ON THE EFFICIENCY OF MODIFIED EXPONENTIAL DUAL TO RATIO-PRODUCT-CUM TYPE ESTIMATOR UNDER STRATIFIED RANDOM SAMPLING USING TWO AUXILIARY VARIABLES

ABSTRACT

Separate and Combined dual to ratio, product, cum estimators of the population mean under stratified random sampling scheme are suggested using the ideas and the analogy of [39, 24, 37] and adopted strategy given by [16, 27, 40, 23] in terms of two auxiliaries variables but in case of simple random sampling. Asymptotic properties of proposed estimators such as BIAS, MSE and MMSE up to first order of approximation are deduced and reported. However, the proposed estimators is also compared using Mathematical illustration. Performance is evaluated and examined with others related estimators consider using empirical study utilized two natural populations and simulated data sets. The statistical package R plus is used for computations. Moreover, Results eventually indicate the superiority of proposed estimators over existing traditional estimators mentioned in study. Therefore Researchers are highly recommended to use proposed estimators in practical application for estimating population mean.

Keywords:— Auxiliary Variable, Ratio Product Cum Type, Stratum, Study Variable, Optimum Values.

1 Introduction

The modified exponential dual to Ratio-Product-Cum Type (RPCT) estimator using stratified random sampling under two auxiliary variables is a statistical method for estimating population parameters. It is designed to be efficient in situations where two auxiliary variables are available and can be used to improve the accuracy of the estimation process. The efficiency of the modified exponential dual to RPCT estimator depends on several factors, including the sample size, the distribution of the population, the correlation between the auxiliary variables and the target variable, and the sampling design. In general, if these factors are favorable, the estimator can be very efficient and provide accurate estimates of the population parameters. One advantage of the modified exponential dual to RPCT estimator is that it can be used with non-normal populations and can still produce accurate estimates. This makes it a valuable tool for researchers who work with populations that do not conform to normal distributions. Overall, the efficiency of the modified exponential dual to RPCT estimator using stratified random sampling under two auxiliary variables will depend on the specific circumstances of the study. It is important to carefully consider the sampling design, the choice of auxiliary variables, and the size of the sample to ensure that the estimator is used in the most effective way possible.

Numerous effort have been carry out to obtain practical solution which gives approximation optimum stratification, namely: Equalization of $W_h\sigma_h$, Equalization of $W_h\mu_h$, Equalization of $W_h\frac{1}{2}[r(y)+f(y)]$, Equalization of cumulative(cum) of $\sqrt{f(y)}$, Equalization of Cum $\sqrt[3]{f(y)}$ and Equalization of $W_h(y_h - y_{h-1})$ all this rules was founded by scholars [9], [18], [11], [3], [10], [36] and [12] respectively. The major concerned here is the Equalization of cumulative(cum)$\sqrt{f(y)}$ which is original proposed by [10] under assumption that the distribution is bounded and the numbers of strata is large it is a simplification of optimum stratification for Neyman allocation and founded to be more efficient if stratifying variable come in class interval or can be formed into classes. in this context reader is referred to [22, Page 126] for more detail.

At the estimation stage, the use of the supplementary variable started with the effort of [8]. [19], [7] who developed two chain-ratio estimators presented a dual estimator, [5] were the pioneer of exponential type estimators, [32] and [6] are the one who break new ground on Dual transformation [2] developed class of ratio estimators with known function of auxiliary variable for estimating finite population variance , [4] suggested Difference-Cum-Ratio estimators for estimating finite population coefficient of Variation in Simple Random Sampling. but there are some many practical situation when the an auxiliary information is quantitative in nature, that is auxiliary variables is available in the form of an attribute such as sex, beaut and sweat so many estimator is proposed on this unlike [21], [25], [31], [29] also with condition when population coefficients variation C_P, kurtosis $\beta_2(\Phi)$, correlation ρ_{pb} both variation C_P kurtosis $\beta_2(\Phi)$ are known is proposed by (1).

Consider $U = (U_1, U_2, U_3, ..., U_N)$ be finite population of size N and it divided into L homogeneous of strata size $N_h (h = 1, 2, ...L)$. A sample n_h is drawn each stratum using simple random sampling without replacement. Let y be the study variable and x and z be the auxiliary variables

$\bar{Y}_h^s = \frac{1}{N_h}\sum_{h=1}^{L} y_{hi}$: h^{th} Population mean of the study variate y in h^{th} stratum.

$\bar{X}_h^s = \frac{1}{N_h}\sum_{h=1}^{L} x_{hi}$: h^{th} Population mean of the auxiliary variate x in h^{th} stratum.

$\bar{Z}_h^s = \frac{1}{N_h}\sum_{h=1}^{L} z_h$: h^{th} Population mean of the auxiliary variate z in h^{th} stratum.

$\bar{Y}^c = \bar{Y} = \frac{1}{N}\sum_{h=1}^{L}\sum_{i=1}^{N_h} y_h = \sum_{h=1}^{L} W_h\bar{Y}_h$: Population mean of the study variate y.

$\bar{X}^c = \bar{X} = \frac{1}{N}\sum_{h=1}^{L}\sum_{i=1}^{N_h} x_h = \sum_{h=1}^{L} W_h\bar{X}_h$: Population mean of the auxiliary variate x.

$\bar{Z}^c = \bar{Z} = \frac{1}{N}\sum_{h=1}^{L}\sum_{i=1}^{N} z_h = \sum_{h=1}^{L} W_h\bar{Z}_h$: Population mean of the auxiliary variate z.

$\bar{y}_h^s = \frac{1}{n_h}\sum_{h=1}^{L} y_h$: h^{th} sample mean of the study variate y in h^{th} stratum.

$\bar{x}_h^s = \frac{1}{n_h}\sum_{h=1}^{L} x_h$: h^{th} sample mean of the auxiliary variate x in h^{th} stratum.

$\bar{z}_h^s = \frac{1}{n_h}\sum_{h=1}^{L} z_h$: h^{th} sample mean of the auxiliary variate z in h^{th} stratum.

$\bar{x}_*^s$ dual separate transform variable of the auxiliary variate y in h^{th} stratum.

$\bar{z}_*^s$ dual separate transform variable of the auxiliary variate y in h^{th} stratum.

$\bar{x}_*^c$ dual combined transform variable of the auxiliary variate y in h^{th} stratum.

$\bar{z}_c^*$ dual combined transform variable of the auxiliary variate y in h^{th} stratum.

$S_{yh}^2 = \sum_{L}^{N_h}(y_{hi} - \bar{Y}_h)^2 (N_h - 1)^{-1}$ population variance in each h^{th} stratum y

$S_{xh}^2 = \sum_{L}^{N_h}(x_{hi} - \bar{X}_h)^2 (N_h - 1)^{-1}$ population variance in each h^{th} stratum x

$S_{zh}^2 = \sum_{L}^{N_h}(z_{hi} - \bar{Z}_h)^2 (N_h - 1)^{-1}$ population variance in each h^{th} stratum z

$S_{yxh} = \sum_{L}^{N_h}(x_{hi} - \bar{X}_h)(y_{hi} - \bar{Y}_h)(N_h - 1)^{-1}$ population variance in each h^{th} stratum x and y.

$S_{yzh} = \sum_{L}^{N_h}(y_{hi} - \bar{Y}_h)(z_{hi} - \bar{Z}_h)(N_h - 1)^{-1}$ population variance in each h^{th} stratum y and z

$S_{xzh} = \sum_{L}^{N_h}(x_{hi} - \bar{X}_h)(z_{hi} - \bar{Z}_h)(N_h - 1)^{-1}$ population variance in each h^{th} stratum x and z

1.1 STRATIFIED RANDOM SAMPLING BASED ON RATIO AND PRODUCT ESTIMATOR

Usual separate, combined and product ratio estimators envisaged by [15] is $\hat{\bar{Y}}_{11}^{ST}$ and [13] are $\hat{\bar{Y}}_{12}^{ST}$ $\hat{\bar{Y}}_{13}^{ST}$ all in population $\bar{Y}$ in stratified random sampling estimators are defined respectively as

$$\hat{\bar{Y}}_{11}^{ST} = \sum_{h=1}^{L} W_h \bar{y}_h \left(\frac{\bar{X}_h^s}{\bar{x}_h^s} \right) \tag{1}$$

$$\hat{\bar{Y}}_{12}^{ST} = \bar{y}_c \left(\frac{\sum_{h=1}^{L} W_h \bar{X}_h^c}{\sum_{h=1}^{L} W_h \bar{x}_h^c} \right) \tag{2}$$

$$\hat{\bar{Y}}_{13}^{ST} = \bar{y}_c \left(\frac{\sum_{h=1}^{L} W_h \bar{z}_h^c}{\sum_{h=1}^{L} W_h \bar{Z}_h^c} \right) \tag{3}$$

