DISCRETE ERLANG MIXED DISTRIBUTIONS AND THEIR PROPERTIES

Abstract: Discrete Erlang mixtures have been constructed and their raw and central moments derived in terms of moments of the mixing distributions. Cumulants obtained from the cumulant generating function were also used in deriving the moments. The posterior distribution and posterior moments were also presented. Bayesian estimation was used in parameter estimation of the mixed Erlang distributions. Some methods and special functions used in the study are the exponential series, logarithmic series, geometric series, Modified Bessel function of the first kind and the Touchard polynomials. The discrete mixing distributions used are the geometric, Poisson and logarithmic.

Key Words: Discrete Erlang mixtures, moments, cumulant, cumulant generating function, posterior distribution, Poisson, geometric, logarithmic

1. Introduction

The gamma distribution is a continuous probability function with two parameters, namely, the shape parameter α which is a positive real number, and the rate parameter θ which is also a positive real number. (The scale parameter is sometimes used in place of the rate, which is the reciprocal of the rate parameter). The Erlang distribution is a special case of the gamma distribution, where the the shape parameter $\alpha = n$ is a positive integer.

The Erlang distribution was introduced by Agner K Erlang, when he applied it in analyzing the number of telephone calls made concurrently to switching station operators. It is used to model events that occur in a given interval of time, where the shape parameter predicts the number of events while the rate/scale predicts the time interval between these events. It has a wide applicability due to its relation to the exponential and Poisson distributions.

The exponential distribution models time between consecutive events while the Erlang distribution is used in describing time intervals between any two events. The Erlang (n, θ) is the distribution of a sum of n independent exponentially distributed variables, each with parameter θ.

The Poisson distribution is used to model events that occur within a given time interval, while the waiting times between occurrences of the events are Erlang distributed. Its relation to the Erlang distribution is through the Poisson process.

Over time the Erlang distribution has been modified by using mixing weights to form finite mixtures and by varying the shape parameter to form discrete Erlang mixtures and the rate/scale parameter to model continuous Erlang mixtures. The mixed distributions have a wider applicability compared to the Erlang distribution. [1], [4] and [9] are among people who derived finite Erlang mixtures, while [8], [5] and [6] worked on continuous Erlang mixtures.

The focus of this work is on discrete Erlang mixed distributions, which are obtained by mixing the Erlang distribution with discrete mixing distributions. [10] showed that the Erlang mixture can be used in the approximation of any non-negative continuous

distribution. [7] evinced that the order statistics of independent mixed Erlang random variables belong to the same distribution class of Erlang mixtures. [3] used mixtures of the Erlang distribution in moment based approximation. They conducted numerical experiments on the mixed Erlang approximation method, where the model was seen to provide an overall good fit. [11] demonstrated that a large number of distributions are of the discrete mixed Erlang type. They showed that the Laplace transform of the Erlang mixture can be expressed in terms of the probability generating function of the mixing distribution. They also discussed special cases of the Erlang mixture, which include the exponential distribution, the Erlang distribution and the non-central chi-square distribution. [12] derived distributional properties of a class of multivariate mixed Erlang distributions with different scale parameters. [2] presented the equilibrium function, among other properties, of the mixed Erlang distribution.
The outline of the paper is as follows: The Erlang mixed distribution and its properties have been defined in general in section 2, and particular cases of the mixed distributions have been obtained in sections 3, 4 and 5 using the geometric, Poisson and logarithmic mixing distributions respectively.

2. Definitions, notations and Terminologies

- The probability density function of the conditional (Erlang) distribution is;

$$f(t|n) = \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1}, \quad t > 0; \lambda > 0, n = 1, 2, 3, .. \tag{2.1}$$

  where n is the shape parameter and λ is the rate parameter.

- The mixed Erlang distribution is thus;

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} P_n \\ &= \lambda e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} P_n \\ &= \lambda e^{-\lambda t} E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right) \end{aligned} \tag{2.2}$$

  where P_n is a discrete mixing distribution.

