Re-investigation of the Foundation of Moment of Inertia by Means of Zero and Minus One Factorial.

[1]

Abstract

The objective of this paper is to re-investigate the foundation of the moment of inertia of bodies by means of a natural system that uses zero and the factorial of minus one in a consistent way. The method pursued is to intensively inquire into the concept of nothing and then employ the new ideas acquired in rebuilding a foundational understanding of moments of inertia.

keywords: Moment of inertia, Infinite summations, Nothing Zero, Minus one factorial

1 Introduction

The moment of inertia, a fundamental concept in mechanics, has been extensively studied and applied in various fields, including engineering, physics, and astronomy. However, the traditional understanding of this concept is based on a simplistic model that neglects the intricate nature of reality. The infinite summation inherent in the moment of inertia has long been a subject of concern, as it seems to contradict the finite values observed in physical systems.

The objective of this paper is to reexamine the foundation of the moment of inertia, addressing the long-standing paradox and providing a novel understanding of this fundamental concept. We aim to develop a new arithmetic framework that incorporates the concepts of zero and nothing, enabling the reconciliation of infinite summations with finite values.

Our approach is rooted in a rigorous analysis of the nature of infinity and the factorial of minus one. By exploring the nuances of zero and nothing, we

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establish a coherent arithmetic framework that facilitates the computation of infinite summations in a finite and meaningful way. This framework enables us to redefine the moment of inertia, providing a novel foundation for understanding the rotational behavior of objects.

The paper is structured as follows: Section 2 deals with the concepts of zero and nothing, while Section 3 explores the infinite number $(-1)!$. Section 4 develops the arithmetic of zero and $(-1)!$, and Section 5 applies this framework to the sum of powers of natural numbers. Finally, Section 6 presents a new foundation for moments of inertia, reconciling the infinite with the finite.

2 Zero and Absolute Nothing

The discussions of all the various undertakings to apprehend the essence of nothing would make a long narrative [1]. The subject of nothing was discussed in the works of the Indian mathematicians, Bramagupta and Bhaskara, but was taken seriously after the foundation of the edifice of the calculus was questioned, and two kinds of nothings, absolute nothing and zero denoted 0, were noticed.

In 1716 John Craig claimed that the number zero 0 cannot be the absolute nothing,

> . . . for infinite number of absolute nothings cannot make 1, but by 0 is understood an infinitely small part, as in the calc. diff. dx is an infinitely small part of x, so that dx is as 0 to x. Not that dx is absolutely nothing, for it is divisible into an infinite number of parts each of which is ddx.

To further elucidate his ideas, he continued:

> But then it may be inquired what is the quotient that arises from the division of 1 by absolute nothing. I say there is no quotient because there is no division. Therefore, it is a mistake to say the quotient is 1 or unity undivided which is demonstrably false, neither is the quotient = 0. For properly speaking there is no quotient and there it is an error to assign any.

Here is what was written in a great Dictionary of Mathematics. We have excerpted some alluring comment made by Davies and Peck from this Dictionary:

> It is in consequence of confounding the 0 arising from dividing a by infinity with the absolute nothing that so much confusion has been created in the discussions that have grown out of this subject.

The intellectual minds may find it difficult to make out a lucid distinction between the sources of the two kinds of nothings and so confound one for the other. The grave consequence is the entrance of paradoxes in the use of nothings in calculations. We must be careful, then, not to beg the question by assuming that the two nothings are the same. It is, therefore, worth our while to examine in detail and very carefully the difference between these nothings.

2.1 Absolute Nothing

Absolute nothing means the *complete absence of quantity*. In the simplification of expressions or equations, the expression $+a - a$ vanishes, i.e it ceases to exist, and what remains is absolute nothing; the minuend and the subtrahend cancel out each other. The symbol $+a$ implies add a and the symbol $-a$ take away a. Thus the expression $+a - a$ can be interpreted as 'add a and then take it away'. Frankly speaking, the expression $+a - a$ is unworthy of any symbolic representation. There is nothing existing initially indicating a blank until a is added. The addition of a is something introduced which is later removed by taking the a away. The system returns to its initial state of blank, the existence of nothing. Thus the expression $+a - a$ does not require a symbol for its representation.

The expression $(c+1)^2 - c^2 - 1$ which equals $c^2 + 2c + 1 - c^2 - 1$, becomes, collecting like terms, the expression $c^2 - c^2 + 2c + 1 - 1$. This expression is rewritten as $2c$. The omission of the expressions $c^2 - c^2$ and $1 - 1$ is justified because the minuends cancel the subtrahends and *vice versa*.

Consider the problem of solving the equation $2x - 10 = 3 + x$. To remove x from the right–hand side of the equation, add $-x$ to both sides. The equation becomes $2x - 10 - x = 3 + x - x$ which becomes $x - 10 = 3$. To remove -10 from the left–hand side, add $+10$ to both sides of the equation. Thus, we have $x - 10 + 10 = 3 + 10$ which becomes $x = 13$.

2.2 Zero

Zero is a number equivalent in value to *absolute nothing*. It may arise in the process of substituting a numerical value for a variable in a mathematical expression. For every constant a there is exactly one assignable number $-a$ of the variable x such that in the evaluation of $x+a$ at $x = -a$, we have $-a+a = 0$ where 0, called zero, is a numerical value equivalent to the absolute nothing. Similarly, for every constant $-a$ there is exactly one assignable number a of the variable x such that in the evaluation of $x - a$ at $x = a$, we have $a - a = 0$.

A more thorough mathematical study of 0 shows that new and really unexpected conclusions can be drawn. The 0 is merely a symbol standing for the **non-vanishing** expression $a - a$, the difference of a and itself. When we set $x = a$ in the difference $x - a$ we get the expression $a - a$, the difference of a and itself. The difference $a - a$ has a value equivalent to nothing but the expression itself *can never vanish*; the two a's are conserved. The ever existing nature of the expression stems from the fact that the two a' s do not cancel out each other. In fact, the two a's remain in the expression as they are; only their difference is conceived. For this reason, the expression $a - a$ requires a single symbol to represent it. Since the difference of the two a's is nothing we denote the expression with the usual symbol 0 for zero. Inasmuch 0 represents an existing or non-vanishing expression, it can divide itself, in which the result is unity, viz

$$\frac{0}{0} = 1.$$

If we interpret this 0 in the sense that is usually assigned to it, we come to the ridiculous conclusion that a *difference subtraction* is the same as a *takeaway subtraction.*

3 Minus One Factorial

3.1 Minus One Factorial and Zero as Reciprocals of Each Other

Directly associated with the understanding on zero is the fact about minus one factorial, written as $(-1)!$. The picture of 0 could not be complete without the description of $(-1)!$, its reciprocal. In the description of the reciprocal, we are led to a number which cannot find a complete fulfilment in any property of real numbers.