Approximated biases and MSEs of the estimators in 1, 2 and 3 are give as

$$Bias(\hat{\bar{Y}}_{11}^{ST}) = \sum_{h=1}^{L} W_h \lambda_h \left(R_{xh} S_{xh}^2 - S_{yxh} \right) \bar{X}_h^{-1} \tag{4}$$

$$Bias(\hat{\bar{Y}}_{12}^{ST}) = \sum_{h=1}^{L} W_h \lambda_h \left(R_{xc} S_{xh}^2 - S_{yxh} \right) \bar{X}^{-1} \tag{5}$$

$$Bias(\hat{\bar{Y}}_{13}^{ST}) = \sum_{h=1}^{L} W_h \lambda_h S_{yzh} \bar{Z}^{-1} \tag{6}$$

where $R_{xh} = \frac{\bar{Y}_h^s}{\bar{X}_h^s}$, $R_{zh} = \frac{\bar{Y}_h^s}{\bar{Z}_h^s}$ and $R_{xc} = \frac{\bar{Y}^c}{\bar{X}^c}$ $R_{zc} = \frac{\bar{Y}^c}{\bar{Z}^c}$

$$MSE(\hat{\bar{Y}}_{11}^{ST}) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{xh}^2 S_{xh}^2 + 2 R_{xh} S_{yxh} \right) \tag{7}$$

$$MSE(\hat{\bar{Y}}_{12}^{ST}) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{xc}^2 S_{xh}^2 - 2 R_{xc} S_{yxh} \right) \tag{8}$$

$$MSE(\hat{\bar{Y}}_{13}^{ST}) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{zc}^2 S_{zh}^2 + 2 R_{zc} S_{yzh} \right) \tag{9}$$

Motivated by Singh(1967), [16] had suggested ratio cum ratio , product cum product, ratio cum product and product cum ratio estimators of population mean are given in eq 10, 11, 12 and 13 in stratified random sampling $\bar{Y}$ as follow

$$\hat{\bar{Y}}_{21}^{ST} = \bar{y}_c \left(\frac{\sum_{h=1}^{L} W_h \bar{X}_h^c}{\sum_{h=1}^{L} W_h \bar{x}_h^c} \right) \left(\frac{\sum_{h=1}^{L} W_h \bar{Z}_h^c}{\sum_{h=1}^{L} W_h \bar{z}_h^c} \right) \tag{10}$$

$$\hat{\bar{Y}}_{22}^{ST} = \bar{y}_c \left(\frac{\sum_{h=1}^{L} W_h \bar{x}_h^c}{\sum_{h=1}^{L} W_h \bar{X}_h^c} \right) \left(\frac{\sum_{h=1}^{L} W_h \bar{z}_h^c}{\sum_{h=1}^{L} W_h \bar{Z}_h^c} \right) \tag{11}$$

$$\hat{\bar{Y}}_{23}^{ST} = \bar{y}_c \left(\frac{\sum_{h=1}^{L} W_h \bar{X}_h^c}{\sum_{h=1}^{L} W_h \bar{x}_h^c} \right) \left(\frac{\sum_{h=1}^{L} W_h \bar{z}_h^c}{\sum_{h=1}^{L} W_h \bar{Z}_h^c} \right) \tag{12}$$

$$\hat{\bar{Y}}_{24}^{ST} = \bar{y}_c \left(\frac{\sum_{h=1}^{L} W_h \bar{x}_h^c}{\sum_{h=1}^{L} W_h \bar{X}_h^c} \right) \left(\frac{\sum_{h=1}^{L} W_h \bar{Z}_h^c}{\sum_{h=1}^{L} W_h \bar{z}_h^c} \right) \tag{13}$$

The biases and MSEs of the estimators $\hat{\bar{Y}}_{21}^{ST}$, $\hat{\bar{Y}}_{22}^{ST}$, $\hat{\bar{Y}}_{23}^{ST}$ and $\hat{\bar{Y}}_{24}^{ST}$ under stratified random sampling are given below in eq 14, 15, 16, 17 and 18, 19, 20, 21 respectively

$$Bias\left(\hat{\bar{Y}}_{21}^{ST}\right) = \sum_{h=1}^{L} W_h \lambda_h \left(R_{xc}^2 S_{xh}^2 + R_{zc}^2 S_{zh}^2 + R_{xc} R_{zc} S_{xzh} - R_{xc} S_{yxh} - R_{zc} S_{yzh}\right) \bar{Y}^{-1} \tag{14}$$

$$Bias\left(\hat{\bar{Y}}_{22}^{ST}\right) = \sum_{h=1}^{L} W_h \lambda_h \left(R_{xc} R_{zc} S_{xzh} + R_{xc} S_{yxh} + R_{zc} S_{yzh}\right) \bar{Y}^{-1} \tag{15}$$

$$Bias\left(\hat{\bar{Y}}_{23}^{ST}\right) = \sum_{h=1}^{L} W_h \lambda_h \left(R_{xc}^2 S_{xh}^2 - R_{xc} R_{zc} S_{xzh} - R_{xc} S_{yxh} + R_{zc} S_{yzh}\right) \bar{Y}^{-1} \tag{16}$$

$$Bias\left(\hat{\bar{Y}}_{24}^{ST}\right) = \sum_{h=1}^{L} W_h \lambda_h \left(R_{zc}^2 S_{zh}^2 - R_{xc} R_{zc} S_{xzh} + R_{xc} S_{yxh}\right) \bar{Y}^{-1} \tag{17}$$

$$MSE(\hat{\bar{Y}}_{21}^{ST}) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{xc}^2 S_{xh}^2 + R_{zc}^2 S_{zh}^2 + 2R_{xc} R_{zc} S_{xzh} - 2R_{xc} S_{yxh} - 2R_{zc} S_{yzh}\right) \tag{18}$$

$$MSE(\hat{\bar{Y}}_{22}^{ST}) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{xc}^2 S_{xh}^2 + R_{zc}^2 S_{zh}^2 + 2R_{xc} R_{zc} 2S_{xzh} + 2R_{xc} S_{yxh} + 2R_{zc} S_{yzh}\right) \tag{19}$$

$$MSE(\hat{\bar{Y}}_{23}^{ST}) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{xc}^2 S_{xh}^2 + R_{zc}^2 S_{zh}^2 - 2R_{xc} R_{zc} S_{xzh} - 2R_{xc} S_{yxh} + 2R_{zc} S_{yzh}\right) \tag{20}$$

$$MSE(\hat{\bar{Y}}_{24}^{ST}) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{xc}^2 S_{xh}^2 + R_{zc}^2 S_{zh}^2 + 2R_{xc}R_{zc}S_{xzh} + 2R_{xc}S_{yxh} - 2R_{zc}S_{yzh} \right) \tag{21}$$

[38] proposed separate estimator of population mean using multi-auxiliary variate under post stratification the proposed estimator is defined be as

$$\hat{\bar{Y}}_{31}^{ST} = \sum_{h=1}^{L} W_h \bar{y}_h \left(\vartheta_h \frac{\bar{X}_h^s}{\bar{x}_h^s} + (1-\vartheta_h) \frac{\bar{z}_h^s}{\bar{Z}_h^s} \right) \tag{22}$$

where ϑ of the estimator 22 is the constant parameter to minimized MSE of the estimator $\hat{\bar{Y}}_{31}^{ST}$, the equation of the bias and MSE up to first term approximation are given as below.

$$Bias(\hat{\bar{Y}}_{31}^{ST}) = \sum_{h=1}^{L} W_h \lambda_h \left(\vartheta_h R_{xh}^2 S_{xh}^2 - \vartheta_h R_{xh} S_{yxh} - (1-\vartheta_h) R_{zh} S_{yzh} \right) \bar{Y}_h^{-1} \tag{23}$$

$$MSE(\hat{\bar{Y}}_{31}^{ST}) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{xh}^2 S_{xh}^2 + (1-\vartheta_h)^2 R_{zh}^2 S_{zh}^2 - 2(1-\vartheta_h) R_{xh} R_{zh} S_{xzh} - 2\vartheta_h R_{xh} S_{yxh} + 2(1-\vartheta_h) R_{zh} S_{yzh} \right) \tag{24}$$

$$\vartheta_h min = \left(\frac{R_{zh} S_{zh}^2 - R_{xh} R_{zh} S_{xzh} + R_{xh} S_{yxh} + R_{zh} S_{yzh}}{R_{xh}^2 S_{xh}^2 + R_{zh}^2 S_{zh}^2} \right) \tag{25}$$

1.2 STRATIFIED RANDOM SAMPLING BASED ON EXPONENTIAL RATIO AND PRODUCT ESTIMATOR

[28] defined exponential ratio product type estimator which is on stratified random sampling motivated from 5 which is on simple random sampling of the population mean $\bar{Y}$. The Ratio exponential estimator in stratified random sampling is given by

$$\hat{\bar{Y}}_{41}^{ST} = \bar{y}_c \exp\left(\frac{\sum_{h=1}^{L} W_h \left(\bar{X}_h^c - \bar{x}_h^c \right)}{\sum_{h=1}^{L} W_h \left(\bar{X}_h^c + \bar{x}_h^c \right)} \right) \tag{26}$$

$$\hat{\bar{Y}}_{42}^{ST} = \bar{y}_c \exp\left(\frac{\sum_{h=1}^{L} W_h \left(\bar{z}_h^c - \bar{Z}_h^c \right)}{\sum_{h=1}^{L} W_h \left(\bar{z}_h^c + \bar{Z}_h^c \right)} \right) \tag{27}$$

Biases and MSE of the modified ratio exponential estimator $\bar{Y}_{41}$ and product exponential estimator $\bar{Y}_{42}$ under stratified random sampling are given as

$$Bias\left(\hat{\bar{Y}}_{41}^{ST}\right) = \sum_{h=1}^{L} W_h \lambda_h \left(\frac{3}{8} R_{xc} S_{xh}^2 - \frac{1}{2} S_{yxh} \right) \bar{X}^{-1} \tag{28}$$

$$Bias\left(\hat{\bar{Y}}_{42}^{ST}\right) = \sum_{h=1}^{L} W_h \lambda_h \left(-\frac{1}{8} R_{zc} S_{zh}^2 + \frac{1}{2} S_{yzh} \right) \bar{Z}^{-1} \tag{29}$$

$$MSE(\hat{\bar{Y}}_{41}^{ST}) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{xc}^2 \frac{S_{xh}^2}{4} - R_{xc} S_{yxh} \right) \tag{30}$$

$$MSE(\hat{\bar{Y}}_{42}^{ST}) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{zc}^2 \frac{S_{zh}^2}{4} + R_{zc} S_{yzh} \right) \tag{31}$$