- The r^{th} moment of the Erlang mixture is given by;

$$\begin{aligned}
E(T^r) &= EE(T^r|n) \\
&= E\int_0^\infty t^r f(t|n)dt \\
&= E\int_0^\infty t^r \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} dt \\
&= E\left(\frac{\lambda^n}{\Gamma n}\int_0^\infty t^{n+r-1} e^{-\lambda t} dt\right) \\
&= E\left(\frac{\lambda^n}{\Gamma n}\frac{\Gamma(n+r)}{\lambda^{n+r}}\right) \\
E(T^r) &= \frac{1}{\lambda^r} E\left(\frac{\Gamma(n+r)}{\Gamma n}\right) \qquad (2.3)
\end{aligned}$$

Raw moments and central moments of the mixed Erlang distribution in terms of moments of the mixing distribution are;

- Raw moments

$$E(T) = \frac{1}{\lambda}E(n) \qquad (2.4)$$

$$E(T^2) = \frac{1}{\lambda^2}E\left(\frac{\Gamma(n+2)}{\Gamma n}\right) = \frac{1}{\lambda^2}E[n(n+1)] = \frac{1}{\lambda^2}[E(n^2)+E(n)] \qquad (2.5)$$

$$E(T^3) = \frac{1}{\lambda^3}E\left(\frac{\Gamma(n+3)}{\Gamma n}\right) = \frac{1}{\lambda^3}E[n(n+1)(n+2)] = \frac{1}{\lambda^3}[E(n^3)+3E(n^2)+2E(n)] \qquad (2.6)$$

$$\begin{aligned}
E(T^4) &= \frac{1}{\lambda^4}E\left(\frac{\Gamma(n+4)}{\Gamma n}\right) = \frac{1}{\lambda^4}E[n(n+1)(n+2)(n+3)] \\
&= \frac{1}{\lambda^4}[E(n^4)+6E(n^3)+11E(n^2)+6E(n)] \qquad (2.7)
\end{aligned}$$

- Central moments

  i. Variance

$$\begin{aligned}
\mu_2 &= E\left[T-E(T)\right]^2 = E(T^2) - [E(T)]^2 \\
&= \frac{1}{\lambda^2}[E(n^2)+E(n)] - \frac{1}{\lambda^2}[E(n)]^2 \\
&= \frac{1}{\lambda^2}\left\{E(n^2)+E(n)-[E(n)]^2\right\} \\
&= \frac{1}{\lambda^2}\left\{Var(n)+E(n)\right\} \qquad (2.8)
\end{aligned}$$

ii. Third moment

$$
\begin{aligned}
\mu_3 &= E\left[T - E(T)\right]^3 = E(T^3) - 3E(T^2)E(T) + 2[E(T)]^3 \\
&= \frac{1}{\lambda^3}[E(n^3) + 3E(n^2) + 2E(n)] - \frac{3}{\lambda^3}[E(n^2) + E(n)]E(n) + \frac{2}{\lambda^3}[E(n)]^3 \\
&= \frac{1}{\lambda^3}\left\{E(n^3) + 3E(n^2) + 2E(n) - 3E(n^2)E(n) - 3[E(n)]^2 + 2[E(n)]^3\right\} \\
&= \frac{1}{\lambda^3}\left\{E[n - E(n)]^3 + 3Var(n) + 2E(n)\right\} \qquad (2.9)
\end{aligned}
$$

iii. Fourth moment

$$
\begin{aligned}
\mu_4 &= E\left[T - E(T)\right]^4 = E(T^4) - 4E(T^3)E(T) + 6E(T^2)[E(T)]^2 - 3[E(T)]^4 \\
&= \frac{1}{\lambda^4}[E(n^4) + 6E(n^3) + 11E(n^2) + 6E(n)] - \frac{4}{\lambda^4}[E(n^3) + 3E(n^2) + \\
&\quad 2E(n)]E(n) + \frac{6}{\lambda^4}[E(n^2) + E(n)][E(n)]^2 - \frac{3}{\lambda^4}[E(n)]^4 \\
&= \frac{1}{\lambda^4}\{E(n^4) + 6E(n^3) + 11E(n^2) + 6E(n) - 4E(n^3)E(n) - 12E(n^2)E(n) - \\
&\quad 8[E(n)]^2 + 6E(n^2)[E(n)]^2 + 6[E(n)]^3 - 3[E(n)]^4\} \\
&= \frac{1}{\lambda^4}\{E[n - E(n)]^4 + 6E[n - E(n)]^3 + 6Var(n)E(n) + 11Var(n) + 3[E(n)]^2 + 6E(n)\} \\
&= \frac{1}{\lambda^4}\{E[n - E(n)]^4 + 6E[n - E(n)]^3 + Var(n)[6E(n) + 11] + 3[E(n)]^2 + 6E(n)\} \\
&\qquad (2.10)
\end{aligned}
$$

