CONTINUOUS ERLANG MIXTURES AND THEIR RELATION TO EXPONENTIAL MIXTURES AND POISSON MIXTURES

Abstract: This study provides a novel method for obtaining Erlang mixtures from a mixed Poisson process. The study solved the basic differential equations of the Poisson process to obtain the Poisson distribution. The waiting time distribution in a Poisson process is illustrated as an Erlang distribution. The study also presented the Erlang mixture as the first passage time distribution in the mixed Poisson process, and it was expressed using both the direct method and the method of moments. Moreover, these two ways of inferring a mathematical identity have been equated. The exponential mixture and Poisson mixture are explained as special cases of the Erlang mixture. A practical example is given, using type II gamma distribution mixtures. Properties of the mixtures, such as raw moments and probability generating function, are analyzed.

Keywords: Erlang mixture, exponential mixture, Poisson mixture, Poisson process, first passage time distribution

1. Introduction

The Erlang distribution is a special case of the gamma distribution, where the shape parameter, $\alpha = n$, is a positive integer. It was introduced by Agner K Erlang, when he applied it in analyzing the number of telephone calls made concurrently to switching station operators. It is used to model events that occur in a given interval of time, with the shape parameter predicting the number of events and the rate/scale predicting the time interval between these events. It has a wide applicability due to its relation to the exponential and Poisson distributions. The exponential distribution models time between consecutive events, while the Erlang distribution is used in describing time intervals between any two events. The Erlang $(n,\, \theta)$ is the distribution of a sum of n independent exponentially distributed variables, each with parameter θ. The Poisson distribution is used to model events that occur within a given time interval, while the waiting times between occurrences of the events are Erlang distributed.

A mixed distribution is a combination of two or more distributions, known as the mixture components. It is used to model populations with sub-populations, with the mixture components representing the sub-populations. Mixed distributions are constructed to address overdispersion and other limitations that basic distributions fail to address in modeling real lifetime data. Continuous mixtures are among the three types of mixtures, the other two being finite and discrete. Pearson [13] was the pioneer of mixed distributions when he constructed a finite mixture from two normal distributions with different means and variances. Greenwood and Yule [4] initiated continuous mixtures when they mixed the Poisson and gamma distributions to form the negative binomial distribution. Continuous Erlang mixtures were introduced by McNolty [9]. He used the Rayleigh, scaled beta, gamma, Maxwell- Boltzman and the random Bessel variate mixing distributions in constructing the mixed distributions.

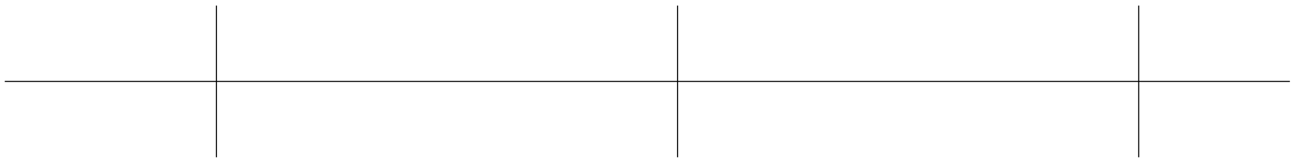
Kang [7] and Jordanova and Stehlík [5] are among other people who studied the continuous Erlang mixtures. Kang [7] derived extreme value distributions of Erlang mixtures and proved that they depended on those of their mixing distributions. Jordanova and Stehlík [5] analyzed properties of the Erlang-Pareto I distribution. Jordanova et al. [6] expressed the Erlang-Pareto I distribution in terms of the incomplete gamma function. They also obtained the exponential-Pareto distribution from its CDF and the Poisson-Pareto mixture by direct integration. They, however, didn't show the connection between the three mixtures.
Sarguta [15] constructed continuous Poisson mixtures using various mixing distributions. Sankaran [14] and Bhati et al. [1] presented the Poisson-Lindley and Poisson-transmuted exponential distributions respectively. None of them linked the mixed distributions to either the Erlang mixtures or exponential mixtures. Wakoli [16] linked exponential mixtures to Poisson mixtures by showing that a sum of hazard functions of exponential mixtures results to a convolution of compound Poisson distributions. Ottieno and Wakoli [12], Ottieno and Wakoli [11], Walhin and Paris [17], Nadarajah and Kotz [10], Frangos and Karlis [2] and Maceda [8] are among other authors who also studied Poisson and exponential mixtures.
The objective of this work is to show the relationship between continuous Erlang mixtures, exponential mixtures, and Poisson mixtures. The Erlang distribution and Erlang mixture have been shown to be waiting time distributions in a Poisson process and a mixed Poisson process respectively. Properties of the mixed distributions obtained include the mean and the PGF.
The outline of the rest of the paper is as follows: Distributions arising from a Poisson process and a mixed Poisson process have been discussed is section 2. The connection between Erlang mixtures and exponential mixtures has been demonstrated in section 3. In section 4, the relation of the Erlang mixtures to the Poisson mixtures has been shown. In section 5, the Erlang-Type II gamma mixture has been studied as an example of the mixed Erlang distribution, and its special cases, the Exponential-Type II gamma mixture and the Poisson-Type II gamma mixture, have been presented in sections 6 and 7 respectively. Section 8 provides a conclusion of the paper.

