frac{[\lambda t(1-p)]^{n-1}}{(n-1)!} = pe^{\lambda t(1-p)} \tag{3.2}$$

and, $$E\left(n^r \frac{(\lambda t)^{n-1}}{(n-1)!}\right) = p\sum_{n=1}^{\infty} n^r \frac{[\lambda t(1-p)]^{n-1}}{(n-1)!}$$
$$= \frac{p}{\lambda t(1-p)} \sum_{n=1}^{\infty} n^{r+1} \frac{[\lambda t(1-p)]^n}{n!} = \frac{p}{\lambda t(1-p)} e^{\lambda t(1-p)} T_{r+1}[\lambda t(1-p)] \tag{3.3}$$

where $T_r(x) = e^{-x}\sum_{k=0}^{\infty} \frac{k^r x^k}{k!} = \sum_{k=0}^{r} S(r,k)x^k$ is the Touchard polynomials and $S(r,k) = \sum_{j=0}^{k} \frac{(-1)^{k-j} j^r}{(k-j)!j!}$ is the Stirling number of the second kind.

a) The Erlang-geometric distribution is thus;

$$f(t) = \lambda p e^{-\lambda p t}, \quad t = 0, 1, 2, ...; 0 < p < 1, \lambda > 0 \tag{3.4}$$

b) The moment generating function of the mixed distribution is;

$$
\begin{aligned}
M_t(s) &= \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda - s}\right)^n p(1-p)^{n-1} \\
&= \frac{p\lambda}{\lambda - s} \sum_{n=1}^{\infty} \left[\frac{\lambda(1-p)}{\lambda - s}\right]^{n-1} \\
&= \frac{p\lambda}{\lambda - s} \frac{1}{1 - \frac{\lambda(1-p)}{\lambda - s}} = \frac{p\lambda}{p\lambda - s} \qquad (3.5)
\end{aligned}
$$

and the cumulant generating function is thus;

$$
M_t(s) = \log\left(\frac{p\lambda}{p\lambda - s}\right) = \log(p\lambda) - \log(p\lambda - s) \qquad (3.6)
$$

c) The raw moments of the geometric distribution are;

$$
\begin{aligned}
E(n) &= \frac{1}{p} && (3.7) \\
E(n^2) &= \frac{2(1-p)}{p^2} + \frac{1}{p} && (3.8) \\
E(n^3) &= \frac{6(1-p)^2}{p^3} + \frac{6(1-p)}{p^2} + \frac{1}{p} && (3.9) \\
E(n^4) &= \frac{24(1-p)^3}{p^4} + \frac{36(1-p)^2}{p^3} + \frac{14(1-p)}{p^2} + \frac{1}{p} && (3.10)
\end{aligned}
$$

and the central moments are therefore;

$$
\begin{aligned}
Var(n) &= \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2} && (3.11) \\
E[n - E(n)]^3 &= \frac{6(1-p)^2}{p^3} + \frac{6(1-p)}{p^2} + \frac{1}{p} - \frac{3(2-p)}{p^3} + \frac{2}{p^3} = \frac{2 - 3p + p^2}{p^3} && (3.12) \\
E[n - E(n)]^4 &= \frac{24(1-p)^3}{p^4} + \frac{36(1-p)^2}{p^3} + \frac{14(1-p)}{p^2} + \frac{1}{p} - \frac{4}{p^4}(6 - 12p + 6p^2 + 6p - 6p^2 + p^2) + \\
&\quad \frac{6}{p^4}(2-p) - \frac{3}{p^4} = \frac{9 - 18p + 10p^2 - p^3}{p^4} && (3.13)
\end{aligned}
$$

d) Hence, the moments and cumulants of the Erlang-geometric distribution are given by;

$$
\begin{aligned}
E(T) = K_1(t) &= \frac{1}{\lambda p} && (3.14) \\
Var(T) = K_2(t) &= \frac{1}{\lambda^2}\left\{\frac{1-p}{p^2} + \frac{1}{p}\right\} = \frac{1}{(p\lambda)^2} && (3.15) \\
\mu_3 = K_3(t) &= \frac{1}{\lambda^3}\left\{\frac{2 - 3p + p^2}{p^3} + \frac{3(1-p)}{p^2} + \frac{2}{p}\right\} = \frac{2}{(p\lambda)^3} && (3.16) \\
\mu_4 &= \frac{1}{\lambda^4}\left\{\frac{9 - 18p + 10p^2 - p^3}{p^4} + \frac{6(2 - 3p + p^2)}{p^3} + \frac{1-p}{p^2}\left[\frac{6}{p} + 11\right] + \frac{3}{p^2} + \frac{6}{p}\right\} = \frac{9}{(p\lambda)^4} && (3.17)
\end{aligned}
$$