- The moment generating function of the Erlang mixture is given by

$$
\begin{aligned}
M_t(s) &= E(e^{ts}) = EE(e^{ts}|n) \\
&= E\int_0^\infty e^{ts} f(t|n)dt \\
&= E\int_0^\infty e^{ts}\frac{\lambda^n}{\Gamma n}e^{-\lambda t}t^{n-1}dt \\
&= E\left(\frac{\lambda^n}{\Gamma n}\int_0^\infty t^{n-1}e^{-(\lambda - s)t}dt\right) \\
&= E\left(\frac{\lambda^n}{\Gamma n}\frac{\Gamma n}{(\lambda - s)^n}\right) = E\left(\frac{\lambda}{\lambda - s}\right)^n \qquad (2.11)
\end{aligned}
$$

and hence the cumulant generating function is

$$
K_t(s) = \mathsf{log}M_t(s) = \mathsf{log}E\left(\frac{\lambda}{\lambda - s}\right)^n \qquad (2.12)
$$

The r^{th} cumulant of the mixed distribution, $K_r(t)$, is the r^{th} derivative of the cumulant generating function at $s = 0$, and the first, second and third cumulants are the expected value, second and third central moments respectively.

$$K_t'(s) = \frac{E\left[\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right]}{E\left(\frac{\lambda}{\lambda-s}\right)^n} \quad \text{and} \quad K_1(t) = K_t'(0) = \frac{1}{\lambda}E(n) \tag{2.13}$$

$$K_t''(s) = \frac{E\left(\frac{\lambda}{\lambda-s}\right)^n E\left[\frac{n(n+1)\lambda^n}{(\lambda-s)^{n+2}}\right] - \left\{E\left[\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right]\right\}^2}{\left[E\left(\frac{\lambda}{\lambda-s}\right)^n\right]^2} \quad \text{and}$$

$$K_2(t) = K_t''(0) = \frac{1}{\lambda^2}\left\{E(n^2) + E(n) - [E(n)]^2\right\} \tag{2.14}$$

$$K_t'''(s) = \frac{\left[E\left(\frac{\lambda}{\lambda-s}\right)^n\right]\left\{E\left(\frac{\lambda}{\lambda-s}\right)^n E\left[\frac{n(n+1)(n+2)\lambda^n}{(\lambda-s)^{n+3}}\right] - E\left[\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right] E\left[\frac{n(n+1)\lambda^n}{(\lambda-s)^{n+2}}\right]\right\} -}{\left[E\left(\frac{\lambda}{\lambda-s}\right)^n\right]^2}$$

$$\frac{2E\left[\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right]\left\{E\left(\frac{\lambda}{\lambda-s}\right)^n E\left[\frac{n(n+1)\lambda^n}{(\lambda-s)^{n+2}}\right] - \left[E\left(\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right)\right]^2\right\}}{} \quad \text{and}$$

$$K_3(t) = K_t'''(0) = \frac{1}{\lambda^3}\left\{E(n^3) + 3E(n^2) + 2E(n) - 3E(n^2)E(n) - 3[E(n)]^2 + 2[E(n)]^3\right\} \tag{2.15}$$

- The Posterior distribution is given by

$$\begin{aligned} g(n|T) &= \frac{f(t|n)P_n}{f(t)} \\ &= \frac{\frac{\lambda^n}{\Gamma n}e^{-\lambda t}t^{n-1}P_n}{\lambda e^{-\lambda t}E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} = \frac{\frac{(\lambda t)^{n-1}}{(n-1)!}P_n}{E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} \end{aligned} \tag{2.16}$$

where $f(t|n)$ is the likelihood function which is the Erlang distribution and P_n is the prior distribution.
The posterior r^{th} moment is given by

$$\begin{aligned} E(n^r|T) &= \sum_{n=1}^{\infty} n^r g(n|t) \\ &= \frac{\sum_{n=1}^{\infty} n^r \frac{(\lambda t)^{n-1}}{(n-1)!}P_n}{E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} = \frac{E\left(\frac{n^r(\lambda t)^{n-1}}{(n-1)!}\right)}{E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} \end{aligned} \tag{2.17}$$

and the posterior mean is

$$E(n|T) = \frac{E\left(\frac{n(\lambda t)^{n-1}}{(n-1)!}\right)}{E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} \tag{2.18}$$

- The posterior mean $E(n|T)$ is the Bayes estimator of the parameter n, assuming squared error loss function.