2. Distributions arising from a Poisson process and a mixed Poisson process

There are two approaches to deriving distributions arising from mixed Poisson processes. The first one is based on a Poisson process with a randomized rate. The other approach is based on a pure birth process.
The Poisson process is a special case of a pure birth process. Solving the basic difference-differential equations for a Poisson process results to a Poisson distribution. The waiting time distribution for an n^{th} event to occur in a Poisson process has been shown to be an Erlang distribution. The first passage time distribution based on randomization has been expressed in two forms. Mathematical identities based on these two forms have also been determined. The first passage time distribution of the mixed Poisson process has been proven to be an Erlang mixture. The r^{th} moment of the Erlang mixture has also been analyzed in this section.

- **Solving the basic difference-differential equation for a Poisson process using the probability generating function (PGF) technique**

  Consider the following diagram



Time	0	t	$t+\Delta t$
Population size	$X(0)$	$X(t)$	$X(t+\Delta t)$

Let $X(t)$ be the population size in time interval t and $p_n(t) = \text{Prob}\{X(t) = n\}$.
The basic difference-differential equations for a pure birth process are given by;

$$\begin{aligned} p_0'(t) &= -\lambda_0 p_0(t) \\ p_n'(t) &= -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t), \quad n = 1, 2, 3, ... \end{aligned} \tag{2.1}$$

where $p_n'(t) = \frac{d}{dt} p_n(t)$ and λ_n is the birth rate in time interval Δt when the population size is n in time interval t.
For a Poisson process, $\lambda_n = \lambda$ for all n. Thus we have;

$$p_0'(t) = -\lambda p_0(t) \tag{2.2}$$

$$\text{and} \quad p_n'(t) = -\lambda p_n(t) + \lambda p_{n-1}(t), \quad n = 1, 2, 3, .. \tag{2.3}$$

Multiplying equation (2.3) by S^n and then summing the result over n, we obtain;

$$\begin{aligned} \sum_{n=1}^{\infty} p_n'(t) S^n &= -\lambda \sum_{n=1}^{\infty} p_n(t) S^n + \lambda \sum_{n=1}^{\infty} p_{n-1}(t) S^n \\ &= -\lambda \sum_{n=1}^{\infty} p_n(t) S^n + \lambda S \sum_{n=1}^{\infty} p_{n-1}(t) S^{n-1} \end{aligned} \tag{2.4}$$

$$\text{Define} \quad G(s,t) = \sum_{n=0}^{\infty} p_n(t) S^n = p_0(t) + \sum_{n=1}^{\infty} p_n(t) S^n = \sum_{n=1}^{\infty} p_{n-1}(t) S^{n-1}$$

$$\Longrightarrow \quad \frac{\delta}{\delta t} G(s,t) = p_0'(t) + \sum_{n=1}^{\infty} p_n'(t) S^n$$

Equation (2.4) becomes;

$$\begin{aligned} \frac{\delta}{\delta t} G(s,t) - p_0'(t) &= -\lambda \left[G(s,t) - p_0(t)\right] + \lambda S G(s,t) \\ &= -\lambda G(s,t) + \lambda p_0(t) + \lambda S G(s,t) \end{aligned} \tag{2.5}$$