[27] motivated by Singh (1965,1967) Introduced various Exponential Ratio cum Ratio, product cum product, Ratio cum product and product cum Ratio type estimators of finite population mean in stratified random sampling

$$\hat{\bar{Y}}_{51}^{ST} = \bar{y}_c exp \left(\frac{\sum_{h=1}^{L} W_h \left(\bar{X}_h^c - \bar{x}_h^c \right)}{\sum_{h=1}^{L} W_h \left(\bar{X}_h^c + \bar{x}_h^c \right)} \right) exp \left(\frac{\sum_{h=1}^{L} W_h \left(\bar{Z}_h^c - \bar{z}_h^c \right)}{\sum_{h=1}^{L} W_h \left(\bar{Z}_h^c + \bar{z}_h^c \right)} \right) \tag{32}$$

$$\hat{\bar{Y}}_{52}^{ST} = \bar{y}_c exp \left(\frac{\sum_{h=1}^{L} W_h \left(\bar{x}_h^c - \bar{X}_h^c \right)}{\sum_{h=1}^{L} W_h \left(\bar{x}_h^c + \bar{X}_h^c \right)} \right) exp \left(\frac{\sum_{h=1}^{L} W_h \left(\bar{z}_h^c - \bar{Z}_h^c \right)}{\sum_{h=1}^{L} W_h \left(\bar{z}_h^c + \bar{Z}_h^c \right)} \right) \tag{33}$$

$$\hat{\bar{Y}}_{53}^{ST} = \bar{y}_c exp \left(\frac{\sum_{h=1}^{L} W_h \left(\bar{X}_h^c - \bar{x}_h^c \right)}{\sum_{h=1}^{L} W_h \left(\bar{X}_h^c + \bar{x}_h^c \right)} \right) exp \left(\frac{\sum_{h=1}^{L} W_h \left(\bar{z}_h^c - \bar{Z}_h^c \right)}{\sum_{h=1}^{L} W_h \left(\bar{z}_h^c + \bar{Z}_h^c \right)} \right) \tag{34}$$

$$\hat{\bar{Y}}_{54}^{ST} = \bar{y}_c exp \left(\frac{\sum_{h=1}^{L} W_h \left(\bar{x}_h^c - \bar{X}_h^c \right)}{\sum_{h=1}^{L} W_h \left(\bar{x}_h^c + \bar{X}_h^c \right)} \right) exp \left(\frac{\sum_{h=1}^{L} W_h \left(\bar{Z}_h^c - \bar{z}_h^c \right)}{\sum_{h=1}^{L} W_h \left(\bar{Z}_h^c + \bar{z}_h^c \right)} \right) \tag{35}$$

The expressions for biases and mean square errors(MSEs) of the estimators in 32, 33, 34 and 35 up to second degree approximation are as follow

$$Bias \left(\hat{\bar{Y}}_{51}^{ST} \right) = \sum_{h=1}^{L} W_h \lambda_h \left(\frac{3}{8} R_{xc}^2 S_{xh}^2 + \frac{3}{8} R_{zc}^2 S_{zh}^2 + \frac{1}{4} R_{xc} R_{zc} S_{xzh} - \frac{1}{2} R_{xc} S_{yxh} - \frac{1}{2} R_{zc} S_{yzh} \right) \bar{Y}^{-1} \tag{36}$$

$$Bias \left(\hat{\bar{Y}}_{52}^{ST} \right) = \sum_{h=1}^{L} W_h \lambda_h \left(-\frac{1}{8} R_{xc}^2 S_{xh}^2 - \frac{1}{8} R_{zc}^2 S_{zh}^2 + \frac{1}{4} R_{xc} R_{zc} S_{xzh} + \frac{1}{2} R_{xc} S_{yxh} + \frac{1}{2} R_{zc} S_{yzh} \right) \bar{Y}^{-1} \tag{37}$$

$$Bias \left(\hat{\bar{Y}}_{53}^{ST} \right) = \sum_{h=1}^{L} W_h \lambda_h \left(-\frac{3}{8} R_{xc}^2 S_{xh}^2 - \frac{1}{8} R_{zc}^2 S_{zh}^2 - \frac{1}{4} R_{xc} R_{zc} S_{xzh} - \frac{1}{2} R_{xc} S_{yxh} + \frac{1}{2} R_{zc} S_{yzh} \right) \bar{Y}^{-1} \tag{38}$$

$$Bias \left(\hat{\bar{Y}}_{54}^{ST} \right) = \sum_{h=1}^{L} W_h \lambda_h \left(-\frac{1}{8} R_{xc}^2 S_{xh}^2 - \frac{3}{8} R_{zc}^2 S_{zh}^2 - \frac{1}{4} R_{xc} R_{zc} S_{xzh} + \frac{1}{2} R_{xc} S_{yxh} - \frac{1}{2} R_{zc} S_{yzh} \right) \bar{Y}^{-1} \tag{39}$$

$$MSE \left(\hat{\bar{Y}}_{51}^{ST} \right) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + \frac{1}{4} R_{xc}^2 S_{xh}^2 + \frac{1}{4} R_{zc}^2 S_{zh}^2 + \frac{1}{2} R_{xc} R_{zc} S_{xzh} - R_{xc} S_{yxh} - R_{zc} S_{yzh} \right) \tag{40}$$

$$MSE \left(\hat{\bar{Y}}_{52}^{ST} \right) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + \frac{1}{4} R_{xc}^2 S_{xh}^2 + \frac{1}{4} R_{zc}^2 S_{zh}^2 + \frac{1}{2} R_{xc} R_{zc} S_{xzh} + R_{xc} S_{yxh} + R_{zc} S_{yzh} \right) \tag{41}$$

$$MSE \left(\hat{\bar{Y}}_{53}^{ST} \right) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + \frac{1}{4} R_{xc}^2 S_{xh}^2 + \frac{1}{4} R_{zc}^2 S_{zh}^2 - \frac{1}{2} R_{xc} R_{zc} S_{xzh} - R_{xc} S_{yxh} + R_{zc} S_{yzh} \right) \tag{42}$$

$$MSE \left(\hat{\bar{Y}}_{54}^{ST} \right) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + \frac{1}{4} R_{xc}^2 S_{xh}^2 + \frac{1}{4} R_{zc}^2 S_{zh}^2 - \frac{1}{2} R_{xc} R_{zc} S_{xzh} + R_{xc} S_{yxh} - R_{zc} S_{yzh} \right) \tag{43}$$

1.3 STRATIFIED RANDOM SAMPLING BASED ON DUAL RATIO AND PRODUCT ESTIMATOR

[17] Obtained dual to combined ratio and product estimators $\hat{\bar{Y}}_{61}$ and $\hat{\bar{Y}}_{62}$ as

$$\hat{\bar{Y}}_{61}^{ST} = \bar{y}_c \left(\frac{\sum_{h=1}^{L} W_h \bar{x}_h^*}{\sum_{h=1}^{L} W_h \bar{X}_h} \right) \tag{44}$$

$$\hat{\bar{Y}}_{62}^{ST} = \bar{y}_c \left(\frac{\sum_{h=1}^{L} W_h \bar{Z}_h}{\sum_{h=1}^{L} W_h \bar{z}_h^*} \right) \tag{45}$$

Here are the biases and Means square error of the estimators up first degree of approximations respectively

$$Bias\left(\hat{\bar{Y}}_{61}^{ST}\right) = \sum_{h=1}^{L} W_h \lambda_h \left(-g_c S_{yxh}\right) \bar{X}^{-1} \tag{46}$$

$$Bias\left(\hat{\bar{Y}}_{62}^{ST}\right) = \sum_{h=1}^{L} W_h \lambda_h \left(R_{zc} g_c^2 S_{zh}^2 + g_c S_{yzh}\right) \bar{Z}^{-1} \tag{47}$$

$$MSE\left(\hat{\bar{Y}}_{61}^{ST}\right) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{xc}^2 g_c^2 S_{xh}^2 - 2 g_c R_{xc} S_{yxh}\right) \tag{48}$$

$$MSE\left(\hat{\bar{Y}}_{62}^{ST}\right) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{zc}^2 g_c^2 S_{zh}^2 + 2 g_c R_{zc} S_{yzh}\right) \tag{49}$$

Singh, 1995[as cited in 30, pg. 740] Introduced an improved separate ratio to dual estimator of population mean $\bar{Y}$ in stratified random sampling

$$\hat{\bar{Y}}_{71}^{ST} = \sum_{h=1}^{L} W_h \bar{y}_h \left(\frac{\bar{x}_*^s}{\bar{X}_h^s} \right) \tag{50}$$

The bias and mean square error according to first order of approximation are stated in equations 51 and 52.

$$Bias(\hat{\bar{Y}}_{71}^{ST}) = \sum_{h=1}^{L} W_h \lambda_h \left(-g_h S_{yxh}\right) \bar{X}_h^{-1} \tag{51}$$

$$MSE(\hat{\bar{Y}}_{71}^{ST}) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 - R_{xh}^2 g_h^2 S_{xh}^2 + 2 R_{xh} g_h S_{yxh}\right) \tag{52}$$