3. Erlang-Geometric distribution

The geometric mixing distribution is

$$P_n = p(1-p)^{n-1}, \quad n = 1, 2, 3, ...; 0 < p < 1 \tag{3.1}$$

and $$E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right) = \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} p(1-p)^{n-1}$$
$$= p\sum_{n=1}^{\infty} \frac{[\lambda t(1-p)]^{n-1}}{(n-1)!} = pe^{\lambda t(1-p)} \tag{3.2}$$

and $$E\left(n^r \frac{(\lambda t)^{n-1}}{(n-1)!}\right) = p\sum_{n=1}^{\infty} n^r \frac{[\lambda t(1-p)]^{n-1}}{(n-1)!}$$
$$= \frac{p}{\lambda t(1-p)} \sum_{n=1}^{\infty} n^{r+1} \frac{[\lambda t(1-p)]^n}{n!} = \frac{p}{\lambda t(1-p)} e^{\lambda t(1-p)} T_{r+1}[\lambda t(1-p)] \tag{3.3}$$

where $T_r(x) = e^{-x}\sum_{k=0}^{\infty} \frac{k^r x^k}{k!} = \sum_{k=0}^{r} S(r,k)x^k$ is the Touchard polynomials and $S(r,k) = \sum_{j=0}^{k} \frac{(-1)^{k-j} j^r}{(k-j)!j!}$ is the Stirling number of the second kind.

a) The Erlang-Geometric distribution is thus

$$f(t) = \lambda p e^{-\lambda p t}, \quad t = 0, 1, 2, ...; 0 < p < 1, \lambda > 0 \tag{3.4}$$

b) The moment generating function of the mixed distribution is

$$M_t(s) = \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda - s}\right)^n p(1-p)^{n-1}$$
$$= \frac{p\lambda}{\lambda - s} \sum_{n=1}^{\infty} \left[\frac{\lambda(1-p)}{\lambda - s}\right]^{n-1}$$
$$= \frac{p\lambda}{\lambda - s} \frac{1}{1 - \frac{\lambda(1-p)}{\lambda - s}} = \frac{p\lambda}{p\lambda - s} \tag{3.5}$$

and the cumulant generating function is thus

$$M_t(s) = \log\left(\frac{p\lambda}{p\lambda - s}\right) = \log(p\lambda) - \log(p\lambda - s) \tag{3.6}$$

c) The raw moments of the geometric distribution are

$$E(n) = \frac{1}{p} \tag{3.7}$$

$$E(n^2) = \frac{2(1-p)}{p^2} + \frac{1}{p} \tag{3.8}$$

$$E(n^3) = \frac{6(1-p)^2}{p^3} + \frac{6(1-p)}{p^2} + \frac{1}{p} \tag{3.9}$$

$$E(n^4) = \frac{24(1-p)^3}{p^4} + \frac{36(1-p)^2}{p^3} + \frac{14(1-p)}{p^2} + \frac{1}{p} \tag{3.10}$$

and the central moments are therefore,

$$Var(n) = \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2} \tag{3.11}$$

$$E[n-E(n)]^3 = \frac{6(1-p)^2}{p^3} + \frac{6(1-p)}{p^2} + \frac{1}{p} - \frac{3(2-p)}{p^3} + \frac{2}{p^3} = \frac{2-3p+p^2}{p^3} \tag{3.12}$$

$$E[n-E(n)]^4 = \frac{24(1-p)^3}{p^4} + \frac{36(1-p)^2}{p^3} + \frac{14(1-p)}{p^2} + \frac{1}{p} - \frac{4}{p^4}(6-12p+6p^2+6p-6p^2+p^2)+ \frac{6}{p^4}(2-p) - \frac{3}{p^4} = \frac{9-18p+10p^2-p^3}{p^4} \tag{3.13}$$

d) The moments and cumulants of the Erlang-Geometric distribution are thus

$$E(T) = K_1(t) = \frac{1}{\lambda p} \tag{3.14}$$

$$Var(T) = K_2(t) = \frac{1}{\lambda^2}\left\{\frac{1-p}{p^2} + \frac{1}{p}\right\} = \frac{1}{(p\lambda)^2} \tag{3.15}$$

$$\mu_3 = K_3(t) = \frac{1}{\lambda^3}\left\{\frac{2-3p+p^2}{p^3} + \frac{3(1-p)}{p^2} + \frac{2}{p}\right\} = \frac{2}{(p\lambda)^3} \tag{3.16}$$

