Analysis of the Hilbertian Properties of the Sobolev Space $H(\Omega)$

Abstract

This paper thoroughly examines the Hilbertian properties of the Sobolev space $H^1(\Omega)$. Based on the Lebesgue space $L^2(\Omega)$, $H^1(\Omega)$ is of particular importance in the analysis of functions with weak derivatives. The focus is on the Hilbertian structure of this space, which allows for the definition of a specific inner product and a rigorous verification of the associated completeness. These fundamental characteristics facilitate a detailed study of convergence, continuity, and orthogonality of functions in $H^1(\Omega)$, thereby enhancing its relevance for solving various complex mathematical problems.

This paper aims to address gaps identified in the existing literature, where the explicit verification of these essential properties is sometimes omitted, thus compromising mathematical rigor and the applicability of results in various contexts.

Keywords :Hilbert space, Sobolev space $H^1(\Omega)$,Lebesgue space $L^2(\Omega)$,distribution space $\mathcal{D}'(\Omega)$,inner product, completeness, convergence and literature review.

Introduction

The Sobolev space $\boldsymbol{H}(\boldsymbol{\Omega})$ stands out as a functional framework intrinsically linked to the Lebesgue space $L^2(\Omega)$, making it ideal for modeling functions with weak derivatives. However, its full potential is only realized when endowed with a Hilbertian structure. This particular structuring endows $\boldsymbol{H}(\boldsymbol{\Omega})$ with fundamental properties inherited from classical Hilbert spaces, such as a rigorous definition of an inner product and the completeness of the associated norm.

According to the extensive literature on the Sobolev spaces $\boldsymbol{H}(\boldsymbol{\Omega})$, notably documented in references [1] and [3], it is common for some essential aspects of the norm and inner product to be inadequately verified. Such omissions compromise mathematical rigor and restrict the applicability of results in various scientific and technical fields. We have observed that the authors of references [1] and [3], while stating properties often taken for granted, frequently neglect to provide a detailed and exhaustive proof of the important aspects of $\boldsymbol{H}(\boldsymbol{\Omega})$, particularly concerning the thorough verification of norms and inner products.

In light of these observations, this paper aims to address this oversight by focusing specifically on the rigorous verification of the fundamental properties of norms, inner products, and completeness in Sobolev spaces $\boldsymbol{H}(\boldsymbol{\Omega})$. This approach seeks to clarify and deepen the understanding of Hilbertian structures within these functional spaces, thereby establishing a solid theoretical foundation for their use and development in a variety of applied and fundamental research contexts.

Sobolev spaces $\boldsymbol{H}(\Omega)$

Space$\mathcal{D}'(\Omega)$

Definition 1.1 . Let be Ωan open de $\mathbb{R}^n$ and $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support. We call distribution Ton Ωany linear and continuous form on $\mathcal{D}(\Omega)$.

$(i)$$T$is linear if : an application $T$of $\mathcal{D}(\Omega)$in $\mathbb{R}(ou\ \mathbb{C})$corresponds to a function $\varphi \in \mathcal{D}(\Omega)$, a number noted $\langle T,\varphi\rangle$such that , for all $\varphi_1,\varphi_2 \in \mathcal{D}(\Omega)$and $\alpha,\beta \in \mathbb{C}$, we have:
$\langle T,\alpha\varphi_1+\beta\varphi_2\rangle = \alpha\langle T,\varphi_1\rangle + \beta\langle T,\varphi_2\rangle$.

$(ii)$$T$is continuous if : the sequence (φ_k)converges in $\mathcal{D}(\Omega)$towards φ, then the sequence $(\langle T,\varphi_k\rangle)$converges in the usual sense towards $\langle T,\varphi\rangle$.

We thus designate the space of distributions on$\Omega$by$\mathcal{D}'(\Omega)$, which is the topological dual of the space $\mathcal{D}(\Omega)$.

Proposition 1.2. The space $\mathcal{D}'(\Omega)$is a Banach space.

Evidence. $\mathcal{D}'(\Omega)$being a topological dual of$\mathcal{D}(\Omega)$,it follows that, $\mathcal{D}'(\Omega)$becomes a Banach space.■

Definition 1.3. We call the derivative Tofa distribution $T$the linear and continuous form defined by:$\langle T',\varphi\rangle = -\langle T,\varphi'\rangle, \forall\varphi \in \mathcal{D}\ \ (\Omega)$

In general, the order derivative $n$of the distribution $T$is defined by the relation:

$\langle T^{(n)},\varphi\rangle = (-1)^n\langle T,\varphi^{(n)}\rangle, \forall\varphi \in \mathcal{D}\ \ (\Omega)$.