Substituting equation (2.2) into equation (2.5) yields;

$$
\begin{aligned}
\frac{\delta}{\delta t}G(s,t) + \lambda p_0(t) &= -\lambda G(s,t) + \lambda p_0(t) + \lambda S G(s,t) \\
\frac{\delta}{\delta t}G(s,t) &= -\lambda(1-S)G(s,t) \\
\frac{1}{G(s,t)}\frac{\delta}{\delta t}G(s,t) &= -\lambda(1-S) \\
\frac{\delta}{\delta t}\mathsf{ln}G(s,t) &= -\lambda(1-S) \\
\mathsf{ln}G(s,t) &= -\lambda(1-S)t + C \\
G(s,t) &= e^{-(1-S)\lambda t}e^{C} \qquad (2.6)
\end{aligned}
$$

Letting the initial condition be $X(0) = 0 \implies p_0(0) = 1$ and $p_n(0) = 0$ for $n \neq 0$. At $t = 0$, equation (2.6) becomes $G(s,0) = e^C$.
But by definition;

$$
G(s,t) = \sum_{n=0}^{\infty} p_n(t)S^n = p_0(t) + \sum_{n=1}^{\infty} p_n(t)S^n
$$

$$
\text{and therefore,} \quad G(s,0) = p_0(0) + \sum_{n=1}^{\infty} p_n(0)S^n = 1 + 0 = 1
$$

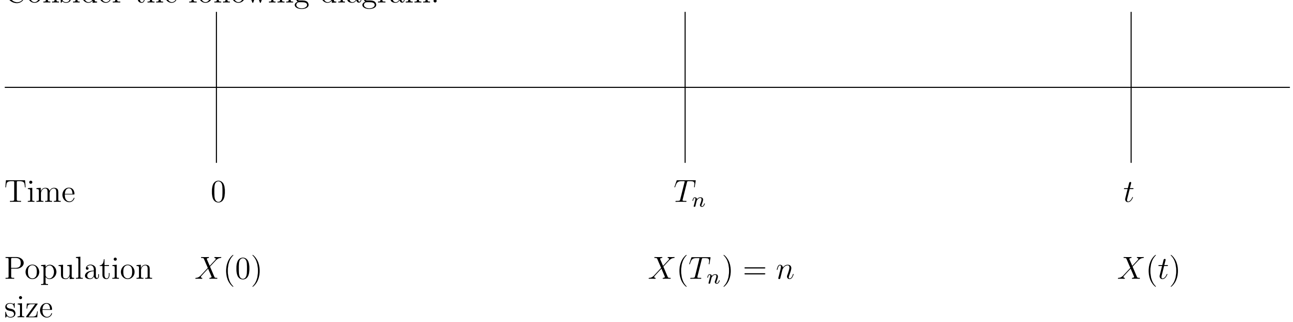
$$
\text{Hence,} \quad G(s,0) = e^C = 1
$$

$$
\text{and,} \quad G(s,t) = e^{-\lambda t(1-s)} \qquad (2.7)
$$

which is the probability generating function of a Poisson distribution with parameter λt.

- **Waiting time distribution for a Poisson process**

Consider the following diagram.



Let T_n be the first time the population is of size n, that is, $X(T_n) = n$.

$$\begin{aligned} T_n < t &\implies X(T_n) \leq X(t), \quad \text{that is,} \quad n \leq X(t) \\ T_n = t &\implies X(T_n) = X(t), \quad \text{that is,} \quad n = X(t) \\ T_n \leq t &\implies X(T_n) \leq X(t) \implies n \leq X(t) \\ \text{therefore; } \text{Prob}(T_n \leq t) &= \text{Prob}(X(t) \geq X(T_n)) = \text{Prob}(X(t) \geq n) \end{aligned}$$

$$\begin{aligned} \text{Let} \quad F_n(t) &= \text{Prob}(T_n \leq t) \\ \text{Since} \quad p_n(t) &= \text{Prob}(X(t) = n), \quad \text{then,} \\ F_n(t) &= \text{Prob}(X(t) \geq n) \\ &= 1 - \text{Prob}(X(t) < n) \\ &= 1 - \text{Prob}(X(t) \leq n-1) \end{aligned}$$

$$F_n(t) = 1 - \sum_{j=0}^{n-1} p_j(t) \tag{2.8}$$

$$\text{and therefore,} \quad f_n(t) = \frac{d}{dt} F_n(t) = -\sum_{j=0}^{n-1} \frac{d}{dt} p_j(t) \tag{2.9}$$