$$\mu_4 = \frac{1}{\lambda^4}\left\{\frac{9-18p+10p^2-p^3}{p^4} + \frac{6(2-3p+p^2)}{p^3} + \frac{1-p}{p^2}\left[\frac{6}{p}+11\right] + \frac{3}{p^2} + \frac{6}{p}\right\} = \frac{9}{(p\lambda)^4} \tag{3.17}$$

e) The posterior distribution is

$$\begin{aligned} g(n|T) &= \frac{\frac{(\lambda t)^{n-1}}{(n-1)!}p(1-p)^{n-1}}{pe^{\lambda t(1-p)}} \\ &= \frac{e^{-\lambda t(1-p)}[\lambda t(1-p)]^{n-1}}{(n-1)!} \end{aligned} \tag{3.18}$$

which is Poisson$\sim [\lambda t(1-p)]$.
The posterior r^{th} moment is

$$E(n^r|T) = \frac{\frac{p}{\lambda t(1-p)}e^{\lambda t(1-p)}T_{r+1}[\lambda t(1-p)]}{pe^{\lambda t(1-p)}} = \frac{T_{r+1}[\lambda t(1-p)]}{\lambda t(1-p)} \tag{3.19}$$

The posterior mean is hence given by,

$$E(n|T) = \frac{T_2[\lambda t(1-p)]}{\lambda t(1-p)} = \lambda t(1-p) + 1 \tag{3.20}$$

4. Erlang-Poisson distribution

The Poisson mixing distribution is

$$P_n = \frac{e^{-\theta}\theta^n}{n!}, \qquad n = 0,1,2,...; 0 < \theta < 1, \tag{4.1}$$

$$\begin{aligned}
\text{and} \quad E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right) &= \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}\frac{e^{-\theta}\theta^n}{n!} \\
&= \theta e^{-\theta}\sum_{n=1}^{\infty} \frac{(\lambda\theta t)^{n-1}}{n!(n-1)!}\frac{(-1)^{n-1}}{(-1)^{n-1}} \\
&= \theta e^{-\theta}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-\lambda\theta t)^{\frac{2(n-1)}{2}}}{n!(n-1)!} \\
&= \frac{\theta e^{-\theta}}{i\sqrt{\lambda\theta t}}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\sqrt{-\lambda\theta t}\right)^{2n-1}}{n!(n-1)!} \\
&= \frac{\theta e^{-\theta}}{i\sqrt{\lambda\theta t}}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\frac{2i\sqrt{\lambda\theta t}}{2}\right)^{2n-1}}{n!(n-1)!} \\
&= \frac{\theta e^{-\theta}}{i\sqrt{\lambda\theta t}}\dot{\tau}_1\left(2i\sqrt{\lambda\theta t}\right)
\end{aligned} \tag{4.2}$$

where $\dot{\tau}_\rho(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\rho+1)}\left(\frac{x}{2}\right)^{2k+\rho}$ is the Modified Bessel function of the first kind.

a) The Erlang-Poisson mixture is thus;

$$f(t) = \frac{\lambda\theta e^{-(\lambda t+\theta)}}{i\sqrt{\lambda\theta t}}\dot{\tau}_1(2i\sqrt{\lambda\theta t}), \quad t = 0,1,2,...; 0 < \theta < 1, \lambda > 0 \tag{4.3}$$

b) The moment generating function of the mixture is

$$\begin{aligned}
M_t(s) &= \sum_{n=0}^{\infty}\left(\frac{\lambda}{\lambda - s}\right)^n\frac{e^{-\theta}\theta^n}{n!} \\
&= e^{-\theta}\sum_{n=0}^{\infty}\left(\frac{\lambda\theta}{\lambda - s}\right)^n\frac{1}{n!} \\
&= e^{-\theta\left(1-\frac{\lambda}{\lambda-s}\right)} = e^{\frac{\theta s}{\lambda - s}}
\end{aligned} \tag{4.4}$$

and the cumulant generating function is thus;

$$K_t(s) = \ln\left(e^{\frac{\theta s}{\lambda-s}}\right) = \frac{\theta s}{\lambda - s} \tag{4.5}$$

c) The raw moments of the Poisson distribution are;