Let our appeal be to combinatorics. We compute as follows.

$$\begin{aligned} x+1 &= x+1 \\ &= \frac{(x+1)x!}{x!} \\ &= \frac{(x+1)!}{x!} \end{aligned}$$

Letting $x=-1$, we get

$$0 = \frac{0!}{(-1)!}.$$

Taking $0! = 1$ we write

$$0 = \frac{1}{(-1)!}$$

and hence

$$(-1)! = \frac{1}{0};$$

minus one factorial is the reciprocal of zero and *vice versa.*

3.2 Duality Property of Zero

From our last result, we give another definition to 0, namely

$$0 = \frac{1}{(-1)!}.$$

This is a very remarkable definition of zero, denoted 0. Zero is a value representing nothing yet it is equal to the quotient of something divided by something. The relation $0 = 1/(-1)!$ can be understood only as referring to the fact that 0 has two natures, its *nothing nature* seen in the whole number 0 and its *something nature* seen in its fractional number $1/(-1)!$.

The evaluation of the expression $x + 1$ at $x = -1$ makes it clear that 0 is equivalent in value to absolute nothing. It is not sufficient, however, to regard it as a mere nothing. If 0 is thought of as *absence of quantity*, then it is indeed hard to understand what is here meant by the result $0 = 1/(-1)!$. But if 0 is not absolute nothing; if it is an existent, then may we understand and with confidence hold to the vital fact that the fraction $1/(-1)!$ does not vanish and become void. The fraction in question must be honestly considered as a non-vanishing expression. It cannot disappear into absolute nothing, nor can it be deleted or omitted whenever it appears.

The fraction $1/(-1)!$, though equal to 0, carries with it the idea of existence. For it is absurd to regard as absolute nothing the fraction of two quantities.

3.3 Absolute Nothing and Zero Distinguished

The nothing from the take-away subtraction $a - a$ on the one hand and nothing 0 from the division of a by infinity on the other are of quite different kinds.

In the zero 0, we have a number, which seems at first to clash with absolute nothing, and which teaches us, lest we should *confound the nothings*, not to deal with the zero 0 and absolute nothing as interchangeable. Absolute nothing is absence of quantity and cannot therefore be expressed as ratio of two quantities. The zero 0, unlike absolute nothing, is a ratio, that of unity to $(-1)!$. Thus whereas absolute nothing is blank, the zero 0 is an existent, for it is absurd for a ratio to cease existence. We cannot conceive how the ratio of 1 to $(-1)!$ can become a blank.

3.4 Numerosity of Minus One Factorial

The scientific community *traditionally* accept $(-1)!$ as undefined. This was arrived at due to Martin Ohm's analysis of $1/0$. The result of this analysis was to place division by zero under the modern definition. This *new* concept swept the expanse of the scientific community and through the decades has become so embedded in human thought that to question its validity is immediate heresy.

The object of this subsection is to show that $(-1)!$ is defined and demands the right of citizenship in the realm of numbers.

3.4.1 Sum of Harmonic Series

The infinite series, a mathematical construct of paramount significance, burst into prominence during the era of calculus's conception, courtesy of the illustrious minds of Isaac Newton and Gottfried Wilhelm Leibniz. However, preceding this epochal moment, a select few had delved into the realm of infinite series, among them the erudite Pietro Mengoli of Bologna. In his magnum opus, "Novae Quadraturae Arithmeticae" (1650), Mengoli probed the mysteries of infinite series, showcasing his acumen by demonstrating the divergence of the harmonic series.

The harmonic series, an infinite sequence of reciprocals of natural numbers, had long fascinated mathematicians:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

Mengoli's ingenuity lay in dividing the terms of this series into an infinite number of groups, each aggregating to a value exceeding unity. This tour de force revealed the harmonic series' divergence, a finding that would later influence the development of calculus.

Our aim here is to demonstrate that $\ln(-1)!$, the natural logarithm of minus one factorial, is the sum of the harmonic series, *viz*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \ln(-1)!. \tag{1}$$

The possibility of such a relation as (1) is suggested by inspecting the Taylor series expansion of $\ln(x+1)$,

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots ,$$

and letting $x = -1$. Accomplishing these, we obtain the following:

$$\ln 0 = -1 - \frac{1}{2} - \frac{1}{3} - \cdots$$
$$-\ln 0 = 1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

Employing $\ln(-1)! = -\ln 0$, we arrived at the required result

$$\ln(-1)! = 1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

3.4.2 Confirmation of the Sum of Harmonic Series

Let us now turn to the derivation of a famous formula in analysis in order to show to the reader that $\ln(-1)!$ is a *true number*. There is a very interesting formula discovered by Euler in his 1776 paper, which presents a beautiful means of computing γ. This formula is

$$1 - \gamma = \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k}$$

where γ is Euler's constant and $\zeta(k)$ is zeta constant.

Euler derived the Maclaurin series expansion for $\ln(x!)$, which reads

$$\ln(x!) = -\gamma x + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} x^k, \qquad |x| < 1.$$

If we let $x = -1$ so that $|x| = 1$, an infringement of the proviso $|x| < 1$, then we obtain the result

$$\ln(-1)! = \gamma + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k}.$$

We have already established that the natural logarithm of $(-1)!$ is the sum of the harmonic series, $\sum_{k=1}^{\infty} \frac{1}{k}$. If we then replace $\ln(-1)!$ with the sum $\sum_{k=1}^{\infty} \frac{1}{k}$, we procure for ourselves

$$\sum_{k=1}^{\infty} \frac{1}{k} = \gamma + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k}$$

which results in

$$1 + \sum_{k=2}^{\infty} \frac{1}{k} - \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} = \gamma$$

which in turn furnishes our required formula

$$1 + \sum_{k=2}^{\infty} \frac{1 - \zeta(k)}{k} = \gamma$$

or

$$1 - \gamma = \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k}.$$

By this counter example alone, all objections to the doctrine of the numerosity of $(-1)!$ and hence $1/0$ are removed. Further instances that demonstrate that $\ln(-1)!$ is the sum of the divergent harmonic series are found in the the papers [20], [21], [22].