e) The posterior distribution is,

$$g(n|T) = \frac{\frac{(\lambda t)^{n-1}}{(n-1)!}p(1-p)^{n-1}}{pe^{\lambda t(1-p)}}$$
$$= \frac{e^{-\lambda t(1-p)}[\lambda t(1-p)]^{n-1}}{(n-1)!} \tag{3.18}$$

which is Poisson$\sim [\lambda t(1-p)]$.
The posterior r^{th} moment is,

$$E(n^r|T) = \frac{\frac{p}{\lambda t(1-p)}e^{\lambda t(1-p)}T_{r+1}[\lambda t(1-p)]}{pe^{\lambda t(1-p)}} = \frac{T_{r+1}[\lambda t(1-p)]}{\lambda t(1-p)} \tag{3.19}$$

The posterior mean is hence given by,

$$E(n|T) = \frac{T_2[\lambda t(1-p)]}{\lambda t(1-p)} = \lambda t(1-p) + 1 \tag{3.20}$$

4. Erlang-Poisson distribution

The Poisson mixing distribution is;

$$P_n = \frac{e^{-\theta}\theta^n}{n!}, \quad n = 0, 1, 2, ...; 0 < \theta < 1, \tag{4.1}$$

and, $$E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right) = \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}\frac{e^{-\theta}\theta^n}{n!}$$
$$= \theta e^{-\theta}\sum_{n=1}^{\infty} \frac{(\lambda\theta t)^{n-1}}{n!(n-1)!}\frac{(-1)^{n-1}}{(-1)^{n-1}}$$
$$= \theta e^{-\theta}\sum_{n=1}^{\infty} (-1)^{n-1}\frac{(-\lambda\theta t)^{\frac{2(n-1)}{2}}}{n!(n-1)!}$$
$$= \frac{\theta e^{-\theta}}{i\sqrt{\lambda\theta t}}\sum_{n=1}^{\infty} (-1)^{n-1}\frac{\left(\sqrt{-\lambda\theta t}\right)^{2n-1}}{n!(n-1)!}$$
$$= \frac{\theta e^{-\theta}}{i\sqrt{\lambda\theta t}}\sum_{n=1}^{\infty} (-1)^{n-1}\frac{\left(\frac{2i\sqrt{\lambda\theta t}}{2}\right)^{2n-1}}{n!(n-1)!}$$
$$= \frac{\theta e^{-\theta}}{i\sqrt{\lambda\theta t}}\dot{\tau}_1\left(2i\sqrt{\lambda\theta t}\right) \tag{4.2}$$

where $\dot{\tau}_\rho(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\rho+1)}\left(\frac{x}{2}\right)^{2k+\rho}$ is the Modified Bessel function of the first kind.

a) The Erlang-Poisson mixture is thus;

$$f(t) = \frac{\lambda\theta e^{-(\lambda t+\theta)}}{i\sqrt{\lambda\theta t}}\dot{\tau}_1(2i\sqrt{\lambda\theta t}), \quad t = 0, 1, 2, ...; 0 < \theta < 1, \lambda > 0 \tag{4.3}$$

b) The moment generating function of the mixture is;

$$
\begin{aligned}
M_t(s) &= \sum_{n=0}^{\infty} \left(\frac{\lambda}{\lambda - s}\right)^n \frac{e^{-\theta}\theta^n}{n!} \\
&= e^{-\theta} \sum_{n=0}^{\infty} \left(\frac{\lambda\theta}{\lambda - s}\right)^n \frac{1}{n!} \\
&= e^{-\theta\left(1-\frac{\lambda}{\lambda-s}\right)} = e^{\frac{\theta s}{\lambda - s}} \qquad (4.4)
\end{aligned}
$$

and the cumulant generating function is therefore;

$$K_t(s) = \ln\left(e^{\frac{\theta s}{\lambda-s}}\right) = \frac{\theta s}{\lambda - s} \qquad (4.5)$$

c) The raw moments of the Poisson distribution are;