1.4 STRATIFIED RANDOM SAMPLING BASED ON DUAL EXPONENTIAL RATIO AND PRODUCT ESTIMATOR

[34] used dual transformation to [28] suggested ratio and product type exponential estimator as

$$\hat{\bar{Y}}_{81}^{ST} = \bar{y}_c exp \left(\frac{\sum_{h=1}^{L} W_h (\bar{x}_*^c - \bar{X}_h^c)}{\sum_{h=1}^{L} W_h (\bar{x}_*^c + \bar{X}_h^c)} \right) \tag{53}$$

$$\hat{\bar{Y}}_{82}^{ST} = \bar{y}_c exp \left(\frac{\sum_{h=1}^{L} W_h (\bar{Z}_h^c - \bar{z}_*^c)}{\sum_{h=1}^{L} W_h (\bar{Z}_h^c + \bar{z}_*^c)} \right) \tag{54}$$

Final, the mean squared error and bias of the suggest ratio type exponential estimators $\hat{\bar{Y}}_{81}^{ST}$ and $\hat{\bar{Y}}_{82}^{ST}$ up to first order approximations are given as

$$Bias(\hat{\bar{Y}}_{81}^{ST}) = -\sum_{h=1}^{L} W_h \lambda_h g_c \left(S_{yxh} + \frac{3}{4} R_{xc} g_c S_{xh}^2 \right) \bar{X}^{-1} \quad (55)$$

$$Bias(\hat{\bar{Y}}_{82}^{ST}) = \sum_{h=1}^{L} W_h \lambda_h g_c \left(S_{yzh} + \frac{5}{8} R_{zc} g_c S_{zh}^2 \right) \bar{Z}^{-1} \quad (56)$$

$$MSE(\hat{\bar{Y}}_{81}^{ST}) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + \frac{1}{4} R_{xc}^2 g_c^2 S_{xh}^2 - R_{xc} g_c S_{yxh} \right) \quad (57)$$

$$MSE(\hat{\bar{Y}}_{82}^{ST}) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + \frac{1}{4} R_{xc}^2 g_c^2 S_{zh}^2 + R_{zc} g_c S_{yzh} \right) \quad (58)$$

[23] proposed dual to Ratio cum product type exponential estimator of finite population mean in stratified random sampling but estimator was wrongly stated by [23] to be ”dual to product cum ratio type” and correct bias and mse is obtained in (60) and (61) as follow:

$$\hat{\bar{Y}}_{91}^{ST} = \bar{y}_c \exp \left(\frac{\sum_{h=1}^{L} W_h \left(\bar{X}_h^c - \bar{x}_*^c \right)}{\sum_{h=1}^{L} W_h \left(\bar{X}_h^c + \bar{x}_*^c \right)} \right) exp \left(\frac{\sum_{h=1}^{L} W_h \left(\bar{z}_*^c - \bar{Z}_h^c \right)}{\sum_{h=1}^{L} W_h \left(\bar{z}_*^c + \bar{Z}_h^c \right)} \right) \quad (59)$$

To the first degree of approximation Bias and mean square error of the suggested ratio type estimators are defined below respectively

$$Bias\left(\hat{\bar{Y}}_{91}^{ST}\right) = \sum_{h=1}^{L} W_h \lambda_h \left(\frac{3}{8} R_{xc}^2 g_c^2 S_{xh}^2 - \frac{1}{8} R_{zc}^2 g_c^2 S_{zh}^2 - \frac{1}{4} R_{xc} R_{zc} g_c^2 S_{xzh} + \frac{1}{2} R_{xc} g_c S_{yxh} - \frac{1}{2} R_{zc} g_c S_{yzh} \right) \bar{Y}^{-1} \quad (60)$$

$$MSE\left(\hat{\bar{Y}}_{91}^{ST}\right) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + \frac{1}{4} R_{xc}^2 g_c^2 S_{xh}^2 + \frac{1}{4} R_{zc}^2 g_c^2 S_{zh}^2 - \frac{1}{2} R_{xc} R_{zc} g_c^2 S_{xzh} + R_{xc} g_c S_{yxh} - R_{zc} g_c S_{yzh} \right) \quad (61)$$

2 Proposed Estimators

The ideas and the analogy of [39, 24, 37] and adopted strategy given by [16, 27, 40, 23] in terms of two auxiliaries variables but in case of simple random sampling lead to postulate the proposed estimators based on stratified random sampling such as:

Separate Estimator 62 is proposed as

$$\hat{\bar{Y}}_{PCRDOSE}^{*,ST} = \sum_{h=1}^{L} W_h \bar{y}_h exp \left(\frac{\bar{X}_h^s}{\alpha_h \bar{x}_{*h}^s + (1-\alpha_h) \bar{X}_h^s} - \frac{\alpha_h \bar{z}_{*h}^s + (1-\alpha_h) \bar{Z}_h^s}{\bar{Z}_h^s} \right) \quad (62)$$

were

$$\bar{x}_*^s = (N_h^s \bar{X}_h^s - n_h^s \bar{x}_h)/(N_h^s - n_h^s) \quad g_h = n_h^s \left(N_h^s - n_h^s\right)^{-1}$$

$$\bar{z}_*^s = (N_h^s \bar{Z}_h^s - n_h^s \bar{z}_h^s)/(N_h^s - n_h^s) \quad \lambda_h = (1 - f_h)/n_h^s$$

Combined Estimator 63is proposed as

$$\hat{\bar{Y}}^{*,ST}_{PCRDOCE} = \bar{y}_c exp\left(\frac{\sum_{h=1}^{L} W_h \bar{X}_h^c}{\alpha_c \sum_{h=1}^{L} W_h \bar{x}_{*h}^c + (1-\alpha_c)\sum_{h=1}^{L} W_h \bar{X}_h^c} - \frac{\alpha_c \sum_{h=1}^{L} W_h \bar{z}_{*h}^c + (1-\alpha_c)\sum_{h=1}^{L} W_h \bar{Z}_h^c}{\sum_{h=1}^{L} W_h \bar{Z}_h^c}\right) \tag{63}$$

$$\bar{x}_*^c = \left(N\bar{X}^c - n\bar{x}^c\right)(N-n)^{-1} \quad g_c = n\,(N-n)^{-1}$$
$$\bar{z}_*^c = \left(N\bar{Z}^c - n\bar{z}^c\right)(N-n)^{-1} \quad \lambda_h = (1-f_h)\,n_h^{-1}$$

where α is a suitable chosen constant to be determined such that MSE of the proposed estimator in (62) and (63) is minimum.

3 Properties(Bias and MSE) of Proposed Estimators

In this section, biases, MSEs and MMSEs of the proposed estimator will be establish using Tailor series expansion up to second degree of expansion.

3.1 Bias, MSE and MMSE of the Proposed Separate Estimator

To obtain mean square error and bias let define the followings notations:$e_{yh} = \frac{\bar{y}_h}{\bar{Y}_h} - 1$,$e_{xh} = \frac{\bar{x}_h}{\bar{X}_h} - 1$, $e_{zh} = \frac{\bar{z}_h}{\bar{Z}_h} - 1$,$e_{yc} = \frac{\bar{y}_c}{\bar{Y}} - 1$, $e_{xc} = \frac{\bar{x}_c}{\bar{X}} - 1$ and $e_{zc} = \frac{\bar{z}_c}{\bar{Z}} - 1$

$$e_{i,j,k(c)} = \sum_{h=1}^{L} W_h^{i+j+k} \frac{E[(\bar{y}_h - \bar{Y}_h)^i (\bar{x}_h - \bar{X}_h)^j (\bar{z}_h - \bar{Z}_h)^k]}{\bar{Y}^i \bar{X}^j \bar{Z}^k} \tag{64}$$

The expectation of error terms of proposed estimator (62)
$E(e_{yh}) = E(e_{xh}) = E(e_{zh}) = 0$, $E(e_{xh}^2) = \lambda_h \frac{S_{xh}^2}{\bar{X}_n^2}$, $E(e_{yh}^2) = \lambda_h \frac{S_{yh}^2}{\bar{Y}_h^2}$, $E(e_{zc}^2) = \lambda_h \frac{S_{zh}^2}{\bar{Z}^{c2}}$, $E(e_{xc}^2) = \lambda_h \frac{S_{xh}^2}{\bar{X}^{c2}}$, $E(e_{yh}e_{xh}) = \lambda_h \frac{S_{yxh}}{\bar{X}_h \bar{Y}_h}$, $E(e_{xh}e_{zh}) = \lambda_h \frac{S_{xzh}}{\bar{X}_h \bar{Z}_h}$, $E(e_{yh}e_{zh}) = \lambda_h \frac{S_{yzh}}{\bar{X}_h \bar{Z}_h}$
The expectation of error terms of proposed estimator (63)
$E(e_{yc}) = E(e_{xc}) = E(e_{zc}) = 0$, $E(e_{xc}^2) = \sum_{h=1}^{L} W_h^2 \lambda_h \frac{S_{xh}^2}{\bar{X}^{c2}}$, $E(e_{yc}^2) = \sum_{h=1}^{L} W_h^2 \lambda_h \frac{S_{yh}^2}{\bar{Y}^{c2}}$, $E(e_{zc}^2) = \sum_{h=1}^{L} W_h^2 \lambda_h \frac{S_{zh}^2}{\bar{Z}^{c2}}$, $E(e_{xc}^2) = \sum_{h=1}^{L} W_h^2 \lambda_h \frac{S_{xh}^2}{\bar{X}^{c2}}$, $E(e_{yc}e_{xc}) = \sum_{h=1}^{L} W_h^2 \lambda_h \frac{S_{yxh}}{\bar{X}^c \bar{Y}^c}$, $E(e_{xc}e_{zc}) = \sum_{h=1}^{L} W_h^2 \lambda_h \frac{S_{xzh}}{\bar{X}^c \bar{Z}^c}$, $E(e_{yc}e_{zc}) = \sum_{h=1}^{L} W_h^2 \lambda_h \frac{S_{yzh}^2}{\bar{X}^c \bar{Z}^c}$
Express equation 62 in terms of error e_{xh}, e_{yh}, e_{zh} and simplify the results accordingly

$$\hat{\bar{Y}}^{*ST}_{PCRDOSE} = \sum_{h=1}^{L} W_h \bar{y}_h exp\left(\frac{\bar{X}_h^s}{\bar{X}_h^s(1+\alpha_h g_h) - \alpha_h g_h \bar{x}_h^s} - \frac{\bar{Z}_h^s(1+\alpha_h g_h) - \alpha_h g_h \bar{z}_h^s}{\bar{Z}_h^s}\right) \tag{65}$$