As a helpful number in the determination of the infinite sum of the divergent harmonic series, $(-1)!$ becomes to us more and more real. It helps us to understand old facts and leads us to new ones. The attribution of divergent infinite series to it is one step farther in the development of the mathematics of infinite series.

3.5 On Sum of Infinite Number of Unities

We assume the number 1 called a unit or simply unity to be defined. We next define the numbers $2, 3, 4, \ldots$. The most natural mode of forming these numbers is to begin with joining 1 to itself and to this summation another 1. Continuing in this manner, we obtain collections of units as follows

$$1 = 1, \qquad 1 + 1 = 2, \qquad 1 + 1 + 1 = 3,$$

and in general

$$\underbrace{1 + 1 + 1 + \cdot + 1}_{n \text{ ones}} = n. \tag{2}$$

It is always possible to add 1 to any number however great the number is. As there are no limits to the extensions of numbers, we observe that this pattern can be continued indefinitely, i.e. without stopping. It is, however, further observed that the ultimate stage of this pattern is the infinite summation $1+1+1+\ldots$.

We now have to define the operation

$$1+1+1+\ldots \tag{3}$$

which consists in adding the number 1 infinite number of times.

In the previous section our attention was focused on the infinite sum of the divergent harmonic series. It is possible that, from the Taylor series adduced in the previous section, we can form another divergent series which proves so important in finding the sum of the infinite number of ones. If we differentiate both sides of that Taylor series, we obtain the new Taylor series:

$$\frac{1}{x+1} = 1 - x + x^2 - x^3 + \cdots .$$

This is rewritten as

$$\frac{x!}{(x+1)!} = 1 - x + x^2 - x^3 + \cdots .$$

We remember how we set $x = -1$ in the Taylor series in the previous section and obtain the sum of the harmonic series. We do the same thing here, i.e set $x = -1$, and get the value of the infinite sum of the infinite series of unities, *viz*

Theorem 1.

$$(-1)! = 1 + 1 + 1 + 1 + \cdots .$$

The content of this result is much richer than we have been able to indicate. It conceals a depth revealed only by careful investigation.

3.6 On the Cardinal Number of the Infinite Set of Natural Numbers

It has been shown in the theory of infinite set developed by Georg Cantor that any set that can be placed in a one-to-one correspondence with the natural numbers will have the same number of elements (cardinal number) as the set of natural numbers. It turns out that many infinite set of numbers have cardinal numbers $\aleph_0$.

Let set **S** consists of summations of unities, and set **N** the set of natural numbers with cardinal number $\aleph_0$, *viz*

$$\mathbf{N} = \{1, 2, 3, \ldots, \aleph_0\}.$$

Now, consider the one-to-one correspondence:

$$\begin{aligned} \text{Set } \mathbf{S} &\leftrightarrow \text{Set } \mathbf{N} \\ 1 &\leftrightarrow 1 \\ 1{+}1 &\leftrightarrow 2 \\ 1{+}1{+}1 &\leftrightarrow 3 \\ 1{+}1{+}1{+}1 &\leftrightarrow 4 \\ 1{+}1{+}1{+}1{+}1 &\leftrightarrow 5 \end{aligned}$$

and so forth.To each summation of unities in set **S** there corresponds its sum, a natural number in set **N** such that the two sets are equally unending, and in exactly the same way. Hence, both sets have the same cardinal number, namely, $\aleph_0$. Since the final element of set **S** is $1 + 1 + 1 + \cdots$, it thus follow that

$$1 + 1 + 1 + 1 + \cdots \leftrightarrow \aleph_0$$

and hence

$$1 + 1 + 1 + 1 + \cdots = \aleph_0.$$

It turns out that since

$$1 + 1 + 1 + 1 + \cdots = (-1)!,$$

the cardinality $\aleph_0$ of the set of natural numbers is equal to minus one factorial, *viz*

$$\aleph_0 = (-1)!.$$

Hence, the the set

$$\mathbf{N} = \{1, 2, 3, \ldots, \aleph_0\}$$

becomes

$$\mathbf{N} = \{1, 2, 3, \ldots, (-1)!\}.$$

So, to the question, "'how many numbers are there in the sequence of natural numbers $1, 2, 3, \ldots, n, \ldots$?"', we answer "there are $(-1)!$ of them". This new view proves very successful and fruitful in the further development of this work. How is it that this fact remained for so long obscured?

4 Arithmetic of Zeros and Minus One Factorial

4.1 Terminology

The zero 0 is the unit zero. It reciprocal $(-1)!$ is the unit infinity.

A zero of order n is a zero of the form $k \cdot 0^n$ where n is a positive integer and k is any finite number. The number k is called the finite part of the zero. The expression 0^n is called the unit zero part of the zero. The following zero is of order 5 and has a finite part of 3: $3 \cdot 0^5$. Here are some more examples of zeros $4 \cdot 0, 0^2, \pi \cdot 0, -6 \cdot 0^4$.

The n th order of infinity is of the form $k \cdot (-1)!^n$ where n is a positive integer and k is any finite number. The number k is called the finite part of the infinite number. The expression $(-1)!^n$ is called the unit infinity part of the infinite number. The following infinite number is of order 3 and has a finite part of 2: $2 \cdot (-1)!^3$.

4.2 Arithmetic of Zeros

To create an exact and consistent arithmetic of zero has acquired an extraordinary celebrity from the fact that none has been created, but there is no reason to doubt that it is possible.

4.2.1 Equality of Zeros

The zeros $a \cdot 0^m$ and $b \cdot 0^n$ are equal if and only if $a = b$ and $m = n$.