$$E(n) = \theta \qquad (4.6)$$

$$E(n^2) = \theta^2 + \theta = \theta(\theta + 1) \qquad (4.7)$$

$$E(n^3) = \theta^3 + 3\theta^2 + \theta = \theta(\theta^2 + 3\theta + 1) \qquad (4.8)$$

$$E(n^4) = \theta^4 + 6\theta^3 + 7\theta^2 + \theta = \theta(\theta^3 + 6\theta^2 + 7\theta + 1) \qquad (4.9)$$

and the central moments are hence given by;

$$Var(n) = \theta^2 + \theta - \theta^2 = \theta \qquad (4.10)$$

$$E[n - E(n)]^3 = \theta^3 + 3\theta^2 + \theta - 3\theta(\theta^2 + \theta) + 2\theta^3 = \theta \qquad (4.11)$$

$$E[n - E(n)]^4 = \theta^4 + 6\theta^3 + 7\theta^2 + \theta - 4\theta(\theta^3 + 3\theta^2 + \theta) + 6\theta^2(\theta^2 + \theta) - 3\theta^4 = 3\theta^2 + \theta \qquad (4.12)$$

d) Moments and cumulants of the Erlang-Poisson distribution are thus;

$$E(T) = K_1(t) = \frac{\theta}{\lambda} \qquad (4.13)$$

$$Var(T) = K_2(t) = \frac{1}{\lambda^2}(\theta + \theta) = \frac{2\theta}{\lambda^2} \qquad (4.14)$$

$$\mu_3 = K_3(t) = \frac{1}{\lambda^3}(\theta + 3\theta + 2\theta) = \frac{6\theta}{\lambda^3} \qquad (4.15)$$

$$\mu_4 = \frac{1}{\lambda^4}[3\theta^2 + \theta + 6\theta + \theta(6\theta + 11) + 3\theta^2 + 6\theta] = \frac{12}{\lambda^4}(\theta^2 + 2) \qquad (4.16)$$

e) The posterior distribution is

$$g(n|T) = \frac{\frac{(\lambda t)^{n-1}}{(n-1)!}\frac{e^{-\theta}\theta^n}{n!}}{\frac{\theta e^{-\theta}}{i\sqrt{\lambda\theta t}}\dot{\tau}_1\left(2i\sqrt{\lambda\theta t}\right)} = \frac{i\sqrt{\lambda\theta t}(\lambda\theta t)^{n-1}}{\dot{\tau}_1\left(2i\sqrt{\lambda\theta t}\right)n!(n-1)!} \qquad (4.17)$$

The posterior r^{th} moment is

$$E(n^r|T) = \frac{i\sqrt{\lambda\theta t}}{\dot{\tau}_1\left(2i\sqrt{\lambda\theta t}\right)} \sum_{n=1}^{\infty} n^r \frac{(\lambda\theta t)^{n-1}}{n!(n-1)!} \qquad (4.18)$$

and the posterior mean is

$$E(n|T) = \frac{i\sqrt{\lambda\theta t}}{\dot{\tau}_1\left(2i\sqrt{\lambda\theta t}\right)}\sum_{n=1}^{\infty}\frac{(\lambda\theta t)^{n-1}}{(n-1)!(n-1)!} = i\sqrt{\lambda\theta t}\frac{\dot{\tau}_0(2i\sqrt{\lambda\theta t})}{\dot{\tau}_1(2i\sqrt{\lambda\theta t})} \tag{4.19}$$

5. Erlang-logarithmic distribution

The logarithmic mixing distribution is given by;

$$P_n = \frac{p^n}{-n\log(1-p)}, \quad n = 1, 2, 3, ...; 0 < p < 1 \tag{5.1}$$

and thus, $$E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right) = \sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}\frac{p^n}{-n\log(1-p)}$$
$$= \frac{1}{-\lambda t\log(1-p)}\sum_{n=1}^{\infty}\frac{(\lambda tp)^n}{n!}$$
$$= \frac{-(e^{\lambda tp}-1)}{\lambda t\log(1-p)} = \frac{1-e^{\lambda tp}}{\lambda t\log(1-p)} \tag{5.2}$$

and, $$E\left(\frac{n^r(\lambda t)^{n-1}}{(n-1)!}\right) = \frac{1}{-\lambda t\log(1-p)}\sum_{n=1}^{\infty}\frac{n^r(\lambda tp)^n}{n!} = \frac{e^{\lambda tp}T_r(\lambda tp)}{-\lambda t\log(1-p)} \tag{5.3}$$