Now assume that $\mid e_{xh} \mid < 1$ so that $(1-\alpha_h g_h e_{xh})^{-1}$ are expandable. Expanding the right hand side up to second degree approximation.

$$\hat{\bar{Y}}^{*ST}_{PCRDOSE} = \sum_{h=1}^{L} W_h \bar{y}_h exp\left(\frac{\bar{X}_h^s}{\bar{X}_h^s(1+\alpha_h g_h) - \alpha_h g_h (1+e_{xh})\bar{X}_h^s} - \frac{\bar{Z}_h^s(1+\alpha_h g_h) - \alpha_h g_h (1+e_{zh})\bar{Z}_h^s}{\bar{Z}_h^s}\right) \tag{66}$$

$$\hat{\bar{Y}}^{*ST}_{PCRDOSE} = \sum_{h=1}^{L} W_h\bar{y}_h exp\left[(1-\alpha_h g_h e_{xh})^{-1} - (1-\alpha_h g_h e_{zh})\right] \tag{67}$$

Apply binomial expansion to expand LHS of (67) we obtain the result of RHS as follow

$$(1-\alpha_h\, g_h e_{xh})^{-1} = 1+\alpha_h\, g_h e_{xh} + \alpha_h^2\, g_h^2 e_{zh}^2 + ... \tag{68}$$

Neglect the higher terms power of 2 or more than and add the result to R.H.S. of (67)

$$\hat{\bar{Y}}^{*ST}_{PCRDOSE} = \sum_{h=1}^{L} W_h\bar{y}_h exp\left(\alpha_h\, g_h e_{xh} + \alpha_h g_h e_{xh} + \alpha_h^2\, g_h^2 e_{zh}^2\right) \tag{69}$$

Taking the exponential of equation (69) using Taylor's series expansion we have

$$\hat{\bar{Y}}^{*ST}_{PCRDOSE} = \sum_{h=1}^{L} W_h\bar{y}_h \left(1+\alpha_h\, g_h e_{xh} + \alpha_h\, g_h e_{zh} + \alpha_h^2\, g_h^2 e_{xh}^2 + \frac{(\alpha_h\, g_h e_{xh} + \alpha_h\, g_h e_{zh} + \alpha_h^2\, g_h^2 e_{xh}^2)^2}{2}\right) \tag{70}$$

Expand the L.H.S of equation (70) and neglate the higher terms greater than 2 we obtain R.H.S of the same equation below

$$(\alpha_h\, g_h e_{xh} + \alpha_h\, g_h e_{zh} + \alpha_h^2\, g_h^2 e_{xh}^2)^2 = \left(\alpha_h^2 g_h^2 e_{xh}^2 + 2\alpha_h^2 g_h^2 e_{xh} e_{zh} + \alpha_h^2 g_h^2 e_{zh}^2\right) \tag{71}$$

Substitute the R.H.S. 0f (71) into equation (70) place of L.H.S. equation (3.3.5) arrange according to magnitude power of error terms

$$\hat{\bar{Y}}^{*ST}_{PCRDOSE} = \sum_{h=1}^{L} W_h\bar{y}_h \left(1+\alpha_h g_h e_{xh} + \alpha_h g_h e_{zh} + \alpha_h^2 g_h^2 e_{xh}^2 + \frac{\alpha_h^2 g_h^2 e_{xh}^2}{2} + \frac{\alpha_h^2 g_h^2 e_{zh}^2}{2} + \alpha_h^2 g_h^2 e_{xh} e_{zh}\right) \tag{72}$$

Subtract $\bar{Y}_h^s$ from both side of equation **??**

$Bias(\hat{\bar{Y}}^{*ST}_{PCRDOSE}) = E(\bar{Y}_h^s - \hat{\bar{Y}}^{*}_{PCRDOSE})$

$$Bias\left(\hat{\bar{Y}}^{*ST}_{PCRDOSE}\right) = \bar{Y}_h \sum_{h=1}^{L} W_h \left(e_{yh} + \alpha_h g_h e_{xh} + \alpha_h g_h e_{zh} + \frac{3\alpha_h^2 g_h^2 e_{xh}^2}{2} + \alpha_h^2 g_h^2 e_{xh} e_{zh} + \frac{\alpha_h^2 g_h^2 e_{zh}^2}{2} + \alpha_h g_h e_{xh} e_{yh} + \alpha_h g_h e_{zh} e_{yh}\right) \tag{73}$$

To obtain the Bias we take expectation errors terms of (73) and used the results of 64.

$$Bias\left(\hat{\bar{Y}}^{*ST}_{PCRDOSE}\right) = \bar{Y}_h \sum_{h=1}^{L} W_h \left(E(e_{yh}) + \alpha_h g_h E(e_{xh}) + \frac{3\alpha_h^2 g_h^2 E(e_{xh}^2)}{2} + \alpha_h g_h E(e_{zh}) + \alpha_h^2 g_h^2 E(e_{xh} e_{zh}) + \frac{\alpha_h^2 g_h^2 E(e_{zh}^2)}{2} + \alpha_h g_h E(e_{xh} e_{yh}) + \alpha_h g_h E(e_{zh} e_{yh})\right) \tag{74}$$

$$Bias\left(\hat{\bar{Y}}^{*ST}_{PCRDOSE}\right) = \bar{Y}_h^s \sum_{h=1}^{L} W_h \lambda_h \left(\frac{3}{2}\alpha_h^2 g_h^2 \frac{S_{xh}^2}{\bar{X}_h^{s2}} + \alpha_h^2 g_h^2 \frac{S_{zh}^2}{2\bar{Z}_h^{s2}} + \alpha_h^2 g_h^2 \frac{S_{xzh}}{\bar{X}_h^{s2}\bar{Z}_h^{s2}} + \alpha_h g_h \frac{S_{yxh}}{\bar{X}_h^{s}\bar{Y}_h^{s}} + \alpha_h g_h \frac{S_{yzh}}{\bar{Y}_h^{s}\bar{Z}_h^{s}}\right) \tag{75}$$

$$Bias\left(\hat{\bar{Y}}^{*ST}_{PCRDOSE}\right) = \sum_{h=1}^{L} W_h \lambda_h \left(\frac{3}{2}R_{xh}^2 \alpha_h^2 g_h^2 S_{xh}^2 + \frac{1}{2}R_{zh}^2 \alpha_h^2 g_h^2 S_{zh}^2 + R_{xh}R_{zh}\alpha_h^2 g_h^2 S_{xzh} + R_{xh}\alpha_h g_h S_{yxh} + R_{zh}\alpha_h g_h S_{yzh}\right) / \bar{Y}_h^s \quad (76)$$

Taking only the leading term of eq (73) above square the it, Expand the result up to the first order approximation to get the mean square error (MSE) of $\hat{\bar{Y}}^{*ST}_{PCRDOSE}$ which is derive as follow,

(77)

$$hat\bar{Y}^{*ST}_{PCRDOSE} = \left(\bar{Y}_h^s \left(e_{yh} + \alpha_h g_h e_{xh} + \alpha_h g_h e_{zh}\right)\right)^2 \quad (78)$$

$$MSE\left(\hat{\bar{Y}}^{*ST}_{PCRDOSE}\right) = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{zh}^2 \alpha_h^2 g_h^2 S_{zh}^2 + R_{xh}^2 \alpha_h^2 g_h^2 S_{xh}^2 + 2R_{xh}R_{zh}\alpha_h^2 g_h^2 S_{xzh} + 2R_{xh}\alpha_h g_h S_{yxh} + 2R_{zh}\alpha_h g_h S_{yzh}\right) \quad (79)$$

To obtain minimum mean square error or test for optimality, differentiate (79) with respect to α_h, equate to zero and solve for the α_h, the optimal and minimum mean square erro is attained as in 81 and 82 respectively.