$$E(n) = \theta \tag{4.6}$$
$$E(n^2) = \theta^2 + \theta = \theta(\theta+1) \tag{4.7}$$
$$E(n^3) = \theta^3 + 3\theta^2 + \theta = \theta(\theta^2 + 3\theta + 1) \tag{4.8}$$
$$E(n^4) = \theta^4 + 6\theta^3 + 7\theta^2 + \theta = \theta(\theta^3 + 6\theta^2 + 7\theta + 1) \tag{4.9}$$

and the central moments are hence given by

$$Var(n) = \theta^2 + \theta - \theta^2 = \theta \tag{4.10}$$

$$E[n - E(n)]^3 = \theta^3 + 3\theta^2 + \theta - 3\theta(\theta^2 + \theta) + 2\theta^3 = \theta \tag{4.11}$$

$$E[n - E(n)]^4 = \theta^4 + 6\theta^3 + 7\theta^2 + \theta - 4\theta(\theta^3 + 3\theta^2 + \theta) + 6\theta^2(\theta^2 + \theta) - 3\theta^4 = 3\theta^2 + \theta \tag{4.12}$$

d) Moments and cumulants of the Erlang-Poisson distribution are thus;

$$E(T) = K_1(t) = \frac{\theta}{\lambda} \tag{4.13}$$

$$Var(T) = K_2(t) = \frac{1}{\lambda^2}(\theta + \theta) = \frac{2\theta}{\lambda^2} \tag{4.14}$$

$$\mu_3 = K_3(t) = \frac{1}{\lambda^3}(\theta + 3\theta + 2\theta) = \frac{6\theta}{\lambda^3} \tag{4.15}$$

$$\mu_4 = \frac{1}{\lambda^4}[3\theta^2 + \theta + 6\theta + \theta(6\theta + 11) + 3\theta^2 + 6\theta] = \frac{12}{\lambda^4}(\theta^2 + 2) \tag{4.16}$$

e) The posterior distribution is

$$g(n|T) = \frac{\frac{(\lambda t)^{n-1}}{(n-1)!}\frac{e^{-\theta}\theta^n}{n!}}{\frac{\theta e^{-\theta}}{i\sqrt{\lambda\theta t}}\dot{\tau}_1\left(2i\sqrt{\lambda\theta t}\right)} = \frac{i\sqrt{\lambda\theta t}(\lambda\theta t)^{n-1}}{\dot{\tau}_1\left(2i\sqrt{\lambda\theta t}\right)n!(n-1)!} \tag{4.17}$$

The posterior r^{th} moment is

$$E(n^r|T) = \frac{i\sqrt{\lambda\theta t}}{\dot{\tau}_1\left(2i\sqrt{\lambda\theta t}\right)}\sum_{n=1}^{\infty} n^r \frac{(\lambda\theta t)^{n-1}}{n!(n-1)!} \tag{4.18}$$

and the posterior mean is

$$E(n|T) = \frac{i\sqrt{\lambda\theta t}}{\dot{\tau}_1\left(2i\sqrt{\lambda\theta t}\right)}\sum_{n=1}^{\infty} \frac{(\lambda\theta t)^{n-1}}{(n-1)!(n-1)!} = i\sqrt{\lambda\theta t}\frac{\dot{\tau}_0(2i\sqrt{\lambda\theta t})}{\dot{\tau}_1(2i\sqrt{\lambda\theta t})} \tag{4.19}$$

5. Erlang-Logarithmic distribution

The logarithmic mixing distribution is given by;

$$P_n = \frac{p^n}{-n\mathsf{log}(1-p)}, \quad n = 1, 2, 3, ...; 0 < p < 1 \tag{5.1}$$

and thus
$$E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right) = \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} \frac{p^n}{-n\mathsf{log}(1-p)}$$
$$= \frac{1}{-\lambda t\mathsf{log}(1-p)} \sum_{n=1}^{\infty} \frac{(\lambda tp)^n}{n!}$$
$$= \frac{-(e^{\lambda tp}-1)}{\lambda t\mathsf{log}(1-p)} = \frac{1-e^{\lambda tp}}{\lambda t\mathsf{log}(1-p)} \tag{5.2}$$

and
$$E\left(\frac{n^r(\lambda t)^{n-1}}{(n-1)!}\right) = \frac{1}{-\lambda t\mathsf{log}(1-p)} \sum_{n=1}^{\infty} \frac{n^r(\lambda tp)^n}{n!} = \frac{e^{\lambda tp}T_r(\lambda tp)}{-\lambda t\mathsf{log}(1-p)} \tag{5.3}$$