Proposition 1.4. If $(T_k)$is a sequence of distributions converging$\mathcal{D}'(\Omega)$ to the distribution T,then the sequence$\left(\frac{\partial T_k}{\partial x_i}\right)$ converges in $\mathcal{D}'(\Omega)$to $\frac{\partial T}{\partial x_i}$.

Proof.[6, page 66]■

Space$L^2(\Omega)$

Definition 2.1. Let be Ωan open of $\mathbb{R}^n$. We define $L^2(\Omega)$as the set of functions $f:\Omega \to \mathbb{R}$ such that $f$is integrable and we set:

$$L^2(\Omega) = \left\{f:\Omega \to \mathbb{R}: f \mid \int_\Omega |f(x)|^2 dx < +\infty\right\}$$

Definition 2.2. Scalar product and norm

We provide $L^2(\Omega)$the scalar product $\langle .,.\rangle_{L^2(\Omega)}$and the norm $\|.\|_{L^2(\Omega)}$defined by:

$$\langle f,g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx, \forall (f,g) \in [L^2(\Omega)]^2$$

and

$$\|f\|_{L^2(\Omega)} = \sqrt{\langle f,f\rangle_{L^2(\Omega)}} = \left(\int_\Omega f^2(x)dx\right)^{1/2}, \forall f \in L^2(\Omega)$$

Proposition 2.3. The space $\left(L^2(\Omega), \langle .\,,.\rangle_{L^2(\Omega)}\right)$is a Euclidean space.

Proof. It is enough to show that $\langle .\,,.\rangle_{L^2(\Omega)}$is a dot product on $L^2(\Omega)$.

(i). Bilinearity

For all $f,g,h \in L^2(\Omega)$andfor all$a,b \in \mathbb{R}$, we have :

$$\langle af+bg,h\rangle_{L^2(\Omega)} = \int_\Omega [af(x)+bg(x)]h(x)dx$$

$$= \int_\Omega [af(x)h(x)+bg(x)h(x)]dx$$

$$= a\int_\Omega f(x)h(x)dx + b\int_\Omega g(x)h(x)dx$$

$$= a\langle f,h\rangle_{L^2(\Omega)} + b\langle g,h\rangle_{L^2(\Omega)}$$

Andalso:

$$\langle f,ag+bh\rangle_{L^2(\Omega)} = \int_\Omega f(x)[ag(x)+bh(x)]dx$$

$$= \int_\Omega [af(x)g(x)+bf(x)h(x)]dx$$

$$= a\langle f,g\rangle_{L^2(\Omega)} + b\langle f,h\rangle_{L^2(\Omega)}$$

This proves the bilinearity of$\langle .\,,.\rangle_{L^2(\Omega)}$

(ii). Symmetry of$\langle .\,,.\rangle_{L^2(\Omega)}$

For all$f,g \in L^2(\Omega)$,we have :

$$\langle f,g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx$$

$$= \int_\Omega g(x)h(x)dx$$

$$= \langle g,f\rangle_{L^2(\Omega)}$$

So, the symmetry is verified .

(iii).Positivity defined

For all $f \in L^2(\Omega)$, we have :

$$\langle f, f\rangle_{L^2(\Omega)} = \int_\Omega [f(x)]^2 dx$$

As $[f(x)]^2$ is always positive or zero , the integral

$\int_\Omega [f(x)]^2 dx \geq 0$. So, $\langle f, f\rangle_{L^2(\Omega)} \geq 0$.

Moreover,if $\langle f, f\rangle_{L^2(\Omega)} = 0$,then $[f(x)]^2 = 0$ almost everywhere on Ω.This implies that $f(x) = 0$ almost everywhere on Ω. So,

$\langle f, f\rangle_{L^2(\Omega)} = 0 \Leftrightarrow f = 0$ almost everywhere on Ω. Therefore, the defined positivity is verified.

Under (i); (ii) and (iii), space $\left(L^2(\Omega), \langle .,.\rangle_{L^2(\Omega)}\right)$has a Euclidean space structure.■

Proposition 2.4. The space $\left(L^2(\Omega), \|.\|_{L^2(\Omega)}\right)$is a Banach space.