4.2.2 Addition and subtraction

For the zeros $a \cdot 0^m$ and $b \cdot 0^m$ of the same order m,

$$a \cdot 0^m + b \cdot 0^m = (a + b) \cdot 0^m$$

and

$$a \cdot 0^m - b \cdot 0^m = (a - b) \cdot 0^m.$$

For instance, take the third order zeros $2 \cdot 0^3$ and $7 \cdot 0^3$. Then,

$$2 \cdot 0^3 + 7 \cdot 0^3 = 9 \cdot 0^3$$

and

$$2 \cdot 0^3 - 7 \cdot 0^3 = -5 \cdot 0^3.$$

The addition or subtraction of zeros of different orders cannot be reduced to any simpler form, and the combination is therefore called *compound zeros*. For example,

$$2 \cdot 0^3 + 7 \cdot 0^5$$

consists of zeros of different orders. All we know of this compound zero is that its value is equivalent to absolute nothing.

4.2.3 Scalar multiplication

The product of any zero, say $a \cdot 0^m$, and any scalar c (number c), is the zero $ca \cdot 0^m$ obtained by multiplying c by the finite part of the zero. For instance, $7 \cdot 0^5$ multiplied by 2 is $14 \cdot 0^5$.

4.2.4 Multiplication of Zeros

For the zeros $a \cdot 0^m$ and $b \cdot 0^n$, we get the multiplication operation

$$a \cdot 0^m \times b \cdot 0^n = ab \cdot 0^{m+n}.$$

For instance,

$$2 \cdot 0^3 \times 7 \cdot 0^4 = 14 \cdot 0^7.$$

Addition, subtraction and multiplication of zeros give a zero.

4.2.5 Division of Zeros

The immortal Brahmagupta asserted in his famous work [6] that

> Zero divided by negative or positive is either zero or is expressed by a fraction with zero as numerator and the finite quantity as denominator. ....

His axiom may be simply expressed as

$$\frac{0}{A} = \left\{0, \frac{0}{A}\right\}.$$

The idea of leaving the expression

$$\frac{0}{A}$$

as indicated, and not replacing it with 0, is to some a very great paradox if not an absurdity.

Bhaskara II was the first to give meaning to Brahmagupta's axiom, for setting

$$A = \frac{1}{B}$$

gives

$$B \cdot 0 = \{0, B \cdot 0\}.$$

From this new result Bhaskara ventured the publication of the following notion: [3], [23]:

> When a number (B) is multiplied by cipher (0), the product ($B \cdot 0$) is cipher (0); but in case any operation remains to be done, cipher (0) is considered to be the multiplier (i.e. the expression $B \cdot 0$ is to be retained), and if cipher (0) also becomes the divisor (in the remaining operation), the number (B) is considered unchanged (i.e. $B \cdot 0/0 = B$ in the remaining operation where 0 is a divisor).

Suppose the product $B \cdot 0$ is the result of an operation, this multiple of 0, i.e. $B \cdot 0$ is equivalent to 0 (nothing). But if an upcoming operation requires the result of the previous operation, the mere 0 to which $B \cdot 0$ is equivalent should not be used, but the original form of expression, that is, the multiple of 0, $B \cdot 0$, should rather be used in the new operation.

In order to exhibit more clearly this principle, we break it into two laws:

> **Law 1**: The law of retention of the multiple of 0, and
> **Law 2**: The law of restoration of the finite multiplicand.

The **law of retention of the multiple of** 0 states that

> If an operation results in a multiple of 0, this multiple should be retained and used in further operation on 0.

The **law of restoration of the finite multiplicand** states that:

> If in this further operation on 0, 0 appears as a divisor and the multiple of 0 from the previous operation acts as a dividend, the 0's cancel out and the result is the finite multiplicand of the multiple of 0.

It should be noted that the two 0's cancelled out are self and the same; it is merely required to cancel them out. This cancellation is in line with Bhaskara's method of dividing numbers in his *Lilavati*:

> If the divisor and the dividend have a common factor, then the common factor can be cancelled and the division is carried out with the remaining factors.

Thus, in figuring with zero, the above principle of Bhaskara for handling 0 is valuable : It is called the **Principle of Impending Operation on Zero**[3]. From this principle, it can be inferred that the unit zero, as already noted, is an existential, capable of dividing itself to give unity, *viz*

$$\frac{0}{0} = 1.$$

This identity $0/0 = 1$ may appear surprising to those familiar with only recent writings in mathematical ideas. It may seem to indicate a crudity of view which has long been antiquated. The tendency in more recent times has been to avoid any reference to division by zero.

Now according to the aforementioned Principle of Bhaskara, zero 0 is equivalent to absolute nothing, the blank, but wears the true nature of quantities so as to be as it were double, and to have two separate characters. Bhaskara expressly acknowledged both natures. This delightful principle, so redolent of the day of Bhaskara, is unfortunately absent from our present day mathematics.

We turn at once to the division property of zero. Based on the identity $0/0 = 1$, we say that for the zeros $a \cdot 0^m$ and $b \cdot 0^n$, we have the division operation

$$a \cdot 0^m \div b \cdot 0^n = \frac{a}{b} \cdot 0^{m-n}.$$

If $m > n$, the quotient is a zero and if $m < n$, the quotient is infinity. If, however, $m = n$, the quotient is the finite number a/b.

To avoid paradoxes in figuring with zeros of higher powers, we extend the principle of Bhaskara to powers of zero: "If in some mathematical calculations, the zero 0 is likely to occur frequently, then, though $a \cdot 0^n$ is nothing where a is a finite number and n is any positive number, one should maintain the form $a \cdot 0^n$ in the rest of the operations until the final operation with 0 is reached. This is because if a finite number is multiplied by a power of zero and divided by the same power of zero, then the result is the finite number".

4.3 Arithmetic of Infinity

Though we can never count infinite (endless) number of things, we can form a true and accurate arithmetic of it. Infinite numbers occur in mathematics, engineering and the sciences and therefore the knowledge of the operations on them is essential. Like finite numbers, infinite numbers can be added, subtracted, multiplied, and divided. Our desire here is, therefore, to put infinity on equal footing as ordinary numbers. Once this has been done, infinity is a perfectly acceptable candidate for mathematical analysis.

4.3.1 Equality of Infinities

Two infinite numbers $a \cdot (-1)!^m$ and $b \cdot (-1)!^n$ are equal if $a = b$ and $m = n$, that is, their finite parts are equal and their order parts are equal.