For a Poisson process;

$$\begin{aligned} F_n(t) &= 1 - \sum_{j=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \\ \text{and thus,} \quad f_n(t) &= -\sum_{j=0}^{n-1} \frac{1}{j!} \frac{d}{dt} e^{-\lambda t}(\lambda t)^j \\ &= -\sum_{j=0}^{n-1} \frac{1}{j!} \left(e^{-\lambda t} j(\lambda t)^{j-1}\lambda - \lambda e^{-\lambda t}(\lambda t)^j \right) \\ &= \lambda e^{-\lambda t} \left(\sum_{j=0}^{n-1} \frac{(\lambda t)^j}{j!} - \sum_{j=0}^{n-1} \frac{(\lambda t)^{j-1}}{(j-1)!} \right) \\ &= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} = \frac{\lambda^n}{\Gamma(n)} e^{-\lambda t} t^{n-1}, \quad t > 0; \lambda > 0, n = 1, 2, 3, .. \end{aligned} \tag{2.10}$$

which is an Erlang (n, λ) distribution.

- **The first passage time distribution for a mixed Poisson Process**
  For a mixed Poisson process where n is fixed and λ is varying, the first passage time distribution is thus given by;

$$f_n(t) = \int_0^\infty \frac{\lambda^n}{\Gamma(n)} e^{-\lambda t} t^{n-1} g(\lambda) d\lambda \tag{2.11}$$

where $g(\lambda)$ is a continuous mixing distribution. This is an Erlang mixture which can be expressed in two ways, namely;

The **direct method**, which is given by;

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}}{\Gamma(n)} \int_0^\infty \lambda^n e^{-\lambda t} g(\lambda) d\lambda \\ &= \frac{t^{n-1}}{\Gamma(n)} E\left(\Lambda^n e^{-t\Lambda}\right) \end{aligned} \tag{2.12}$$

and the **method of moments**, which can be obtained from the direct method as illustrated below.

$$\begin{aligned} f_n(t) &= \frac{t^{n-1}}{\Gamma(n)} E\left(\Lambda^n e^{-t\Lambda}\right) \\ &= \frac{t^{n-1}}{\Gamma(n)} E\left(\Lambda^n \sum_{k=0}^{\infty} \frac{(-\Lambda t)^k}{k!}\right) \\ &= \frac{t^{n-1}}{\Gamma(n)} \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} E\left(\Lambda^{n+k}\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k t^{n+k-1}}{k!\Gamma(n)} E\left(\Lambda^{n+k}\right) \\ let \quad n+k &= j \quad \Longrightarrow \; k = j-n \\ f_n(t) &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma(n)(j-n)!} E(\Lambda^j) \end{aligned} \tag{2.13}$$

Equating (2.12) and (2.13) we obtain the mathematical identity;

$$\begin{aligned} \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma(n)(j-n)!} E(\Lambda^j) &= \frac{t^{n-1}}{\Gamma(n)} E\left(\Lambda^n e^{-t\Lambda}\right) \\ \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} E(\Lambda^j) &= E\left(\Lambda^n e^{-t\Lambda}\right) \end{aligned} \tag{2.14}$$

which has been proven below.

$$\begin{aligned} &\text{let} \quad j-n=k \quad \Longrightarrow \quad j=n+k \\ \sum_{j-n=0}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} E(\Lambda^j) &= \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} E(\Lambda^{n+k}) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} E(\Lambda^{n+k}) \\ &= E\left(\Lambda^n \sum_{k=0}^{\infty} \frac{(-\Lambda t)^k}{k!}\right) = E\left(\Lambda^n e^{-t\Lambda}\right) \end{aligned} \tag{2.15}$$