a) The Erlang-logarithmic distribution is therefore;

$$f(t) = \frac{e^{-\lambda t} - e^{-\lambda t(1-p)}}{t\log(1-p)}, \quad t = 0, 1, 2, ...; \lambda > 0, 0 < p < 1 \tag{5.4}$$

b) The moment generating function of the mixed distribution is;

$$M_t(s) = \sum_{n=1}^{\infty}\left(\frac{\lambda}{\lambda-s}\right)^n\frac{p^n}{-n\log(1-p)}$$
$$= \frac{1}{-\log(1-p)}\sum_{n=1}^{\infty}\left(\frac{\lambda p}{\lambda-s}\right)^n\frac{1}{n}$$
$$= \frac{\log\left[1-\frac{\lambda p}{\lambda-s}\right]}{\log(1-p)} = \frac{\log\left[\lambda(1-p)-s\right]-\log(\lambda-s)}{\log(1-p)} \tag{5.5}$$

and hence the cumulant generating function is given by;

$$K_t(s) = \log\left(\frac{\log\left[\lambda(1-p)-s\right]-\log(\lambda-s)}{\log(1-p)}\right) = \log\left\{\log\left[\lambda(1-p)-s\right]-\log(\lambda-s)\right\} - \log\log(1-p) \tag{5.6}$$

c) The raw moments of the logarithmic distribution are;

$$E(n) = \frac{-p}{(1-p)\mathsf{log}(1-p)} \tag{5.7}$$

$$E(n^2) = \frac{-p^2}{(1-p)^2\mathsf{log}(1-p)} - \frac{p}{(1-p)\mathsf{log}(1-p)} = \frac{-p}{(1-p)^2\mathsf{log}(1-p)} \tag{5.8}$$

$$E(n^3) = \frac{-2p^3}{(1-p)^3\mathsf{log}(1-p)} - \frac{3p^2}{(1-p)^2\mathsf{log}(1-p)} - \frac{p}{(1-p)\mathsf{log}(1-p)} = \frac{-p(p+1)}{(1-p)^3\mathsf{log}(1-p)} \tag{5.9}$$

$$\begin{aligned} E(n^4) &= \frac{-6p^4}{(1-p)^4\mathsf{log}(1-p)} - \frac{12p^3}{(1-p)^3\mathsf{log}(1-p)} - \frac{7p^2}{(1-p)^2\mathsf{log}(1-p)} - \frac{p}{(1-p)\mathsf{log}(1-p)} \\ &= \frac{-p(p^2+4p+1)}{(1-p)^4\mathsf{log}(1-p)} \end{aligned} \tag{5.10}$$

and the central moments are;

$$Var(n) = \frac{-p}{(1-p)^2\mathsf{log}(1-p)} - \frac{p^2}{(1-p)^2[\mathsf{log}(1-p)]^2} = \frac{-p\mathsf{log}(1-p) - p^2}{(1-p)^2[\mathsf{log}(1-p)]^2} \tag{5.11}$$

$$\begin{aligned} E\left[n - E(n)\right]^3 &= \frac{-p(p+1)}{(1-p)^3\mathsf{log}(1-p)} - \frac{3p^2}{(1-p)^3[\mathsf{log}(1-p)]^2} - \frac{2p^3}{(1-p)^3[\mathsf{log}(1-p)]^3} \\ &= \frac{-p(p+1)[\mathsf{log}(1-p)]^2 - 3p^2\mathsf{log}(1-p) - 2p^3}{(1-p)^3[\mathsf{log}(1-p)]^3} \end{aligned} \tag{5.12}$$

$$\begin{aligned} E\left[n - E(n)\right]^4 &= \frac{-p(p^2+4p+1)}{(1-p)^4\mathsf{log}(1-p)} - \frac{4p^2(p+1)}{(1-p)^4[\mathsf{log}(1-p)]^2} - \frac{6p^3}{(1-p)^4[\mathsf{log}(1-p)]^3} - \\ &\quad \frac{3p^4}{(1-p)^4[\mathsf{log}(1-p)]^4} \\ &= \frac{-p(p^2+4p+1)[\mathsf{log}(1-p)]^3 - 4p^2(p+1)[\mathsf{log}(1-p)]^2 - 6p^3\mathsf{log}(1-p) - 3p^4}{(1-p)^4[\mathsf{log}(1-p)]^4} \end{aligned} \tag{5.13}$$