4.3.2 Addition and Subtraction

The sum and difference of two infinite numbers $a \cdot (-1)!^m$ and $b \cdot (-1)!^m$ of the same order are defined by adding or subtracting their finite parts:

$$a \cdot (-1)!^m + b \cdot (-1)!^m = (a + b) \cdot (-1)!^m$$

and

$$a \cdot (-1)!^m - b \cdot (-1)!^m = (a - b) \cdot (-1)!^m.$$

For instance,

$$3 \cdot (-1)!^4 + 2 \cdot (-1)!^4 = 5 \cdot (-1)!^4$$

and

$$3 \cdot (-1)!^4 - 2 \cdot (-1)!^4 = (-1)!^4.$$

The sum or difference of infinite numbers of different powers can be reduced to no simpler form and so called *compound infinity*. For example,

$$3 \cdot (-1)!^7 + 2 \cdot (-1)!^4$$

is a compound infinity as it consists of two infinite numbers of difference orders 7 and 4. All that should be said of this expression is that it is infinitely great.

4.3.3 Scalar Multiplication

The product of the infinite number $a \cdot (-1)!^m$ and the scalar c (number c) is obtained as follows:

$$c \times a \cdot (-1)!^m = ca \cdot (-1)!^m.$$

For example,

$$3 \times 5 \cdot (-1)!^2 = 15 \cdot (-1)!^2.$$

4.3.4 Multiplication of Infinite Numbers

For the infinities $a \cdot (-1)!^m$ and $b \cdot (-1)!^n$, we have

$$a \cdot (-1)!^m \times b \cdot (-1)!^n = ab \cdot (-1)!^{m+n}.$$

As an instance, we have

$$4 \cdot (-1)!^5 \times 2 \cdot (-1)!^6 = 8 \cdot (-1)!^{11}.$$

4.3.5 Division of Infinite Numbers

For the infinite numbers $a \cdot (-1)!^m$ and $b \cdot (-1)!^n$, we have

$$a \cdot (-1)!^m \div b \cdot (-1)!^n = \frac{a}{b} \cdot (-1)!^{m-n}.$$

If $m > n$, the quotient is an infinity and it $m < n$, the quotient is a zero. If, however, $m = n$, the quotient is a finite number (a/b). As an instance, we have

$$4 \cdot (-1)!^5 \div 2 \cdot (-1)!^6 = 8 \cdot (-1)!^{-1} = 8 \cdot 0.$$

4.4 On the Property $0 \cdot (-1)!$

It must be noted as remarkable, looking to the grandeur of the property $0 \cdot (-1)! = 1$ and its apparently direct testimony to the true nature of $(-1)!$, that the product of 0 and $(-1)!$ equals unity. This property is the germ that connects the arithmetic of zeros, finite numbers and infinite numbers.

Let a, b, m and n be any finite numbers. Then

1. $k \cdot 0^m \times l(-1)!^n = kl \cdot 0^{m-n} \quad m > n$
2. $k \cdot 0^m \times l(-1)!^n = kl(-1)!^{n-m} \quad m < n$
3. $k \cdot 0^m \times l(-1)!^m = kl$

5 Sums of Powers of All Natural Numbers

It is a well-established fact, familiar to those versed in the annals of mathematical history, that the renowned Swiss mathematician, Jakob Bernoulli, disseminated his groundbreaking solution to the longstanding problem of determining a general formula for the sums of powers of natural numbers in his seminal work.

In the ancient era, mathematicians such as Archimedes and Euclid endeavored to calculate the areas of various geometric figures, including triangles, circles, and polygons. However, the challenge of finding the area enclosed by a curve, particularly the parabola, remained an open problem.

During the Renaissance, mathematicians like Bonaventura Cavalieri and Johannes Kepler made significant progress in developing methods to calculate the areas under curves. Cavalieri's work, "Geometria Indivisibilibus" (1635), introduced the concept of indivisibles, which laid the groundwork for the development of calculus.

In the 17th century, the problem of finding the area enclosed by a curve became a central focus of mathematical inquiry. The parabola, in particular, drew the attention of prominent mathematicians like Galileo Galilei, René Descartes, and Pierre Fermat.

Jakob Bernoulli's work on the sums of powers of natural numbers was instrumental in addressing this problem. By establishing a general formula for these sums, Bernoulli provided a crucial tool for calculating the areas under curves. Specifically, his formula enabled the determination of the area enclosed by a parabola, a feat that had eluded mathematicians for centuries.

Bernoulli's breakthrough paved the way for later mathematicians, such as Isaac Newton and Gottfried Wilhelm Leibniz, to develop the methods of calculus. Bernoulli, in his *Art Conjectandi* of 1713, showed that

$$\begin{aligned}\sum_{k=1}^{n} k^p &= 1^p + 2^p + 3^p + \cdots + n^p \\ &= \frac{1}{p+1}n^{p+1} + \frac{1}{2}n^p + \frac{n}{2!}B_2 n^{p-1} + \frac{p(p-1)(p-2)}{4!}B_4 n^{p-3} + \cdots\end{aligned}$$

where $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}$, and so on are constitute a sequence of Bernoulli numbers, a sequence of rational numbers that play a pivotal role in numerous mathematical contexts.

Bernoulli's work on the sums of powers of natural numbers, and its connection to finding the area enclosed by a curve, has left an indelible mark on the annals of mathematical history, serving as a testament to his unwavering dedication and unbridled passion for mathematical excellence.

We exhibit a one-to-one correspondence between the natural numbers and their pth powers. Our strategy will be to sketch such a correspondence, showing exactly how each natural number is paired with a unique power of the natural number. In the correspondence

$$\begin{aligned} 1 &\leftrightarrow 1^p \\ 2 &\leftrightarrow 2^p \\ 3 &\leftrightarrow 3^p \\ 4 &\leftrightarrow 4^p \end{aligned}$$

the pairing of the natural number n with n^p continues indefinitely, with neither set containing any element not used up in the pairing process. So, even though the set of powers of natural numbers seems to be a proper subset of the set of natural numbers and thus should have smaller cardinal numbers the above correspondence proves that both sets have the same cardinal number $(-1)!$. Thus, if $(-1)!$ is the last ordinal in the set of natural numbers, $(-1)!^p$ is the last ordinal in the set of powers of the natural numbers so that the set can be properly written as

$$\{1^p, 2^p, 3^p, \ldots, n^p, \ldots (-1)!^p\}.$$

It remains to investigate whether or not the infinite summations

$$1^p, 2^p, 3^p, \ldots \qquad \text{and} \qquad 1^p, 2^p, 3^p, \ldots, (-1)!^p$$

will give the same sum for values of $p > 0$?. For the case $p = 0$, we have obtained the theorem:

Theorem 2. *The summation of unities up to infinity is*

$$1 + 1 + 1 + \cdots = \underbrace{1 + 1 + 1 + \cdots + 1}_{(-1)!\ ones} = (-1)!$$

We consider only the simple case:

$$1 + 2 + 3 + \ldots \tag{4}$$

and

$$1 + 2 + 3 + \ldots + (-1)! \tag{5}$$

Theorem 3. *The summation of all the natural numbers is*

$$1 + 2 + 3 + \cdots + (-1)! = \frac{(-1)!^2}{2} + \frac{(-1)!}{2}$$

and not

$$1 + 2 + 3 + \cdots = (-1)!^2$$

Proof. To create the series (4), we start with the following pattern of numbers

$$\begin{aligned} (1)\,(1) &= 1 \\ (1+x)\,(1+x) &= 1 + 2x + x^2 \\ \left(1+x+x^2\right)\left(1+x+x^2\right) &= 1 + 2x + 3x^2 + 2x^3 + x^4 \\ \left(1+x+x^2+x^3\right)\left(1+x+x^2+x^3\right) &= 1 + 2x + 3x^2 + 4x^3 + 3x^4 + 2x^5 + x^6 \end{aligned}$$

and so on. If we set $x = 1$, we obtain

$$\begin{aligned}(1)(1) &= 1\\ (1+1)(1+1) &= 1+2\cdot 1+1^2\\ \left(1+1+1^2\right)\left(1+1+1^2\right) &= 1+2\cdot 1+3\cdot 1^2+2\cdot 1^3+1^4\\ \left(1+1+1^2+1^3\right)\left(1+1+1^2+1^3\right) &= 1+2\cdot 1+3\cdot 1^2+4\cdot 1^3+3\cdot 1^4+2\cdot 1^5+1^6\end{aligned}$$

and so on. This becomes

$$\begin{aligned}(1)(1) &= 1\\ (1+1)(1+1) &= 1+2+1\\ (1+1+1)(1+1+1) &= 1+2+3+2+1\\ (1+1+1+1)(1+1+1+1) &= 1+2+3+4+3+2+1\end{aligned}$$

and so on. In general, the product

$$(1+1+1+\cdots+1)(1+1+1+\cdots+1)$$

where there are n ones in both factors gives the series

$$1+2+3+\cdots+(n-1)+n+(n-1)+\cdots+3+2+1.$$

The ultimate product of the above pattern is

$$(1+1+1+\cdots)(1+1+1+\cdots)$$

whose expansion should give

$$1+2+3+\cdots+((-1)!-1)+(-1)!+((-1)!-1)+\cdots+3+2+1.$$

Since $1+1+1+\cdots=(-1)!$, we write

$$(-1)!^2 = 1+2+3+\cdots+((-1)!-1)+(-1)!+((-1)!-1)+...+3+2+1\cdots \quad (6)$$

Consider the infinite series expansion

$$\left(\frac{1}{x+1}\right)^2 = 1-2x+3x^2-\cdots.$$

If we set $x=-1$, we get

$$\left(\frac{1}{-1+1}\right)^2 = 1-2(-1)+3(-1)^2-\cdots$$

which becomes

$$\left(\frac{1}{0}\right)^2 = 1+2+3+\cdots$$

which, setting $1/0 = (-1)!$, gives

$$(-1)!^2 = 1 + 2 + 3 + \cdots \tag{7}$$

Thus the series

$$1 + 2 + 3 + \cdots$$

is actually the series

$$1 + 2 + 3 + \cdots + ((-1)! - 1) + (-1)! + ((-1)! - 1) + \cdots + 3 + 2 + 1 \cdots .$$

It may be supposed at first sight that the result (6) is incorrect, but a consideration of the following will alter this assumption. The sum of the first n natural numbers is $(n-1)/2$, that is,

$$1 + 2 + 3 + \cdots + n = \frac{n\,(n+1)}{2}$$

setting $n = (-1)!$ gives

$$1 + 2 + 3 + \cdots + (-1)! = \frac{(-1)!\,((-1)! + 1)}{2}$$

which becomes

$$1 + 2 + 3 + \cdots + (-1)! = \frac{(-1)!^2}{2} + \frac{(-1)!}{2}. \tag{8}$$

It is clear that the results (6) and (7) are different from the result (8). By means of (8) we can show that (6) is valid.

We start with

$$1 + 2 + \cdots + (-1)! + ((-1)! - 1) + \cdots + 2 + 1$$

which can be split into two groups as follows:

$$(1 + 2 + \cdots + (-1)!) + (1 + 2 + \cdots + ((-1)! - 1))\,.$$

The sum of this series is therefore

$$\frac{(-1)!^2}{2} + \frac{(-1)!}{2} + \frac{((-1)! - 1)^2}{2} + \frac{(-1)! - 1}{2}$$

which becomes

$$\frac{(-1)!^2}{2} + \frac{(-1)!}{2} + \frac{(-1)!^2 - 2(-1)! + 1}{2} + \frac{(-1)! - 1}{2}.$$

Simplifying this completely gives

$$(-1)!^2.$$

Observe that in the series

$$1 + 2 + 3 + \cdots,$$

which is actually the series

$$1 + 2 + 3 + \cdots + ((-1)! - 1) + (-1)! + ((-1)! - 1) + \cdots + 3 + 2 + 1,$$

the order of increasing magnitudes of the terms is broken after the $(-1)!$th term. It follows then that the series $1 + 2 + 3 + \cdots$ cannot be interpreted as the summation of all the natural numbers. The summation of all the natural numbers is $1 + 2 + 3 + \cdots + (-1)!$ and its sum is $(-1)!^2/2 + (-1)!/2$. □

It follows from our discussion above that the series.

$$1^p + 2^p + 3^p + \cdots$$

differs from the series.

$$1^p + 2^p + 3^p + \cdots + (-1)!^p.$$

for values of $p > 0$.