The r^{th} moment of the Erlang mixture is given by;

$$\begin{aligned}
E(T^r) &= EE(T^r|\Lambda = \lambda), \quad \text{using conditional expectation} \\
&= E\int_0^\infty t^r f_n(t|\lambda)dt \\
&= E\int_0^\infty t^r \frac{\lambda^n}{\Gamma(n)} e^{-\lambda t} t^{n-1} dt \\
&= E\left(\frac{\lambda^n}{\Gamma(n)}\int_0^\infty t^{n+r-1} e^{-\lambda t} dt\right) \\
&= E\left(\frac{\lambda^n}{\Gamma(n)} \frac{\Gamma(n+r)}{\lambda^{n+r}}\right) = \frac{\Gamma(n+r)}{\Gamma(n)} E\left(\Lambda^{-r}\right) \qquad (2.16)
\end{aligned}$$

Thus, the r^{th} moment of the Erlang mixture has been expressed in terms of the r^{th} moment of the reciprocal of the mixing distribution.

3. The connection between Erlang mixtures and exponential mixtures

The Erlang distribution is a sum of n independent exponential random variables, each with parameter λ, that is, if $X_i \sim$ exponential(λ), then $\sum_{i=1}^n X_i \sim$ Erlang (n, λ). Therefore the exponential mixture is a special case of the Erlang mixture when $n = 1$, as illustrated below.

$$\begin{aligned}
f_n(t) &= \int_0^\infty \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} g(\lambda) d\lambda \\
f_1(t) &= \int_0^\infty \lambda e^{-\lambda t} g(\lambda) d\lambda \qquad (3.1)
\end{aligned}$$

which is the exponential mixture, and can be expressed, using the direct method, as;

$$f_1(t) = E\left(\Lambda e^{-t\Lambda}\right) \qquad (3.2)$$

and, using the method of moments, as

$$f_1(t) = \sum_{j=1}^\infty \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\Lambda^j) \qquad (3.3)$$

The identity is therefore;

$$\sum_{j=1}^\infty \frac{(-1)^{j-1} t^{j-1}}{(j-1)!} E(\Lambda^j) = E\left(\Lambda e^{-t\Lambda}\right) \qquad (3.4)$$

The r^{th} moment is $E(T^r) = r!E\left(\Lambda^{-r}\right)$ and the first moment is thus $E(T) = E\left(\Lambda^{-1}\right)$.

4. The connection Between Erlang mixtures and Poisson mixtures

The Erlang distribution is related to the Poisson distribution through the Poisson process, as shown in section two above. The Poisson mixture is $\frac{t}{n}$ times the Erlang mixture, as demonstrated below.

$$\begin{aligned} f_n(t) &= \int_0^\infty \frac{\lambda^n}{\Gamma(n)} e^{-\lambda t} t^{n-1} g(\lambda) d\lambda \\ &= \frac{n}{t} \int_0^\infty \frac{(\lambda t)^n e^{-\lambda t}}{\Gamma(n+1)} g(\lambda) d\lambda \\ &= \frac{n}{t} \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^n}{n!} g(\lambda) d\lambda = \frac{n}{t} p_n(t) \end{aligned} \tag{4.1}$$

where $p_n(t) = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!} g(\lambda) d\lambda$ is a continuous Poisson mixture.
Therefore, a Poisson mixture is $\frac{t}{n}$ times an Erlang mixture, that is;

$$p_n(t) = \frac{t}{n} f_n(t), \quad n = 1, 2, 3, ... \tag{4.2}$$

The factor $\frac{t}{n}$ transforms a continuous distribution to a discrete distribution.
The Poisson mixture $p_n(t)$ can be expressed, using the direct method, as;

$$\begin{aligned} p_n(t) &= \frac{t}{n} \frac{t^{n-1}}{\Gamma n} E\left(\Lambda^n e^{-t\Lambda}\right) \\ &= \frac{t^n}{n!} E\left(\Lambda^n e^{-t\Lambda}\right) \end{aligned} \tag{4.3}$$

and, using the method of moments, as

$$\begin{aligned} p_n(t) &= \frac{t}{n} \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma n (j-n)!} E(\Lambda^j) \\ &= \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\Lambda^j) \end{aligned} \tag{4.4}$$