Proof. Showing that $\left(L^2(\Omega), \|.\|_{L^2(\Omega)}\right)$is a Banach space amounts to verifying the following two axioms:

(i)Completude de$L^2(\Omega)$

We show that each Cauchy sequence in $L^2(\Omega)$converges to an element of $L^2(\Omega)$. Let it be (f_n) a Cauchy sequence in $L^2(\Omega)$ .This means that for all$\varepsilon > 0$, il exists$k \in \mathbb{N}$ such thatfor all m, $n \geq k$,

$$\|f_m - f_n\|_{L^2(\Omega)} \leq \varepsilon$$

As $(f_n)$is a Cauchy sequence, it is uniformly convergent almost everywhere to a function f.This implies that the sequence $(f_n(x))$converges almost everywhere to $f(x)$when $n$tends to $+\infty$.

Let us show that $f$is in $L^2(\Omega)$. Since $\|f_n - f\|_{L^2(\Omega)} \longrightarrow 0$, when $n$tends to $+\infty$,then (f_n)also converges to $f$norm $L^2(\Omega)$,i.e.

$$\lim_{n\to+\infty} \int_\Omega |f_n(x) - f(x)|^2 dx = 0.$$

Therefore, $f$is in $L^2(\Omega)$, and therefore, $L^2(\Omega)$is a complete vector space.

(ii)Completude deStandard$\|.\|_{L^2(\Omega)}$

It suffices to show that any Cauchy sequence (f_n)in $L^2(\Omega)$converges in $L^2(\Omega)$under this norm. As we have already established, any Cauchy sequence (f_n) in $L^2(\Omega)$ converges uniformly almost everywhere on $\Omega$to a function f, and also converges on $f$to norm $L^2(\Omega)$.

Thus, the standard $\|.\|_{L^2(\Omega)}$on$L^2(\Omega)$is complete.

In conclusion, $\left(L^2(\Omega), \|.\|_{L^2(\Omega)}\right)$is a Banach space.■

Space$H^1(\Omega)$

Definition 3.1. Given Ωan open de $\mathbb{R}^2$, we define the Sobolev space $H^1(\Omega)$by:

$$H^1(\Omega) = \left\{f \in L^2(\Omega): \frac{\partial f}{\partial x_i} \in L^2(\Omega), \forall i = 1,2\right\}$$

With$\frac{\partial f}{\partial x_i}$partial derivatives of $f$taken in the sense of distributions defined by:

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i} dx_1 dx_2 = -\int_\Omega \frac{\partial f}{\partial x_i} \varphi dx_1 dx_2, \forall \varphi \in \mathcal{D}(\Omega)$$

Definition 3.2. The scalar product $\langle .,. \rangle_{H^1(\Omega)}$and the norm $\|.\|_{H^1(\Omega)}$on $H^1(\Omega)$are defined by:

$$\langle f, g \rangle_{H^1(\Omega)} = \int_\Omega \left(fg + \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_2}\right) dx_1 dx_2$$

For all $(f, g) \in [H^1(\Omega)]^2$

and

$$\|f\|_{H^1(\Omega)} = \left(\|f\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_1}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_2}\right\|^2_{L^2(\Omega)}\right)^{1/2}$$

Proposition 3.3. The space $\left(H^1(\Omega), \langle .,. \rangle_{H^1(\Omega)}\right)$is a Euclidean space.

Proof. The space $\left(H^1(\Omega), \langle .,. \rangle_{H^1(\Omega)}\right)$is a Euclidean space if $\langle .,. \rangle_{H^1(\Omega)}$is a dot product on $H^1(\Omega)$x $H^1(\Omega)$. A dot product is a positive definite symmetric bilinear form.

(i).Bilinearity of$\langle .,. \rangle_{H^1(\Omega)}$

For all $f_1, f_2, g_1, g_2 \in H^1(\Omega)$, for all $a, b, c, d \in \mathbb{R}$, we have:

$$\begin{aligned}\langle af_1 + bf_2, cg_1 + dg_2 \rangle_{H^1(\Omega)} \\ = ac\langle f_1, g_1 \rangle_{H^1(\Omega)} + ad\langle f_1, g_2 \rangle_{H^1(\Omega)} + bc\langle f_2, g_1 \rangle_{H^1(\Omega)} + bd\langle f_2, g_2 \rangle_{H^1(\Omega)}\end{aligned}$$