It remains, therefore, to find a closed formula for the series

$$1^p + 2^p + 3^p + \cdots + (-1)!^p$$

Setting $n = (-1)!$ in the Bernoulli's general formula for sums of power already mentioned in this section, we obtain

$$\sum_{k=1}^{(-1)!} k^p = \frac{1}{p+1}(-1)!^{p+1} + \frac{1}{2}(-1)!^p + \frac{p}{2!}B_2(-1)!^{p-1} + \frac{p(p-1)(p-2)}{4!}B_4(-1)!^{p-3} + \cdots.$$

For $p = 1, 2, 3, \ldots$, we have respectively

$$\sum_{k=1}^{(-1)!} k = \frac{1}{2}(-1)!^2 + \frac{1}{2}(-1)!$$

$$\sum_{k=1}^{(-1)!} k^2 = \frac{1}{3}(-1)!^3 + \frac{1}{2}(-1)!^2 + \frac{1}{6}(-1)!$$

$$\sum_{k=1}^{(-1)!} k^3 = \frac{1}{4}(-1)!^4 + \frac{1}{2}(-1)!^3 + \frac{1}{4}(-1)!^2$$

and so forth.

6 A Novel Foundation of Moments of Inertia

6.1 The Problem

The behaviour of rotating object is of considerable importance. It can be applied to rotating car wheels, flywheels, the rotation of high divers and many things.

We can imagine a solid rigid body as made up of many particles of masses $m_1, m_2, m_3, \ldots$ at distances $r_1, r_2, r_3, \ldots$ from the centre of rotation. See Figure 1.

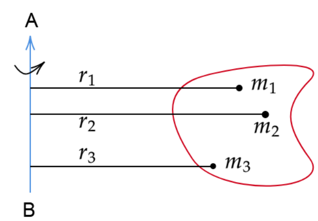

Figure 1: Moment of Inertia

The total rotational kinetic energy of the body will be the sum of the energies of all the particles. Then the total kinetic energy is

$$KE = \frac{1}{2}m_1 w r_1^2 + \frac{1}{2}m_2 w r_2^2 + \frac{1}{2}m_3 w r_3^2 + \ldots.$$

It should be noted that as the body is rigid the angular velocity w is the same for all particles although the linear velocity will be greater for particles further from the axis of rotation. We may write

$$KE = \frac{1}{2}w\sum mr^2.$$

The term

$$\sum mr^2$$

is called the moment of inertia I of the body about the axis of rotation. It is equal to the infinite sum of the moments of the inertia of all the particles about the same axis, that is

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots.$$

Two disturbing questions emerge together out of this subject to reject the evidence of the senses. How are we to talk about the number of particles in the massive body under consideration? The second question which is closely related to the first is 'What is the magnitude of the mass of each particle?' This is hardly a commendable use of the word *particle*. This word would assume that its mass has a finite quantity however small, the very opposite to the meaning that it was intended to convey. It will come to mean that the mass is not zero, but a finite and measurable quantity. If the masses were all finite, the infinite summation

$$m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots$$

would have a sum greater than any finite quantity whatever. By far the most important question on the subject is how can we obtain a finite moment of inertia ? Thus, the aim of this section is to re-examine the foundation of the moment of inertia of bodies by means of a natural system that uses zero and the factorial of minus one in a consistent way.

6.2 A Rigorous Mathematical Approach

The moment of inertia is a fundamental concept in physics and engineering, yet its foundation is often overlooked or misunderstood. We will look into the mathematical intricacies of the moment of inertia, shedding light on the underlying principles.

Consider a material body of mass M. We can divide this body into n parts, each with a mass of $\frac{M}{n}$. The number n must be a natural number, as we can only count the number of parts using natural numbers. As n increases, the masses of the resulting parts decrease, but the total mass remains constant:

$$\begin{aligned} &\frac{M}{1} && = M \\ &\frac{M}{2} + \frac{M}{2} && = M \\ &\frac{M}{3} + \frac{M}{3} + \frac{M}{3} && = M \end{aligned}$$

and so forth. We aforementioned that the ultimate natural number in the sequence

$$1, 2, 3, \dots$$

is $(-1)!$. So if we divide the body in question into $(-1)!$ equal parts, we shall obtain for each part a mass expressed as

$$m = \frac{M}{(-1)!} = M \cdot 0.$$

The first idea about the mass $M \cdot 0$ of each resulting particle of the body which has been divided into $(-1)!$ parts is that the mass is an entity or existential. This is evidently true as the 0 in the zero mass $M \cdot 0$ is an existential as already discussed in a previous section.

We must bear in mind that we cannot sum the infinite series

$$M \cdot 0 + M \cdot 0 + M \cdot 0 + \cdots$$

by merely considering the partial sums. We must sum all the zeros at once. Since there are $(-1)!$ zeros we have

$$M \cdot 0 + M \cdot 0 + M \cdot 0 + \cdots = M \cdot 0 \cdot (-1)! = M.$$

Moments of Inertia of a Finite Rod

We wish to find the moment of inertia of a uniform rod of *no* thickness Let us find the moment of inertia about an axis y passing through one end of the rod and perpendicular to its length as shown in Figure 2. The moment of inertia

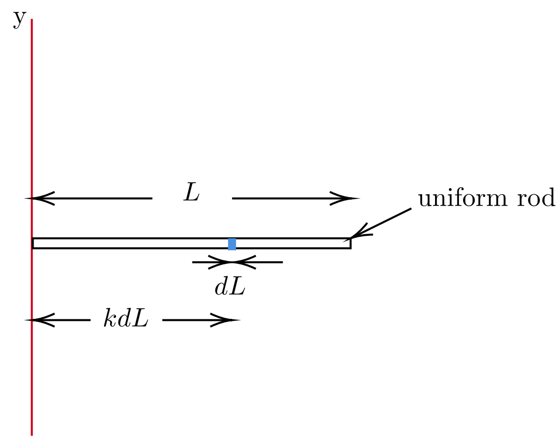

Figure 2: Moment of inertia of a uniform rod

of a point mass is given by $I = mr^2$, but the rod would have to be considered to be an infinite number of equal point masses, and each must be multiplied by the square of its distance from the axis. The rod of length L is divided into $(-1)!$ equal parts. Each part will have a non-vanishing size of dL given as

$$dL = \frac{L}{(-1)!} = L \cdot 0.$$

The zero size portion is not the void but has actual existence, conserving the initial physical properties in such a way that summing all these portions furnishes again the whole length, *viz*

$$L \cdot 0 + L \cdot 0 + L \cdot 0 + \cdots = (-1)! \cdot L \cdot 0 = L.$$

Now, this non-vanishing zero dL is the length between any two consecutive points along the length L. The length r_k between the axis of rotation and the k th point along the length is

$$r_k = kdL = kL \cdot 0.$$

The mass per unit length of the uniform rod is M/L. The mass of the element between the k th and $k+1$ th points along the length of the rod is the mass

$$\begin{aligned} dm &= \frac{M}{L}dL \\ &= \frac{M}{L}L \cdot 0 \\ &= M \cdot 0. \end{aligned}$$