d) Moments and cumulants of the Erlang-logarithmic distribution are therefore;

$$E(T) = K_1(t) = \frac{-p}{\lambda(1-p)\log(1-p)} \tag{5.14}$$

$$Var(T) = K_2(t) = \frac{1}{\lambda^2}\left\{\frac{-p\log(1-p)-p^2}{(1-p)^2[\log(1-p)]^2} - \frac{p}{(1-p)\log(1-p)}\right\} = \frac{-p[p+(2-p)\log(1-p)]}{\lambda^2(1-p)^2[\log(1-p)]^2} \tag{5.15}$$

$$\mu_3 = K_3(t) = \frac{1}{\lambda^3}\left\{\frac{-p(p+1)[\log(1-p)]^2 - 3p^2\log(1-p) - 2p^3}{(1-p)^3[\log(1-p)]^3} - \frac{3p\log(1-p)-p^2}{(1-p)^2[\log(1-p)]^2} - \frac{2p}{(1-p)\log(1-p)}\right. = \frac{-2p^3 - p^2(4-p)\log(1-p) - p(6-6p+2p^2)[\log(1-p)]^2}{\lambda^3(1-p)^3[\log(1-p)]^3} \tag{5.16}$$

$$\mu_4 = \frac{1}{\lambda^4}\left\{\frac{-p(p^2+4p+1)[\log(1-p)]^3 - 4p^2(p+1)[\log(1-p)]^2 - 6p^3\log(1-p) - 3p^4}{(1-p)^4[\log(1-p)]^4} - \frac{6p(p+1)[\log(1-p)]^2 - 3p^2\log(1-p) - 2p^3}{(1-p)^3[\log(1-p)]^3} - \frac{p\log(1-p)-p^2}{(1-p)^2[\log(1-p)]^2}\right.$$
$$\left[-\frac{6p}{(1-p)\log(1-p)} + 11\right] + \frac{3p^2}{(1-p)^2[\log(1-p)]^2} - \frac{6p}{(1-p)\log(1-p)}$$
$$= \frac{-6p(6p^2-6p-p^3+2)[\log(1-p)]^3 - 8p(3p^2+3p+p^3)[\log(1-p)]^2 - 6p(2p^2-p^3)\log(1-p) - 3p^4}{\lambda^4(1-p)^4[\log(1-p)]^4} \tag{5.17}$$

e) The posterior distribution is the zero truncated Poisson (λtp) distribution,

$$g(n|T) = \frac{\frac{(\lambda t)^{n-1}}{(n-1)!}\frac{p^n}{-n\log(1-p)}}{\frac{1-e^{\lambda tp}}{\lambda t\log(1-p)}} = \frac{(\lambda tp)^n}{n!(e^{\lambda tp}-1)} \tag{5.18}$$

The posterior r^{th} moment is

$$E(n^r|T) = \frac{\frac{e^{\lambda tp}T_r(\lambda tp)}{-\lambda t\log(1-p)}}{\frac{1-e^{\lambda tp}}{\lambda t\log(1-p)}} = \frac{T_r(\lambda tp)}{1-e^{-\lambda tp}} \tag{5.19}$$

and the posterior mean is

$$E(n|T) = \frac{T_1(\lambda tp)}{e^{\lambda tp}-1} = \frac{\lambda tp}{1-e^{-\lambda tp}} = \frac{(\lambda tp)e^{\lambda tp}}{e^{\lambda tp}-1} \tag{5.20}$$

6. Conclusion

This research has studied discrete Erlang mixtures using the geometric, Poisson and logarithmic mixing distributions. The posterior distribution of the Erlang-geometric distribution was demonstrated as the Poisson. The Erlang-Poisson mixture and its posterior distribution were expressed in terms of the Modified Bessel function of the first kind. The posterior distribution of the Erlang-logarithmic distribution was shown to be the truncated Poisson distribution, and the posterior moments were expressed as Touchard polynomials. The moments of the mixed distributions were expressed in terms of moments of the mixing distributions. Additionally, the cumulant generating functions of the Erlang mixtures have been obtained from their moment generating functions. Further, moments of the Erlang mixtures have been obtained from their cumulant generating functions as cumulants. Bayesian estimation was applied in parameter estimation, where the posterior means are the Bayes estimators of the Erlang mixtures' parameters.

Construction of discrete Erlang mixed distributions using more mixing distributions and applications of the mixed distributions, are recommendations for further research.

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