a) The Erlang-logarithmic distribution is therefore;

$$f(t) = \frac{e^{-\lambda t} - e^{-\lambda t(1-p)}}{t\mathsf{log}(1-p)}, \quad t = 0, 1, 2, ...; \lambda > 0, 0 < p < 1 \tag{5.4}$$

b) The moment generating function of the mixed distribution is

$$M_t(s) = \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda - s}\right)^n \frac{p^n}{-n\mathsf{log}(1-p)}$$
$$= \frac{1}{-\mathsf{log}(1-p)} \sum_{n=1}^{\infty} \left(\frac{\lambda p}{\lambda - s}\right)^n \frac{1}{n}$$
$$= \frac{\mathsf{log}\left[1 - \frac{\lambda p}{\lambda - s}\right]}{\mathsf{log}(1-p)} = \frac{\mathsf{log}\left[\lambda(1-p) - s\right] - \mathsf{log}(\lambda - s)}{\mathsf{log}(1-p)} \tag{5.5}$$

and hence the cumulant generating function is given by

$$K_t(s) = \mathsf{log}\left(\frac{\mathsf{log}\left[\lambda(1-p) - s\right] - \mathsf{log}(\lambda - s)}{\mathsf{log}(1-p)}\right) = \mathsf{log}\left\{\mathsf{log}\left[\lambda(1-p) - s\right] - \mathsf{log}(\lambda - s)\right\} - \mathsf{loglog}(1-p) \tag{5.6}$$

c) The raw moments of the logarithmic distribution are

$$E(n) = \frac{-p}{(1-p)\mathsf{log}(1-p)} \tag{5.7}$$

$$E(n^2) = \frac{-p^2}{(1-p)^2\mathsf{log}(1-p)} - \frac{p}{(1-p)\mathsf{log}(1-p)} = \frac{-p}{(1-p)^2\mathsf{log}(1-p)} \tag{5.8}$$

$$E(n^3) = \frac{-2p^3}{(1-p)^3\mathsf{log}(1-p)} - \frac{3p^2}{(1-p)^2\mathsf{log}(1-p)} - \frac{p}{(1-p)\mathsf{log}(1-p)} = \frac{-p(p+1)}{(1-p)^3\mathsf{log}(1-p)} \tag{5.9}$$

$$E(n^4) = \frac{-6p^4}{(1-p)^4\mathsf{log}(1-p)} - \frac{12p^3}{(1-p)^3\mathsf{log}(1-p)} - \frac{7p^2}{(1-p)^2\mathsf{log}(1-p)} - \frac{p}{(1-p)\mathsf{log}(1-p)}$$
$$= \frac{-p(p^2+4p+1)}{(1-p)^4\mathsf{log}(1-p)} \tag{5.10}$$

and the central moments are

$$Var(n) = \frac{-p}{(1-p)^2\log(1-p)} - \frac{p^2}{(1-p)^2[\log(1-p)]^2} = \frac{-p\log(1-p) - p^2}{(1-p)^2[\log(1-p)]^2} \tag{5.11}$$

$$\begin{aligned} E\left[n - E(n)\right]^3 &= \frac{-p(p+1)}{(1-p)^3\log(1-p)} - \frac{3p^2}{(1-p)^3[\log(1-p)]^2} - \frac{2p^3}{(1-p)^3[\log(1-p)]^3} \\ &= \frac{-p(p+1)[\log(1-p)]^2 - 3p^2\log(1-p) - 2p^3}{(1-p)^3[\log(1-p)]^3} \end{aligned} \tag{5.12}$$

$$\begin{aligned} E\left[n - E(n)\right]^4 &= \frac{-p(p^2+4p+1)}{(1-p)^4\log(1-p)} - \frac{4p^2(p+1)}{(1-p)^4[\log(1-p)]^2} - \frac{6p^3}{(1-p)^4[\log(1-p)]^3} - \\ &\quad \frac{3p^4}{(1-p)^4[\log(1-p)]^4} \\ &= \frac{-p(p^2+4p+1)[\log(1-p)]^3 - 4p^2(p+1)[\log(1-p)]^2 - 6p^3\log(1-p) - 3p^4}{(1-p)^4[\log(1-p)]^4} \end{aligned} \tag{5.13}$$

d) Moments and cumulants of the Erlang-logarithmic distribution are therefore;