$$= \int_{\Omega} \Big[(af_1 + bf_2)(cg_1 + dg_2) + \frac{\partial}{\partial x_1}(af_1 + bf_2)\frac{\partial}{\partial x_1}(ag_1 + dg_2) + \frac{\partial}{\partial x_2}(af_1 + bf_2)\frac{\partial}{\partial x_2}(cg_1 + dg_2)\Big] dx_1 dx_2.$$

$$= \int_{\Omega} \Big[acf_1g_1 + adf_1g_2 + bcf_2g_1 + bdf_2g_2 + ac\frac{\partial f_1}{\partial x_1}\frac{\partial g_1}{a x_1} + ad\frac{\partial f_1}{\partial x_1}\frac{\partial g_2}{\partial x_1} + bc\ \frac{\partial f_2}{\partial x_1}\frac{\partial g_1}{\partial x_1} + bd\ \frac{\partial f_2}{\partial x_1}.\frac{\partial g_2}{\partial x_1} + ac\frac{\partial f_1}{\partial x_2}\frac{\partial g_1}{\partial x_2} + ad\frac{\partial f_1}{\partial x_2}\frac{\partial g_2}{\partial x_2} + bc\ \frac{\partial f_2}{\partial x_2}\frac{\partial g_1}{\partial x_2} + bd\ \frac{\partial f_2}{\partial x_2}.\frac{\partial g_2}{\partial x_2}\Big] dx_1 dx_2.$$

$$= ac\int_{\Omega} \left(f_1g_1 + \frac{\partial f_1}{\partial x_1}\frac{\partial g_1}{\partial x_1} + \frac{\partial f_1}{\partial x_2}\frac{\partial g_1}{\partial x_2}\right) dx_1 dx_2$$

$$+ad\int_{\Omega} \left(f_1g_2 + \frac{\partial f_1}{\partial x_1}\frac{\partial g_2}{\partial x_1} + \frac{\partial f_1}{\partial x_2}\frac{\partial g_2}{\partial x_2}\right) dx_1 dx_2$$

$$+bc\int_{\Omega} \left(f_2g_1 + \frac{\partial f_2}{\partial x_1}\frac{\partial g_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\frac{\partial g_1}{\partial x_2}\right) dx_1 dx_2$$

$$+bd\int_{\Omega} \left(f_2g_2 + \frac{\partial f_2}{\partial x_1}\frac{\partial g_2}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\frac{\partial g_2}{\partial x_2}\right) dx_1 dx_2$$

$$= ac\langle f_1, g_1\rangle_{H^1(\Omega)} + ad\langle f_1, g_2\rangle_{H^1(\Omega)} + bc\langle f_2, g_1\rangle_{H^1(\Omega)} + bd\langle f_2, g_2\rangle_{H^1(\Omega)}$$

Bilinearityis verified.

(*ii*).Symmetry of$\langle .,.\rangle_{H^1(\Omega)}$

For all $f, g \in H^1(\Omega)$,we have :

$$\langle f, g\rangle_{H^1(\Omega)} = \int_{\Omega} \left(fg + \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_2}\right) dx_1 dx_2$$

$$= \int_{\Omega} \left(gf + \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_1} + \frac{\partial g}{\partial x_2}\frac{\partial f}{\partial x_2}\right) dx_1 dx_2$$

$$= \langle g, f\rangle_{H^1(\Omega)}$$

This ensures the symmetry of$\langle .,.\rangle_{H^1(\Omega)}$.

(*iii*). Positivity defined

for all $f \in H^1(\Omega)$, we have :

$$\langle f, f\rangle_{H^1(\Omega)} = \int_{\Omega} \left(f^2 + {\frac{\partial f}{\partial x_1}}^{2} + {\frac{\partial f}{\partial x_2}}^{2}\right) dx_1 dx_2$$

The quantity $\int_\Omega \left(f^2 + \frac{\partial f}{\partial x_1}^2 + \frac{\partial f}{\partial x_2}^2\right) dx_1 dx_2$ is always non-negative and is zero only if $f = 0$ almost everywhere on Ω. Therefore, because$\langle f, f\rangle_{H^1(\Omega)} \geq 0$ what $\langle f, f\rangle_{H^1(\Omega)} = 0$ if is only if $f = 0$.