$$MSE\left(\hat{\bar{Y}}^{*ST}_{PCRDOSE}\right) min = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 + R_{zh}^2 \alpha_{min}^2 g_h^2 S_{zh}^2 + R_{xh}^2 \alpha_{min}^2 g_h^2 S_{xh}^2 + 2R_{xh}R_{zh}\alpha_{min}^2 g_h^2 S_{xzh} + 2R_{xh}\alpha_{min} g_h S_{yxh} + 2R_{zh}\alpha_{min} g_h S_{yzh}\right) \quad (80)$$

$$\alpha_{min} = \frac{-\sum_{h=1}^{L} W_h^2 \lambda_h (R_{xh}S_{yxh} + R_{zh}S_{yzh})}{g_h \sum_{h=1}^{L} W_h^2 \lambda_h (R_{xh}^2 S_{xh}^2 + R_{zh}^2 S_{zh}^2 + 2R_{xh}R_{zh}S_{xzh})} \quad (81)$$

$$MSE\left(\hat{\bar{Y}}^{*ST}_{PCRDOCE}\right) min = \sum_{h=1}^{L} W_h^2 \lambda_h \left(S_{yh}^2 - \frac{(R_{xh}S_{yxh} + R_{zh}S_{yzh})^2}{(R_{xh}^2 S_{xh}^2 + R_{zh}^2 S_{zh}^2 + 2R_{xh}R_{zh}S_{xzh})}\right) \quad (82)$$

3.2 Biased (bias), MSE and MMSE of the Proposed Combined Estimator

$$\hat{\bar{Y}}^{*ST}_{PCRDOCE} = \bar{y}_{st} exp \left\{ \frac{\sum_{h=1}^{L} W_h \bar{X}_h}{\left[\sum_{h=1}^{L} W_h \bar{X}_h \left(\alpha_c + \alpha_c \frac{n_h}{N_h - n_h}\right) - \sum_{h=1}^{L} W_h \bar{x}_h \left(\alpha_h \frac{n_h}{N_h - n_h}\right)\right] + (1-\alpha_c)\sum_{h=1}^{L} W_h \bar{X}_h} - \frac{\left[\sum_{h=1}^{L} W_h \bar{Z}_h \left(\alpha_c + \alpha_c \frac{n_h}{N_h - n_h}\right) - \sum_{h=1}^{L} W_h \bar{z}_h \left(\alpha_c \frac{n_h}{N_h - n_h}\right)\right] + (1-\alpha_h)\sum_{h=1}^{L} W_h \bar{Z}_h}{\sum_{h=1}^{L} W_h \bar{Z}_h} \right\} \quad (83)$$

Bias of the estimator 63 can be obtain from (83) as follow

$$\hat{\bar{Y}}^{*ST}_{PCRDOCE} = \bar{Y}^c (1 + e_{yc}) exp\left[(1 - \alpha_c g_c e_{xc})^{-1} - (1 - \alpha_c g_c e_{zc})\right] \quad (84)$$

If we assume that $|e_{xc}| < 1$, the expression $(1 - \alpha_c g e_{xc})^{-1}$ inside equation 3.4.6 can be expanded to a convergent infinite series using binomial expansion. Hence

$$(1 - \alpha_c g_c e_{xc})^{-1} = 1 + \alpha_c g_c e_{xc} + \alpha_c^2 g_c^2 e_{zc}^2 + ... \quad (85)$$

Deduct $\bar{Y}$ from both side of equation (86) and take expectation of the we the bias in (87)
$Bias(\hat{\bar{Y}}^{*ST}_{PCRDOCE}) = E(\bar{Y}^c - \hat{\bar{Y}}^{*}_{PCRDOCE})$

$$Bias\left(\hat{\bar{Y}}^{*ST}_{PCRDOCE}\right) = \bar{Y}^c - \bar{Y}^c \left(1 + e_{yc} + \alpha_c g_c e_{xc} + \alpha_c g_c e_{zc} + \alpha_c^2 g_c^2 e_{zc}^2 + \alpha_c^2 g_c^2 e_{xc} e_{zc} + \frac{3}{2}\alpha_c^2 g_c^2 e_{xc}^2 + \alpha_c g_c e_{xc} e_{yc} + \alpha_c g_c e_{yc} e_{zc}\right) \quad (86)$$

$$Bias\left(\hat{\bar{Y}}^{*ST}_{PCRDOCE}\right) = \sum_{h=1}^{L} W_h\lambda_h \left(\frac{3}{2}R_{xc}^2\alpha_c^2 g_c^2 S_{xh}^2 + \frac{1}{2}R_{zc}^2\alpha_c^2 g_c^2 S_{zh}^2 + R_{xc}R_{zc}\alpha_c^2 g_c^2 S_{xzh} + R_{xc}\alpha_c g_c S_{yxh} + R_{xc}\alpha_c g_c S_{yzh}\right)/\bar{Y}^c \quad (87)$$

Now to obtain the MSE of 63, pick the lower terms of error of (**??**), Square them and find their expectation; after substituting the result of expectation of (64) we obtain

$$MSE\left(\hat{\bar{Y}}^{*ST}_{PCRDOCE}\right) = \bar{Y}^{c2}\left(\sum_{h=1}^{L} W_h^2\lambda_h\frac{S_{yh}^2}{\bar{Y}^{c2}} + 2\alpha_c g_c\sum_{h=1}^{L} W_h^2\lambda_h\frac{S_{yxh}}{\bar{X}^c\bar{Z}^c} + 2\alpha_c g_c\sum_{h=1}^{L} W_h^2\lambda_h\frac{S_{yzh}}{\bar{Y}^c\bar{Z}^c}\right.$$
$$\left. + 2\alpha_c^2 g_c^2\sum_{h=1}^{L} W_h^2\lambda_h\frac{S_{xzh}}{\bar{X}^c\bar{Z}^c} + \alpha^2 g_c^2\sum_{h=1}^{L} W_h^2\lambda_h\frac{S_{xh}^2}{\bar{X}^{c2}} + \alpha_c^2 g_c^2\sum_{h=1}^{L} W_h^2\lambda_h\frac{S_{zh}^2}{\bar{Z}^{c2}}\right) \quad (88)$$

$$MSE\left(\hat{\bar{Y}}^{*ST}_{PCRDOCE}\right) = \bar{Y}^{c2}\sum_{h=1}^{L} W_h^2\lambda_h\left(\frac{S_{yh}^2}{\bar{Y}^{c2}} + \alpha_c^2 g_c^2\frac{S_{xh}^2}{\bar{X}^{c2}} + \alpha_c^2 g_c^2\frac{S_{zh}^2}{\bar{Z}^{c2}}\right.$$
$$\left. + 2\alpha_c^2 g_c^2\frac{S_{xzh}}{\bar{X}^c\bar{Z}^c} + 2\alpha_c g_c\frac{S_{yxh}}{\bar{X}^c\bar{Z}^c} + 2\alpha_c g_c\frac{S_{yzh}}{\bar{Y}^c\bar{Z}^c}\right) \quad (89)$$

Finally, noting that

$$MSE\left(\hat{\bar{Y}}^{*ST}_{PCRDOCE}\right) = \sum_{h=1}^{L} W_h^2\lambda_h\left(S_{yh}^2 + R_{xc}^2\alpha_c^2 g_c^2 S_{xh}^2 + R_{zc}^2\alpha_c^2 g_c^2 S_{zh}^2\right.$$
$$\left. + 2R_{xc}R_{zc}\alpha_c^2 g_c^2 S_{xzh} + 2R_{xc}\alpha_c g_c S_{yxh} + 2R_{zc}\alpha_c g_c S_{yzh}\right) \quad (90)$$

From equation (90) differentiate with respect to α_c

$$\alpha_c = -\sum_{h=1}^{L} W_h^2\lambda_h\left(\frac{R_{xc}S_{yxh} + R_{zc}S_{yzh}}{R_{xc}^2 S_{xh}^2 + R_{zc}^2 S_{zh}^2 + 2R_{xc}R_{zc}S_{xzh}}\right)\frac{1}{g_c} \quad (91)$$

$$MSE\left(\hat{\bar{Y}}^{*ST}_{PCRDOCE}\right) = \sum_{h=1}^{L} W_h^2\lambda_h\left[S_{yh}^2 + \alpha_{min}^2 g_c^2\left(R_{xc}^2 S_{xh}^2 + R_{zc}^2 S_{zh}^2 + 2R_{xc}R_{zc}S_{xzh}\right) + 2\alpha_{min} g_c\left(R_{xc}S_{yxh} + R_{zc}S_{yzh}\right)\right] \quad (92)$$

Substituted (91) into (92) we obtain complete equation of minimum mean square error below

$$MSE\left(\hat{\bar{Y}}^{*ST}_{PCRDOCE}\right)min = \sum_{h=1}^{L} W_h^2\lambda_h\left(S_{yh}^2 - \frac{(R_{xc}S_{yxh} + R_{zc}S_{yzh})^2}{(R_{xc}^2 S_{xh}^2 + R_{zc}^2 S_{zh}^2 + 2R_{xc}R_{zc}S_{xzh})}\right) \quad (93)$$

4 Empirical Study

The performance of proposed estimator is always assessed by many author using empirical study, comparing existing estimators over proposed estimators. Twenty existing estimators were compare with two proposed estimators using the following three criteria Bias, Mean Square Error (MSE) and Percentage Relative Efficiency(PRE) by adopted real life data and simulated data sets as given in the following subsection (4.1) and (4.2).