The identity, from equating the two methods, is;

$$\begin{aligned} \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^j}{n!(j-n)!} E(\Lambda^j) &= \frac{t^n}{n!} E\left(\Lambda^n e^{-t\Lambda}\right) \\ \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} E(\Lambda^j) &= E\left(\Lambda^n e^{-t\Lambda}\right) \end{aligned} \tag{4.5}$$

which is the same as equation (2.14).
The probability generating function (PGF) of the Poisson mixture is;

$$\begin{aligned}
G(s,t) &= \sum_{n=0}^{\infty} p_n(t)S^n = \sum_{n=0}^{\infty}\left(\frac{t}{n}f_n(t)\right)S^n \\
&= \sum_{n=0}^{\infty}\left(\frac{t}{n}\int_0^{\infty}\frac{\lambda^n}{\Gamma n}e^{-\lambda t}t^{n-1}g(\lambda)d\lambda\right)S^n = \sum_{n=0}^{\infty}\int_0^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}S^n g(\lambda)d\lambda \\
&= \int_0^{\infty}e^{-\lambda t}\left(\sum_{n=0}^{\infty}\frac{(\lambda t S)^n}{n!}\right)g(\lambda)d\lambda = \int_0^{\infty}e^{-\lambda t}e^{\lambda t S}g(\lambda)d\lambda \\
&= \int_0^{\infty}e^{-(1-s)\lambda t}g(\lambda)d\lambda = E\left({}^{-(1-s)t\wedge}\right) = L_\Lambda[(1-s)t] \qquad (4.6)
\end{aligned}$$

$$\frac{\delta G}{\delta S} = \frac{\delta}{\delta S}E[e^{-t\Lambda}e^{t\Lambda S}] = E[t\Lambda e^{-t\Lambda}e^{t\Lambda S}] \qquad (4.7)$$

$$\frac{\delta^2 G}{\delta S^2} = E[(t\Lambda)^2 e^{-t\Lambda}e^{t\Lambda S}] \qquad (4.8)$$

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$$\frac{\delta^r G}{\delta S^r} = E[(t\Lambda)^r e^{-t\Lambda}e^{t\Lambda S}] \qquad (4.9)$$

at s=1, $$\frac{\delta^r G(s,t)}{\delta S^r} = E[t^r\Lambda^r] \qquad (4.10)$$

which is equal to the r^{th} factorial moment, $E[X(X-1)(X-2)....(X-r+1)] = t^r E(\Lambda^r), \quad r = 1,2,3,...,$ and thus $E(X) = tE(\Lambda)$.

Remark: We notice that the key unifying function in this work is $E[\Lambda^n e^{-t\Lambda}]$, which we can obtain for a given mixing distribution $g(\lambda)$, then deduce the following special cases.

i. $E[\Lambda^j]$ when $n = j$ and $t = 0$

ii. $E[\Lambda^r]$ when $n = r$ and $t = 0$

iii. $E[\Lambda^{-r}]$ when $n = -r$ and $t = 0$

iv. $E[\Lambda e^{-t\Lambda}]$ when $n = 1$

v. $E[e^{-(1-s)t\Lambda}]$ when $n = 0$ and $t = (1-s)t$

5. Erlang-Type II gamma Mixture

The Type II gamma mixing distribution is

$$g(\lambda) = \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{\frac{-\lambda}{\beta}} \lambda^{\alpha-1}, \quad \lambda > 0; \beta > 0, \alpha > 0 \tag{5.1}$$

$$\text{and hence;} \quad E\left(\Lambda^n e^{-t\Lambda}\right) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)} \left(\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}}\right)^\alpha \left(\frac{1}{t+\frac{1}{\beta}}\right)^n \tag{5.2}$$

(see Gathongo [3]).

a) Construction by the direct method results to;

$$f_n(t) = \frac{n}{t}\binom{\alpha+n-1}{n} \left(\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}}\right)^\alpha \left(\frac{t}{t+\frac{1}{\beta}}\right)^n, \quad t > 0; \beta > 0, \alpha > 0, n = 1, 2, 3, ... \tag{5.3}$$

b) By the method of moments we obtain;

$$f_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-1}}{\Gamma(n)(j-n)!} \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} \beta^j \tag{5.4}$$

c) Equating the above two methods gives the identity

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n} t^{j-n}}{(j-n)!} \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} \beta^j = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)} \left(\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}}\right)^\alpha \left(\frac{1}{t+\frac{1}{\beta}}\right)^n \tag{5.5}$$