We therefore conclude that $\langle .,.\rangle_{H^1(\Omega)}$ is a dot product on $H^1(\Omega)$, which proves that $\left(H^1(\Omega), \langle .,.\rangle_{H^1(\Omega)}\right)$is a Euclidean space.■

Noticed. The norm $\|.\|_{H^1(\Omega)}$on $H^1(\Omega)$is also called the Sobolev norm and is equivalent to the norm $L^2(\Omega)$on this space.

Proposition 3.4. The space $\left(H^1(\Omega), \|.\|_{H^1(\Omega)}\right)$is a standardized space.

Proof. To show that$\left(H^1(\Omega), \|.\|_{H^1(\Omega)}\right)$is a norm space, it suffices to prove that $\|.\|_{H^1(\Omega)}$is a norm on $H^1(\Omega)$.

The standard $\|.\|_{H^1(\Omega)}$is defined by:

$$\|f\|_{H^1(\Omega)} = \left(\|f\|_{L^2(\Omega)}^2 + \left\|\frac{\partial f}{\partial x_1}\right\|_{L^2(\Omega)}^2 + \left\|\frac{\partial f}{\partial x_2}\right\|_{L^2(\Omega)}^2\right)^{1/2}$$

We must verify the following three properties for it $\|.\|_{H^1(\Omega)}$to be a norm.

(i). **Separation**

$$\|f\|_{H^1(\Omega)} = 0 \text{ if and only if } f = 0 \text{ almost everywhere on } \Omega.$$

if$\|f\|_{H^1(\Omega)} = 0$, then $\left(\|f\|_{L^2(\Omega)}^2 + \left\|\frac{\partial f}{\partial x_1}\right\|_{L^2(\Omega)}^2 + \left\|\frac{\partial f}{\partial x_2}\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} = 0.$

The square root of a positive radicand is zero when the radicand is zero. So :

$\|f\|_{L^2(\Omega)}^2 = 0$ alors $f = 0$almost everywhere onΩ,

$\left\|\frac{\partial f}{\partial x_1}\right\|_{L^2(\Omega)}^2 = 0$ so $\left\|\frac{\partial f}{\partial x_2}\right\|_{L^2(\Omega)}^2 = 0$ what$\frac{\partial f}{\partial x_1} = 0$almost everywhere on$\Omega$

and

$$\frac{\partial f}{\partial x_2} = 0 \text{almost everywhere on} \Omega.$$

So,$f = 0$ almost everywhere on Ω.

(ii).**Homogeneity**

$$\|\lambda f\|_{H^1(\Omega)} = |\lambda| \|f\|_{H^1(\Omega)}, \text{ for } \lambda \in \mathbb{R}.$$

For λf:

$$\|\lambda f\|_{H^1(\Omega)} = \left(\|\lambda f\|_{L^2(\Omega)}^2 + \left\|\frac{\partial(\lambda f)}{\partial x_1}\right\|_{L^2(\Omega)}^2 + \left\|\frac{\partial(\lambda f)}{\partial x_2}\right\|_{L^2(\Omega)}^2\right)^{1/2}$$

$$= \left(\lambda^2\|f\|_{L^2(\Omega)}^2 + \lambda^2\left\|\frac{\partial f}{\partial x_1}\right\|_{L^2(\Omega)}^2 + \lambda^2\left\|\frac{\partial f}{\partial x_2}\right\|_{L^2(\Omega)}^2\right)^{1/2}$$

$$= |\lambda|\left(\|f\|_{L^2(\Omega)}^2 + \left\|\frac{\partial f}{\partial x_1}\right\|_{L^2(\Omega)}^2 + \left\|\frac{\partial f}{\partial x_2}\right\|_{L^2(\Omega)}^2\right)^{1/2}$$

$$= |\lambda|\|f\|_{H^1(\Omega)}$$

(iii). **Triangle inequality**

$\|f + g\|_{H^1(\Omega)} \leq \|f\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)}$, for $(f, g) \in [H^1(\Omega)]^2$

Let'scalculate:

$$\|f + g\|_{L^2(\Omega)}^2 = \|f + g\|_{L^2(\Omega)}^2 + \left\|\frac{\partial(f+g)}{\partial x_1}\right\|_{L^2(\Omega)}^2 + \left\|\frac{\partial(f+g)}{\partial x_2}\right\|_{L^2(\Omega)}^2$$