4.1 Real Life Application

The data sets are considered to see the efficiency of the proposed estimators with respect to the conventional estimators review in literature, summary or descriptive statistics of the two natural population are reported. The population one was earlier presented by [33], [35] and [23]. The population two is earlier employed by [16], Rajesh and Mukesh(2012), Rajesh etal(2016), [26] and [14]). Description of the two populations are given to details in table 1 and 2 bellow :

Table 1: population one [Source: [20, pg. 228]], Y: Output, X: Fixed Capital, Z: Numbers of Workers

		Stratum 1	Stratum 2
Str Population Size	N_h	5	5
Str Samample Size	n_h	2	3
Str Sample mean y	$\bar{Y}_h$	1925.8	3115.6
Str Sample mean x	$\bar{X}_h$	214.4	333.8
Str Sample mean z	$\bar{Z}_h$	51.8	60.6
Str Standard Dev y	S_{yh}	615.92	340.38
Str Standard Dev x	S_{xh}	74.87	66.35
Str Standard Dev z	S_{zh}	0.75	4.84
Str Standard Dev xy	S_{yxh}	39360.68	22356.5
Str Standard Dev yz	S_{yzh}	411.16	1536.24
Str Standard Dev xz	S_{xzh}	38.08	287.92

Table 2: population two [Source: Six different district in Turkey in the years 2007 (2007)], Y: Number of teachers X: Number of students Z: Number of classes

	Stratum 1	Stratum 2	Stratum 3	Stratum 4	stratum 5	Stratum 6
N_h	127	117	103	170	205	201
n_h	31	21	29	38	22	39
$\bar{Y}_h$	703.74	413.6	573.17	424.66	267.63	393.84
$\bar{X}_h$	20804.59	9211.79	14309.3	9478.85	5569.95	12997.59
$\bar{Z}_h$	498.28	318.33	431.36	311.32	227.2	313.71
S_{yh}	883.835	644.922	1033.467	810.585	403.654	711.723
S_{xh}	30486.751	15180.769	27549.697	18218.931	8497.77	23094.141
S_{zh}	555.5816	365.4776	612.9509281	458.0282	260.8511	397.0481
S_{yxh}	25237153.52	9747942.85.5	28298397.04	14523885	3393591.75	15864573.97
S_{yzh}	480688.2	230092.8	623019.3	364943.4	101539.01	277696.1
S_{xzh}	15914648	5379190.92	16490674.5	8041254	2144057	8857729

Table 3: Statistical Analysis of different estimators based on real population one

S/No.	Estimators	Authors of Recent sep and comb Estimators	POPULATION ONE		
			BIAS	**MSE**	**PRE**
1	$\hat{\bar{Y}}_{11}^{ST}$		161.5036	12276.97	126.36
2	$\hat{\bar{Y}}_{12}^{ST}$	[15],[13]	13.48875	33892.38	45.772
3	$\hat{\bar{Y}}_{13}^{ST}$		180.5041	145961	10.62833
4	$\hat{\bar{Y}}_{21}^{ST}$		7.124364	12944	119.849
5	$\hat{\bar{Y}}_{22}^{ST}$	[16]	19.63428	151966	10.2083
6	$\hat{\bar{Y}}_{23}^{ST}$		-22.71543	11749.19	132.037
7	$\hat{\bar{Y}}_{24}^{ST}$		28.40568	148948	10.4152
8	$\hat{\bar{Y}}_{31}^{ST}$	[38]	4.486646	145961	10.6283
9	$\hat{\bar{Y}}_{41}^{ST}$		-54.88745	10304	150.5552
10	$\hat{\bar{Y}}_{42}^{ST}$	[28]	59.6446	33085.67	46.88804
11	$\hat{\bar{Y}}_{51}^{ST}$		1.147122	10093.5	153.695
12	$\hat{\bar{Y}}_{52}^{ST}$	[27]	4.793144	79604.7	19.4878
13	$\hat{\bar{Y}}_{53}^{ST}$		-14.47166	10549.3	147.054
14	$\hat{\bar{Y}}_{54}^{ST}$		3.610031	77042.5	20.1359
15	$\hat{\bar{Y}}_{61}^{ST}$		-11.25815	19282.5	80.4523
16	$\hat{\bar{Y}}_{62}^{ST}$	[17]	118.4702	34254.81	45.28771
17	$\hat{\bar{Y}}_{71}^{ST}$	Singh, 1995(as cited in [30])	-12.52805	44695.92	34.70836
18	$\hat{\bar{Y}}_{81}^{ST}$		26.08017	14866.7	104.349
19	$\hat{\bar{Y}}_{82}^{ST}$	[34]	31.9425	33246.74	46.66087
20	$\hat{\bar{Y}}_{91}^{ST}$	[23]	11.09063	13042	118.948
21	$\hat{\bar{Y}}_{PCRDOSE}^{*,ST}$		**-3.691608**	**7168.4**	**216.41**
22	$\hat{\bar{Y}}_{PCRDOCE}^{*,ST}$	Proposed Estimators	**-1.834172**	**7631.012**	**203.2918**

Table 4: Statistical Analysis of different estimators based on real population two

S/No.	Estimators	Authors of Recent sep and comb Estimators	POPULATION TWO		
			BIAS	**MSE**	**PRE**
1	$\hat{\bar{Y}}_{11}^{ST}$		4.739442	200.937	37.3094
2	$\hat{\bar{Y}}_{12}^{ST}$	[15],[13]	-4.929927	2324.342	3.225366
3	$\hat{\bar{Y}}_{13}^{ST}$		5.085323	9195.542	0.8153
4	$\hat{\bar{Y}}_{21}^{ST}$		2.935441	130.4	57.4914
5	$\hat{\bar{Y}}_{22}^{ST}$	[16]	8.229396	8996.7	0.83329
6	$\hat{\bar{Y}}_{23}^{ST}$		146.7123	204.29	36.6972
7	$\hat{\bar{Y}}_{24}^{ST}$		20.43594	8808.62	0.85108
8	$\hat{\bar{Y}}_{31}^{ST}$	[38]	0.425800	9195.54	0.81527
9	$\hat{\bar{Y}}_{41}^{ST}$		1.865101	1092.6	6.8615
10	$\hat{\bar{Y}}_{42}^{ST}$	[28]	-0.561662	2276.544	3.293085
11	$\hat{\bar{Y}}_{51}^{ST}$		1.515125	622.236	12.0482
12	$\hat{\bar{Y}}_{52}^{ST}$	[27]	2.368766	5020.75	1.49317
13	$\hat{\bar{Y}}_{53}^{ST}$		-5.710349	670.411	11.1825
14	$\hat{\bar{Y}}_{54}^{ST}$		-0.280461	4880.86	1.53597
15	$\hat{\bar{Y}}_{61}^{ST}$		-1.228824	1324.3	5.66099
16	$\hat{\bar{Y}}_{62}^{ST}$	[17]	43.42974	2251.897	3.329127
17	$\hat{\bar{Y}}_{71}^{ST}$	Singh, 1995(as cited in [30])	-1.519934	40946.5	0.18309
18	$\hat{\bar{Y}}_{81}^{ST}$		4.519089	1734.79	4.32147
19	$\hat{\bar{Y}}_{82}^{ST}$	[34]	5.952849	2240.565	3.345966
20	$\hat{\bar{Y}}_{91}^{ST}$	[23]	0.280461	1721.92	4.35378
21	$\hat{\bar{Y}}_{PCRDOSE}^{*,ST}$		**-0.455423**	**62.391**	**120.16**
22	$\hat{\bar{Y}}_{PCRDOCE}^{*,ST}$	Proposed Estimators	**-1.507569**	**103.16**	**72.675**

Table 3 and 4 show the results of the Biass, MSEs and PREs of the proposed and related existing estimators considered in this study obtained from population one and two respectively. The results revealed that proposed estimators $\hat{\bar{Y}}_{PCRDOSE}^{*,ST}$ and $\hat{\bar{Y}}_{PCRDOCE}^{*,ST}$ have the minimum MSE and higher PRE compared to that of classical $\hat{\bar{Y}}_{11}^{ST}$, $\hat{\bar{Y}}_{12}^{ST}$, $\hat{\bar{Y}}_{13}^{ST}$, $\hat{\bar{Y}}_{21}^{ST}$, $\hat{\bar{Y}}_{22}^{ST}$, $\hat{\bar{Y}}_{23}^{ST}$, $\hat{\bar{Y}}_{24}^{ST}$, $\hat{\bar{Y}}_{31}^{ST}$, $\hat{\bar{Y}}_{41}^{ST}$, $\hat{\bar{Y}}_{42}^{ST}$, $\hat{\bar{Y}}_{51}^{ST}$, $\hat{\bar{Y}}_{52}^{ST}$, $\hat{\bar{Y}}_{53}^{ST}$, $\hat{\bar{Y}}_{54}^{ST}$, $\hat{\bar{Y}}_{61}^{ST}$, $\hat{\bar{Y}}_{62}^{ST}$, $\hat{\bar{Y}}_{71}^{ST}$, $\hat{\bar{Y}}_{81}^{ST}$, $\hat{\bar{Y}}_{82}^{ST}$, and $\hat{\bar{Y}}_{91}^{ST}$, related estimators reviews in the study

