HYPERCYCLICITY CRITERION ON BASIC ELEMENTARY OPERATOR

Abstract: Hypercyclicity criterion has been an important tool in the test of hypercyclicity of different operators. This tool has been used by different mathematicians to show that generalized derivations, left and right multiplication operators , operator algebra and backward shift operators are hypercyclic. In the current paper we show that a basic elementary operator satisfies the hypercyclicity Criterion.

Key words: *Hypercyclicity Criterion, Rank one operator and Basic elementary operator*

1. Introduction

Let K be a separable Hilbert space and A(K) be the set of bounded linear operators on K. Then $A(K)$ is a c* algebra. An elementary operator on A(K) is the operator $M_{T,S}$ defined by $M_{T,S}X = TXS \forall X \in A(K)$ where $T, S \in A(K)$ are fixed.

A subset $J \subseteq A(K)$ is called a Hilbert space ideal of A(K) if;

(i)J is a linear subspace of A(K)

(ii) The norm $\|.\|_J$ is complete in J with $\|S\| \leq \|S\|_J \forall S \in J$ where $\|.\|$ is the norm in A(K) and $\|.\|_J$ is the norm in J For all $D, B \in A(K)$ and $S \in J$, we have $BSD \in J$ and $\|BSD\|_J \leq \|B\|\|S\|\|D\|_J$

(iii)For all $D, B \in A(K)$ and $S \in J$, we have $BSD \in J$ and $\|BSD\|_J \leq \|B\|\|S\|\|D\|_J$

(iv) The rank one operator $x^* \otimes x \in J$ whereby $\|x^* \otimes x\|_J = \|x^*\|\|x\| \forall x^* \in K^*$ and $x \in K$

Recall that rank one operator $x^* \otimes x: K \to K$ is defined by

$(x^* \otimes x)z = \langle x^*, z \rangle x = x^*(z)x$ for $x^* \in K^*$ and $x \in K$ for all $z \in K$.

The space $F(K)$ of all finite rank operators is a linear span of the rank one operators, that is,

$F(K) = K^* \otimes K = \{\sum_{i=1}^{n} x^*{}_i \otimes x_i : x^*{}_i \in K^*$ and $x_i \in K$,n$\geq 1\}$

The Hilbert space ideal J is separable when it contains the finite rank operators as a dense subset w.r.t the norm $\|.\|_J$, i.e $\overline{F(K)}^{\|.\|_J}$ =J

For a separable Hilbert space K, an operator $T \in A(K)$ is hypercyclic if there exists an $x \in X$ such that the orbit of x under T is dense in K, that is $\overline{\{T^n x: n \geq 0\}} = K$.

Kawira et al [7] showed that $A(K)$ is separable for aseparable Hilbert space K and that $M_{T,S}: A(K) \to A(K)$ is hypercyclic.

2. Hypercyclicity Criterion

Let $T \in A(K)$. We say that T satisfies the hypercyclicity criterion if there exists a linear subspace $K_O \subseteq K$, an increasing sequence (n_k) of integers and linear maps $S_{n_k}: K_O \to K, k \geq 1$ such that for any $x \in K_o$, we have ;

(i)$T^{n_k} x \to 0$

(ii)$S_{n_k} x \to 0$

(iii)$T^{n_k} S_{n_k} x \to 0$ as $n \to \infty$

Leon and Montes [9] proved that backward shifts operators satisfies the hypercyclicity criterion. Yousefi and Rezaei [12] showed that an operator T satisfies the hypothesis of the hypercyclicity criterion while De la Rosa [6] showed that a bounded linear operator T satisfies the hypercyclicity criterion.

According to Bayart et al [1] if T satisfies the hypercyclicity criterion then it is hypercyclic. However there exists hypercyclic operators that do not satisfy the hypercyclicity criterion as shown by Rosa and Read [5].

For a separable ideal $J \subset A(K)$, Bonet et al [2] showed that :

$(i) L_A$ is hypercyclic on J iff $A \in A(K)$ satisfies the hypercyclicity criterion.

$(ii) R_B$ is hypercyclic on J iff $B^* \in A(K^*)$ satisfies the hypercyclicity criterion.