The point mass dm which equals $M \cdot 0$ is a non-vanishing mass equivalent to the void. Thus, the moment of inertia is

$$\begin{aligned} I &= \sum_{k=1}^{(-1)!} mr^2 \\ &= \sum_{k=1}^{(-1)!} M \cdot 0(kL \cdot 0)^2 \\ &= \sum_{k=1}^{(-1)!} ML^2k^2 \cdot 0^3 \\ &= ML^2 \cdot 0^3 \sum_{k=1}^{(-1)!} k^2 \end{aligned}.$$

Replacing

$$\sum_{k=1}^{(-1)!} k^2$$

with its actual sum as found in a previous section, we get

$$\begin{aligned} I &= ML^2 \cdot 0^3 \left[\frac{(-1)!^3}{3} + \frac{(-1)!^2}{2} + \frac{(-1)!}{6}\right] \\ &= \frac{ML^2 \cdot 0^3 \cdot (-1)!^3}{3} + \frac{ML^2 \cdot 0^3 \cdot (-1)!^2}{2} + \frac{ML^2 \cdot 0^3 \cdot (-1)!}{6} \\ &= \frac{ML^2}{3} + \frac{ML^2 \cdot 0}{2} + \frac{ML^2 \cdot 0^2}{6} \\ &\equiv \frac{ML^2}{3}. \end{aligned}$$

Moment of Inertia of a Circular Disc

Finally we wish to determine the moment of inertia of a circular disc about an axis through its centre and perpendicular to its plane. Let a be the radius on the disc. The complete division of the radius into equal parts give the part-size of

$$da = \frac{a}{(-1)!} = a \cdot 0.$$

Thus, the distance between any two consecutive points along the radius is the infinitely small $a \cdot 0$. Let us now divide the disc into $(-1)!$ concentric circles. The radius r_k of the k th concentric circle measured from the centre of the circle is $r_k = kda = ka \cdot 0$.

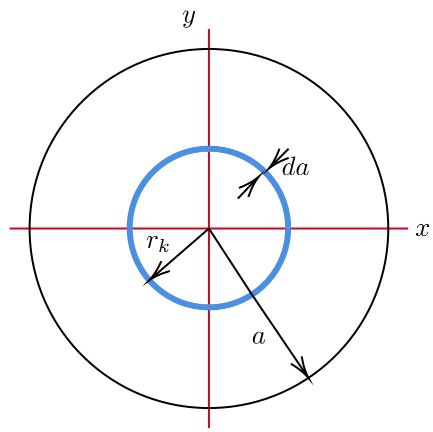

Figure 3: Moment of inertia of a circular disc

Also, the radius of the $k+1$ th concentric circle measured from the same centre of the circle is $r_{k+1} = (k+1)da = (k+1)a \cdot 0$. The area of the element between the k th and the $k+1$ th concentric circles is

$$\begin{aligned} dA &= \pi r_{k+1}^2 - \pi r_k^2 \\ &= \pi \left(r_{k+1}^2 - r_k^2\right) \\ &= \pi \left(r_{k+1} + r_k\right)\left(r_{k+1} - r_k\right) \\ &= \pi \left[(k+1)a \cdot 0 + ka \cdot 0\right]\left[(k+1)a \cdot 0 - ka \cdot 0\right] \\ &= \pi \left[(2k+1)a \cdot 0\right]\left[a \cdot 0\right] \\ &= \pi(2k+1)a^2 \cdot 0^2 \end{aligned}$$

since further operation impends, we retain the expression for dA for further use.

Now the mass per unit area of the disc is $M/\pi a^2$. The mass of the element between the k th and the $k+1$ th concentric circles will be

$$\begin{aligned} m &= \frac{M}{\pi a^2} A \\ &= \frac{M}{\pi a^2} \pi(2k+1)a^2 \cdot 0 \\ &= M(2k+1) \cdot 0. \end{aligned}$$

The moment of inertia is

$$
\begin{aligned}
I &= \sum_{k=1}^{(-1)!} mr_k^2 \\
&= \sum_{k=1}^{(-1)!} M(2k+1)\cdot 0^2 (ka\cdot 0)^2 \\
&= Ma^2\cdot 0^4 \sum_{k=1}^{(-1)!}(2k^3+k^2) \\
&= Ma^2\cdot 0^4\left[2\left(\frac{1}{4}(-1)!^4+\frac{1}{2}(-1)!^3+\frac{1}{4}(-1)!^2\right)+\frac{1}{3}(-1)!^3+\frac{1}{2}(-1)!^2+\frac{1}{6}(-1)!\right] \\
&= Ma^2\cdot 0^4\left[\frac{1}{2}(-1)!^4+(-1)!^3+\frac{1}{2}(-1)!^2+\frac{1}{3}(-1)!^3+\frac{1}{2}(-1)!^2+\frac{1}{6}(-1)!\right] \\
&= Ma^2\cdot 0^4\left[\frac{1}{2}(-1)!^4+(-1)!^3+\frac{1}{3}(-1)!^3+\frac{1}{2}(-1)!^2+\frac{1}{2}(-1)!^2+\frac{1}{6}(-1)!\right] \\
&= Ma^2\cdot 0^4\left[\frac{1}{2}(-1)!^4+\frac{4}{3}(-1)!^3+(-1)!^2+\frac{1}{6}(-1)!\right] \\
&= Ma^2\left[\frac{1}{2}\cdot 0^4\cdot(-1)!^4+\frac{4}{3}\cdot 0^4\cdot(-1)!^3+0^4\cdot(-1)!^2+\frac{1}{6}\cdot 0^4\cdot(-1)!\right] \\
&= Ma^2\left[\frac{1}{2}+\frac{4}{3}\cdot 0+0^2+\frac{1}{6}\cdot 0^3\right] \\
&\equiv \frac{Ma^2}{2}.
\end{aligned}
$$

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