We have: $\|f + g\|_{L^2(\Omega)}^2 = \|f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Omega)}^2 + 2\int_\Omega fg$,

$$\left\|\frac{\partial(f+g)}{\partial x_1}\right\|_{L^2(\Omega)}^2 = \left\|\frac{\partial f}{\partial x_1} + \frac{\partial g}{\partial x_1}\right\|_{L^2(\Omega)}^2,$$

$$\left\|\frac{\partial(f+g)}{\partial x_2}\right\|_{L^2(\Omega)}^2 = \left\|\frac{\partial f}{\partial x_2} + \frac{\partial g}{\partial x_2}\right\|_{L^2(\Omega)}^2.$$

Using Minkowski's inequality for $L^2(\Omega)$, we obtain:

$$\left\|\frac{\partial f}{\partial x_1} + \frac{\partial g}{\partial x_1}\right\|_{L^2(\Omega)} \leq \left\|\frac{\partial f}{\partial x_1}\right\|_{L^2(\Omega)} + \left\|\frac{\partial g}{\partial x_1}\right\|_{L^2(\Omega)},$$

$$\left\|\frac{\partial f}{\partial x_2} + \frac{\partial g}{\partial x_2}\right\|_{L^2(\Omega)} \leq \left\|\frac{\partial f}{\partial x_2}\right\|_{L^2(\Omega)} + \left\|\frac{\partial g}{\partial x_2}\right\|_{L^2(\Omega)}.$$

So :

$$\|f + g\|_{L^2(\Omega)}^2 \leq \left(\|f\|_{L^2(\Omega)}^2 + \left\|\frac{\partial f}{\partial x_1}\right\|_{L^2(\Omega)}^2 + \left\|\frac{\partial f}{\partial x_2}\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} +$$

$$\left(\|g\|_{L^2(\Omega)}^2 + \left\|\frac{\partial g}{\partial x_1}\right\|_{L^2(\Omega)}^2 + \left\|\frac{\partial g}{\partial x_2}\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}$$

So :$\|f + g\|_{H^1(\Omega)} \leq \|f\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)}$

So, $\|.\|_{H^1(\Omega)}$is indeed a standard on $H^1(\Omega)$. Therefore, $\left(H^1(\Omega), \|.\|_{H^1(\Omega)}\right)$is a normalized space. ■

Proposition 3.5. Space $H^1(\Omega)$has a Hilbertian structure.

Proof. We will show that $H^1(\Omega)$is a Hilbert space for the norm$\|.\|_{H^1(\Omega)}$.

(i). According to Proposition 3.3, $\langle .,. \rangle_{H^1(\Omega)}$is a scalar product on$H^1(\Omega)$x $H^1(\Omega)$.

(ii). According to Proposition 3.4, $\|.\|_{H^1(\Omega)}$is a norm on $H^1(\Omega)$.

(iii). Let us show that $H^1(\Omega)$is complete for the norm$\|.\|_{H^1(\Omega)}$

Consider $(f_k) \in H^1(\Omega)$a Cauchy sequence for the norm $\|.\|_{H^1(\Omega)}$.

It means that :For all $\varepsilon > 0$, it exists$k \in \mathbb{N}$ such as for$n, m \geq k$,

$\|f_n - f_m\|_{H^1(\Omega)} \leq \varepsilon$.

That implies that :

$$\|f_n - f_m\|^2_{L^2(\Omega)} + \left\|\frac{\partial f_n}{\partial x_1} - \frac{\partial f_m}{\partial x_1}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial f_n}{\partial x_2} - \frac{\partial f_m}{\partial x_2}\right\|^2_{L^2(\Omega)} \leq \varepsilon^2$$

So :

$\|f_n - f_m\|^2_{L^2(\Omega)} \leq \varepsilon^2$,

$$\left\|\frac{\partial f_n}{\partial x_1} - \frac{\partial f_m}{\partial x_1}\right\|^2_{L^2(\Omega)} \leq \varepsilon^2,$$

$$\left\|\frac{\partial f_n}{\partial x_2} - \frac{\partial f_m}{\partial x_2}\right\|^2_{L^2(\Omega)} \leq \varepsilon^2.$$

Thus, $(f_k)$is a Cauchy sequence for the norm $\|.\|_{L^2(\Omega)}$and $\left(\frac{\partial f_k}{\partial x_i}\right)$is also Cauchy for the norm $\|.\|_{L^2(\Omega)}$for $i = 1,2$.

Now, $L^2(\Omega)$is complete for the standard $\|.\|_{L^2(\Omega)}$.