In this paper we seek to establish whether $M_{T,S}$ satisfies the hypercyclicity criterion. We will consider the C* algebra A(K) which is endowed with a strong operator topology. Also we consider the Hilbert space K consisting of the space of sequences that tend to zero, the weighted backward shift operators on K and the identity operators S on K. Note that $T \in A(K)$ is bounded since $sup_n \|w_n\| < \infty$

3. Main result

Theorem 3.1

Let K be a separable Hilbert space and let A(K) be a strong operator topology C* algebra. Define a basic elementary operator $M_{T,S}$ on $A(K)$ by $M_{T,S}X = TXS, \forall X \in A(K)$ with T,S $\in A(K)$ fixed with T an identity operator on K. Then $M_{T,S}$ satisfies the hypercyclicity criterion.

Proof

Let J be an admissible separable Hilbert space ideal on K and consider a dense linear subspace $K_O \subset K,$ a sequence {n} of positive integers and a sequence of linear maps $w_n: K_0 \to K$such that $\lim_{n\to\infty} W_n\, x = 0$ and $\lim_{n\to\infty} T^n W_n\, x = x$

Consider the rank one operators $K^*\otimes K$ of J. Define linear maps $M_{I,w_n}: K^*\otimes K_o \to K^*\otimes K$ by

$M_{I,w_n}(x^*\otimes x)y = I(x^*\otimes x)W_n y \forall y \in K_O$ and $n \geq 1$

Gilmore [4] showed that $K^*\otimes K$ is dense in J. Then $M_{T,S}$ satisfies the hypercyclicity criterion since

(i)$M^n{}_{T,S}(x^*\otimes x)y = T^n(x^*\otimes x)S^n y$

$= T^n(x^*(S^n y)x)$

$=x^*(S^n y)T^n x$

Thus $\left\|M^n{}_{T,S}(x^*\otimes x)y\right\|_J = \|x^*(S^n y)T^n x\|_J$

$=|x^*(S^n y)|\|T^n x\|$

Taking limits as $n \to \infty$ we get,

$$\lim_{n\to\infty} \left\| M^n{}_{T,S}(x^* \otimes x)y \right\|_J = |x^*(S^n y)|0 = 0$$

Thus $\lim_{n\to\infty} M^n{}_{T,S}(x^* \otimes x)y = 0$

(ii) We have $M_{I,W_n}(x^* \otimes x)y = (x^* \otimes x)W_n y$

$= x^*(W_n y)I(x)$

Thus $\left\| M_{I,W_n}(x^* \otimes x)y \right\|_J = \|x^*(W_n y)Ix\|_J$

$=|x^*(W_n y)|\|x\|$

Taking limits as $n \to \infty$ we get $\lim_{n\to\infty} \left\| M_{I,W_n}(x^* \otimes x)y \right\|_J = 0$

(iii)$M^n{}_{T,S} M_{I,W_n}(x^* \otimes x)y = M^n{}_{T,S} M_{I,W_n}(x^* \otimes x)y$

Thus $M^n{}_{T,S} M_{I,W_n}(x^* \otimes x)y - (x^* \otimes x)y$

$=T^n M_{I,W_n}(x^* \otimes x)S^n y - (x^* \otimes x)y$

$=T^n M_{I,W_n}(x^*(S^n y)x) - x^*(y)x$

$= T^n x^*(S^n y)W_n x - x^*(y)x$

$= x^*(S^n y)(T^n W_n)x - x^*(y)x$

Thus $\left\|{M^n}_{T,S}M_{I,W_n}(x^*\otimes x)y-(x^*\otimes x)y\right\|$=$\|x^*(S^ny)(T^nW_n)x-x^*(y)x\|_J$

Taking limit as$n\to\infty$ we get,

$$\lim_{n\to\infty}\left\|{M^n}_{T,S}M_{I,W_n}(x^*\otimes x)y-(x^*\otimes x)y\right\|=\|x^*(y)x-x^*(y)x\|_J=0$$

Thus ${M^n}_{T,S}M_{I,W_n}(x^*\otimes x)y\to(x^*\otimes x)y$ as $n\to\infty$

Thus all the properties of hypercyclicity criterion are satisfied and so $M_{T,S}$ satisfies the criterion.

4. Conclusion

We have shown that basic elementary operator satisfies the hypercyclicity criterion. Hypercyclicity criterion can also be tested on Jordan elementary operator.

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