$$E(T) = K_1(t) = \frac{-p}{\lambda(1-p)\log(1-p)} \tag{5.14}$$

$$Var(T) = K_2(t) = \frac{1}{\lambda^2}\left\{\frac{-p\log(1-p) - p^2}{(1-p)^2[\log(1-p)]^2} - \frac{p}{(1-p)\log(1-p)}\right\} = \frac{-p[p + (2-p)\log(1-p)]}{\lambda^2(1-p)^2[\log(1-p)]^2} \tag{5.15}$$

$$\begin{aligned} \mu_3 = K_3(t) &= \frac{1}{\lambda^3}\left\{\frac{-p(p+1)[\log(1-p)]^2 - 3p^2\log(1-p) - 2p^3}{(1-p)^3[\log(1-p)]^3} - \frac{3p\log(1-p) - p^2}{(1-p)^2[\log(1-p)]^2} - \right. \\ &\quad \left. \frac{2p}{(1-p)\log(1-p)}\right\} = \frac{-2p^3 - p^2(4-p)\log(1-p) - p(6-6p+2p^2)[\log(1-p)]^2}{\lambda^3(1-p)^3[\log(1-p)]^3} \end{aligned} \tag{5.16}$$

$$\begin{aligned} \mu_4 &= \frac{1}{\lambda^4}\left\{\frac{-p(p^2+4p+1)[\log(1-p)]^3 - 4p^2(p+1)[\log(1-p)]^2 - 6p^3\log(1-p) - 3p^4}{(1-p)^4[\log(1-p)]^4} - \right. \\ &\quad \frac{6p(p+1)[\log(1-p)]^2 - 3p^2\log(1-p) - 2p^3}{(1-p)^3[\log(1-p)]^3} - \frac{p\log(1-p) - p^2}{(1-p)^2[\log(1-p)]^2} \\ &\quad \left.\left[-\frac{6p}{(1-p)\log(1-p)} + 11\right] + \frac{3p^2}{(1-p)^2[\log(1-p)]^2} - \frac{6p}{(1-p)\log(1-p)}\right\} \\ &= \frac{-6p(6p^2 - 6p - p^3 + 2)[\log(1-p)]^3 - 8p(3p^2 + 3p + p^3)[\log(1-p)]^2 -}{\lambda^4(1-p)^4[\log(1-p)]^4} \\ &\quad \frac{6p(2p^2 - p^3)\log(1-p) - 3p^4}{} \end{aligned} \tag{5.17}$$

e) The posterior distribution is the zero truncated Poisson (λtp) distribution,

$$g(n|T) = \frac{\frac{(\lambda t)^{n-1}}{(n-1)!}\frac{p^n}{-n\log(1-p)}}{\frac{1-e^{\lambda tp}}{\lambda t\log(1-p)}} = \frac{(\lambda tp)^n}{n!(e^{\lambda tp} - 1)} \tag{5.18}$$

The posterior r^{th} moment is

$$E(n^r|T) = \frac{\frac{e^{\lambda tp}T_r(\lambda tp)}{-\lambda t\log(1-p)}}{\frac{1-e^{\lambda tp}}{\lambda t\log(1-p)}} = \frac{T_r(\lambda tp)}{1-e^{-\lambda tp}} \tag{5.19}$$

and the posterior mean is

$$E(n|T) = \frac{T_1(\lambda tp)}{e^{\lambda tp}-1} = \frac{\lambda tp}{1-e^{-\lambda tp}} = \frac{(\lambda tp)e^{\lambda tp}}{e^{\lambda tp}-1} \tag{5.20}$$

6. Conclusion

Discrete Erlang mixtures have been derived using the geometric, Poisson and logarithmic mixing distributions. The posterior distribution of the Erlang-geometric distribution was shown to be the Poisson. The Erlang-Poisson mixture and its posterior distribution were expressed in terms of the modified Bessel function of the first kind. The posterior distribution of the Erlang-logarithmic distribution was shown to be the truncated Poisson distribution, and the posterior moments were expressed as Touchard polynomials. The moments of the mixed distributions were expressed in terms of moments of the mixing distributions.The cumulant generating functions of the Erlang mixtures have also been obtained from their moment generating functions. Moments of the Erlang mixtures have also been obtained from their cumulant generating functions as cumulants. Parameter estimation was carried out using Bayesian estimation where the posterior means are the Bayes estimators of the Erlang mixtures' parameters.

Construction of discrete Erlang mixed distributions using more mixing distributions and applications of the mixed distributions are recommended for further research.

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