So :

It exists$f \in L^2(\Omega)$such as$\|f_k - f\|_{L^2(\Omega)}$tends to 0 when$k$tends to$+\infty$, It exists $g_i \in L^2(\Omega)$ such as $\left\| \frac{\partial f_k}{\partial x_i} - g_i \right\|_{L^2(\Omega)}$ tends to 0 when k tends to $+\infty$, for $i = 1{,}2$.

It remains to prove that (f_k) converges to $f \in H^1(\Omega)$ with:

$\|f_k - f\|_{H^1(\Omega)}$ tends to 0 when k tends to $+\infty$.

We have :

$$\int_\Omega f_k \frac{\partial \varphi}{\partial x_i} = -\int_\Omega \frac{\partial f_k}{\partial x_i} \varphi, \forall \varphi \in \mathcal{D}(\Omega), \forall k \in \mathbb{N}$$

By increasing the left side by the Cauchy-Schwarz inequality, we obtain:

$$\left| \int_\Omega f_k \frac{\partial \varphi}{\partial x_i} - \int_\Omega f \frac{\partial \varphi}{\partial x_i} \right| = \left| \int_\Omega (f_k - f) \frac{\partial \varphi}{\partial x_i} \right| \leq \|f_k - f\|_{L^2(\Omega)} \cdot \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^2(\Omega)}$$

As $\|f_k - f\|_{L^2(\Omega)}$ tends to wards 0 by completeness $L^2(\Omega)$, we have:

$\lim_{k \to +\infty} \int_\Omega f_k \frac{\partial \varphi}{\partial x_i} = \int_\Omega f \frac{\partial \varphi}{\partial x_i}$.

Likewise, we obtain:

$$\lim_{k \to +\infty} \int_\Omega \frac{\partial f_k}{\partial x_i} \varphi = \int_\Omega g_i \varphi$$

By passing to the limit, we obtain:

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi, \forall \varphi \in \mathcal{D}(\Omega), \text{for } i = 1{,}2$$

This implies that: $g_i = \frac{\partial f}{\partial x_i}$, where the derivative is taken in the sense of distributions.

So, $f \in H^1(\Omega)$.

In conclusion, (f_k) converges towards f in $L^2(\Omega)$ and $\left(\frac{\partial f_k}{\partial x_i}\right)$ converges towards $g_i = \frac{\partial f}{\partial x_i}$ in $L^2(\Omega)$. Furthermore:

$$\|f_k - f\|_{H^1(\Omega)}^2 = \|f_k - f\|_{L^2(\Omega)}^2 + \left\| \frac{\partial f_k}{\partial x_1} - \frac{\partial f}{\partial x_1} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial f_k}{\partial x_2} - \frac{\partial f}{\partial x_2} \right\|_{L^2(\Omega)}^2.$$

As $\|f_k - f\|_{H^1(\Omega)}^2$ tends to 0 when k tends to $+\infty$,

$\left\| \frac{\partial f_k}{\partial x_1} - \frac{\partial f}{\partial x_1} \right\|_{L^2(\Omega)}^2$ tends to 0 when k tends to $+\infty$,and

$\left\|\frac{\partial f_k}{\partial x_2}-\frac{\partial f}{\partial x_2}\right\|_{L^2(\Omega)}^2$ tends to 0 when $k$tends to $+\infty$,we have:

$\|f_k-f\|_{H^1(\Omega)}$tends to 0 when $k$tends to $+\infty$.

Therefore, (f_k)converges to f in $H^1(\Omega)$.This completes the proof that $H^1(\Omega)$is a Hilbert space. ∎

Conclusion

This article articulates the crucial importance of Hilbertian structure in Sobolevspace $H^1(\Omega)$. By introducing a specific dot product and rigorously verifying the completeness of the associated norm, we have deepened our understanding of the fundamental properties of this functional space. These properties, sometimes neglected in the existing literature documented by [1] and [3], are essential to guarantee the precision of mathematical results in various fields of application, notably in functional analysis and in solving partial differential equations. After identifying negligence in the verification of norms and dot products, this article focused on their clarification and rigorous verification in $H^1(\Omega)$, thus aiming to strengthen mathematical rigor and extend the practical applications of these results. This approach enriches our understanding of Hilbertian structures and encourages efficient use of Sobolev space $H^1(\Omega)$in mathematical investigations. This article opens stimulating perspectives for future research, encouraging further study of Hilbertian properties and their adaptation to solving complex and varied problems.

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