d) The r^{th} moment of the Erlang mixture is

$$E(T^r) = \frac{\Gamma(n+r)}{\Gamma(n)} \frac{\Gamma(\alpha-r)}{\Gamma(\alpha)} \frac{1}{\beta^r} \tag{5.6}$$

and the mean is thus

$$E(T) = \frac{n}{\beta(\alpha-1)} \tag{5.7}$$

6. Exponential-Type II gamma Mixture

a) Construction by the direct method gives;

$$f_1(t) = \frac{\alpha\left(\frac{1}{\beta}\right)^\alpha}{\left(t+\frac{1}{\beta}\right)^{\alpha+1}}, \quad t > 0; \alpha > 0, \beta > 0 \tag{6.1}$$

which is the Lomax distribution with parameters $\left(\alpha, \frac{1}{\beta}\right)$. (see Wakoli [16]).

b) By the method of moments we have;

$$f_1(t) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}t^{j-1}}{(j-1)!} \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)}\beta^j \tag{6.2}$$

c) Equating the above two methods yields the identity

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}t^{j-1}}{(j-1)!} \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)}\beta^j = \frac{\alpha\left(\frac{1}{\beta}\right)^{\alpha}}{\left(t+\frac{1}{\beta}\right)^{\alpha+1}} \tag{6.3}$$

d) The r^{th} moment of the exponential mixture is

$$E(T^r) = \frac{r!}{\beta^r} \frac{\Gamma(\alpha-r)}{\Gamma(\alpha)} \tag{6.4}$$

and the mean is therefore

$$E(T) = \frac{1}{\beta(\alpha-1)} \tag{6.5}$$

7. Poisson-Type II Gamma Mixture

a) Construction by the direct method yields;

$$p_n(t) = \binom{\alpha+n-1}{n}\left(\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}}\right)^{\alpha}\left(\frac{t}{t+\frac{1}{\beta}}\right)^{n}, \quad t>0; \alpha>0, \beta>0 \tag{7.1}$$

which is the negative binomial distribution with parameters α and $\frac{1}{\beta}$. (see Sarguta [15])

b) By the method of moments we obtain;

$$p_n(t) = \sum_{j=n}^{\infty} \frac{(-1)^{j-n}t^j}{n!(j-n)!}\beta^j \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} \tag{7.2}$$

c) Equating the above two methods results in the identity

$$\sum_{j=n}^{\infty} \frac{(-1)^{j-n}t^{j-n}}{(j-n)!}\beta^j\Gamma(\alpha+j) = \Gamma(n+\alpha)\left(\frac{\frac{1}{\beta}}{t+\frac{1}{\beta}}\right)^{\alpha}\left(\frac{1}{t+\frac{1}{\beta}}\right)^{n} \tag{7.3}$$

d) The probability generating function of the Poisson mixture is

$$G(s,t) = \left(\frac{\frac{1}{\beta}}{\frac{1}{\beta} + (1-s)t} \right)^{\alpha} \tag{7.4}$$

e) The r^{th} moment of the Poisson mixture is

$$E(T^r) = (t\beta)^r \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)} \tag{7.5}$$

and the mean is

$$E(T) = \alpha t \beta \tag{7.5}$$

8. Conclusion

This study provides a unique method of obtaining Erlang mixtures from a mixed Poisson process. The study solved the fundamental difference-differential equations for a Poisson process to obtain the Poisson distribution. The waiting time distribution in a Poisson process is demonstrated as the Erlang distribution. The study also presented the Erlang mixture as the first passage time distribution in a mixed Poisson process and expressed it using both the direct method and the method of moments. Further, these two methods were equated to deduce a mathematical identity. The exponential mixture and Poisson mixture are illustrated as special cases of the Erlang mixture. A practical example, using mixtures of Type II gamma distribution, is provided. Properties of the mixtures, such as the r^{th} raw moment and probability generating function, are analyzed.

This study recommends further research into the diverse applications of mixtures and distributions derived from the Poisson process, such as the Erlang and the mixed Erlang distributions. These include, enhancing service efficiency in telecommunications and customer support, improving reliability and maintenance in engineering, optimizing financial risk assessment, analyzing patient survival and treatment effectiveness in healthcare, refining production schedules and inventory management in manufacturing, and developing better environmental monitoring and disaster response strategies among others. In addition, exploring the links between these mixtures and other distributions can lead to more effective models and solutions across various fields.

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