4.2 Simulated Study

After the real life justification of the results, the stimulation studies were also consider to assess the performance of the proposed estimators over other estimator considered in the study. Data of size 1000 units were generated for study population in stratified random. This classified into $3(h = 1,2,3)$ and the error term e_h are normally distributed with mean of zero and standard deviation of one for non-overlapping heterogeneous groups of size 500, 200 and 300 using function defined in Table (4) Samples of sizes 60, 40 and 50 were selected 10,000 times by the SRSWOR method from each stratum respectively. The precision Biases, MSEs and PREs of the considered estimators were computed using eq 94, 95 and 96

$$Bias(\hat{\theta}_s) = \frac{1}{10000}\sum_{s=1}^{10000}\left(\hat{\theta}_s - \bar{Y}\right), \hat{\theta}_s = \bar{y}, \hat{\hat{Y}}_{ij}^{ST}, i,j = \{(1,2,3,4,5,6,7,8,9),(1,2,3,4)\}, \hat{\hat{Y}}_{PCRDOSE}^{*,ST}\ \hat{\hat{Y}}_{PCRDOCE}^{*,ST} \tag{94}$$

$$MSE(\hat{\theta}_s) = \frac{1}{10000}\sum_{s=1}^{10000}\left(\hat{\theta}_s - \bar{Y}\right)^2, \hat{\theta}_s = \bar{y}, \hat{\hat{Y}}_{ij}^{ST}, i,j = \{(1,2,3,4,5,6,7,8,9),(1,2,3,4)\}, \hat{\hat{Y}}_{PCRDOSE}^{*,ST}\ \hat{\hat{Y}}_{PCRDOCE}^{*,ST} \tag{95}$$

$$PRE(\hat{\theta}_s) = \left(\frac{VAR(\hat{\theta}_s)}{MSE(\hat{\theta}_s}\right) \times 100, \hat{\theta}_s = \bar{y}, \hat{\hat{Y}}_{ij}^{ST}, i,j = \{(1,2,3,4,5,6,7,8,9),(1,2,3,4)\}, \hat{\hat{Y}}_{PCRDOSE}^{*,ST}\ \hat{\hat{Y}}_{PCRDOCE}^{*,ST} \tag{96}$$

Table 5: Distributions used to Simulate Population Data

Populations	Auxiliary variable(x)	Auxiliary variable(z)	Study variable(y)
I	$X_h \sim exp(0.2,0.3,0.1)$	$Z_h \sim exp(0.2,0.3,0.1)$	$Y_h = 3X_h + 6Z_h + e_h$
II	$X_h \sim poi(2,3,1)$	$Z_h \sim poi(2,3,1)$	$Y_h = 3X_h + 6Z_h + e_h$
III	$X_h \sim gamma((1.2,1.8),(1.3,1.8),(1.1,1.8))$	$Z_h \sim gamma((1.2,1.8),(1.3,1.8),(1.1,1.8))$	$Y_h = 60X_h + 50Z_h + e_h$
IV	$X_h \sim Chisquare(5,3,1)$	$Z_h \sim Chisquare(5,3,1)$	$Y_h = 60X_h + 50Z_h + e_h$
V	$X_h \sim Geometric(0.2,0.3,0.1)$	$Z_h \sim Geometric(0.2,0.3,0.1)$	$Y_h = 3X_h + 6Z_h + e_h$
VI	$X_h \sim Beta((1,2),(1,3),(1,4))$	$Z_h \sim Beta((1,2),(1,3),(1,4))$	$Y_h = 30X_h + 60Z_h + e_h$
VII	$X_h \sim Weibull((2,1),(3,1),(4,1))$	$Z_h \sim Weibull((2,1),(3,1),(4,1))$	$Y_h = 30X_h + 6Z_h + e_h$
VIII	$X_h \sim Log_N(0.2,0.3,0.1)$	$Z_h \sim Log_N(0.2,0.3,0.1)$	$Y_h = 3X_h + 6Z_h + e_h$

Table 6: Statistical Analysis of different estimators based on Simulated Data

	Exponential			Poisson			Gamma			Chi-Square			Geometric			Beta			Weibull			Log-Normal		
estimator	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
$\hat{\bar{Y}}_{10}^{ST}$	-274.3394	877.1656	100	-26.55745	72.10354	100	-102.35	1068.2	100	-52.2276	276.7463	100	-78.3889	634.2146	100	-40.0412	163.7676	100	105.69	1124.167	100	-32.2405	107.6702	100
$\hat{\bar{Y}}_{11}^{ST}$	-274.8128	848.3692	103.3943	-26.39243	70.27144	102.6072	101.88	1038.69	106.805	-51.9888	270.8413	102.1802	-78.217	612.4341	103.5564	-39.8036	158.8724	103.0812	106.25	1129.613	99.51792	-32.1117	103.7727	103.7558
$\hat{\bar{Y}}_{12}^{ST}$	-274.3113	856.6403	102.396	-26.55264	71.10939	101.3981	102.38	1048.84	101.846	-52.182	272.8444	101.4301	-78.6373	619.0049	102.4571	-39.9924	160.3901	102.1058	105.58	1115.367	100.7769	-32.2759	104.8022	102.7366
$\hat{\bar{Y}}_{13}^{ST}$	-275.5203	7676.113	98.32204	-84.75121	722.9542	99.16823	-299.18	9035.3	98.6313	-151.16	2302.866	98.47763	76.9635	5864.074	97.79753	-123.163	1523.941	99.23088	-106.09	2618.476	98.46149	92.3042	866.2929	97.49899
$\hat{\bar{Y}}_{21}^{ST}$	-275.3555	858.6065	102.1615	-26.38945	71.39358	100.9944	-101.41	1052.48	101.494	-51.917	272	101.745	-77.5404	621.9547	101.9712	-39.6484	161.1702	101.6116	105.88	1131.863	99.32003	-31.721	105.0108	102.5325
$\hat{\bar{Y}}_{22}^{ST}$	91.1735	938.1256	93.36779	26.39646	75.4993	95.91874	101.25	1124.83	95.0188	51.7168	289.7105	95.54438	76.8045	691.012	91.89332	39.8463	171.7442	95.50574	-106.09	1168.725	96.3168	31.5574	118.506	91.28535
$\hat{\bar{Y}}_{23}^{ST}$	274.3792	7549.862	99.96958	84.90047	718.123	99.93711	298.34	8924.02	99.768	150.839	2279.72	100.0869	238.944	5729.411	100.1016	122.941	1517.968	99.68614	160.22	2576.054	100.1217	-91.708	844.9595	99.99388
$\hat{\bar{Y}}_{24}^{ST}$	-276.5694	7597.95	97.29975	-84.75121	726.5095	98.68294	-300.26	1115.32	95.829	-151.341	2315.529	98.47763	-241.565	5936.279	96.61323	-123.503	1531.863	98.71768	-161.02	2637.819	97.73946	-92.7778	878.284	96.16786
$\hat{\bar{Y}}_{31}^{ST}$	275.7948	7628.909	98.93374	84.86384	721.7827	99.43039	299.59	8996.21	97.2273	151.34	2294.932	99.42342	240.518	5864.074	98.79203	123.305	1522.474	99.3911	160.7	2591.784	99.51404	92.5037	859.755	98.27309
$\hat{\bar{Y}}_{41}^{ST}$	-274.1813	864.3138	101.4869	-26.57703	71.46801	100.8893	-102.49	1056.15	101.141	-52.2552	274.3447	100.8754	-78.6638	624.1599	101.6109	-40.0531	161.762	101.2399	-105.59	1117.786	100.7638	-32.3267	105.8489	101.7206
$\hat{\bar{Y}}_{42}^{ST}$	-274.7844	7597.95	99.33351	-84.64182	719.3267	99.66834	-298.53	8960.47	99.455	150.941	2289.123	99.61361	77.8446	5786.999	99.10545	-123.011	1516.557	99.714	-160.45	2594.791	99.36021	-91.9208	853.2051	98.99458
$\hat{\bar{Y}}_{51}^{ST}$	-274.5582	862.5657	101.6926	-26.51885	71.45644	100.9056	-102.14	1055.33	101.22	52.1814	273.3339	101.2485	-78.2754	622.8615	101.8227	-39.92	161.8129	101.208	105.7	1122.2	100.3674	-32.1276	105.494	102.0628
$\hat{\bar{Y}}_{52}^{ST}$	92.10306	900.8361	97.2327	26.54478	73.6649	98.30731	102.08	1091.43	97.9266	52.0927	282.1184	98.11559	77.9215	656.8484	96.67281	40.018	167.2154	98.09238	105.79	1141.313	98.63011	32.0628	112.2298	96.39028
$\hat{\bar{Y}}_{53}^{ST}$	274.0663	7522.581	100.3321	84.59045	716.6259	100.1459	297.97	8889.46	100.746	150.787	2276.544	100.2265	238.711	5708.617	100.4662	122.862	1512.525	100.0449	160.18	2570.661	100.3317	91.5752	840.7621	100.4931
$\hat{\bar{Y}}_{54}^{ST}$	-275.1619	7624.191	98.99162	-63.33711	720.4366	99.51479	-298.91	1085.57	98.4552	-150.995	2293.615	100.2206	-239.996	5810.01	98.71292	-123.142	1518.928	99.55837	-160.57	2600.907	99.12657	-92.0916	856.9695	98.55974
$\hat{\bar{Y}}_{61}^{ST}$	-60.75757	9422.936	9.308834	-84.69365	904.6404	7.970409	67.613	125877	9.62774	-168.492	2842.215	9.736997	-267.715	7178.6	8.834796	-137.804	1899.305	8.622503	-206.72	4278.378	26.32595	-102.612	1054.921	10.20647
$\hat{\bar{Y}}_{62}^{ST}$	1025.168	113470.1	6.651365	316.2041	10282.95	6.972134	1085.7	167125	7.07965	537.649	56543.5	9.736997	918.037	103579.9	5.537012	454.646	21540.9	7.020226	1711.1	276293	0.933136	-352.9	42471.76	1.988679
$\hat{\bar{Y}}_{71}^{ST}$	-208.2759	2524.003	34.75295	-47.97971	230.7687	31.24494	227.3	3006.38	35.5311	-84.5942	716.7549	38.61101	-136.398	1865.123	34.0039	-68.8069	474.8188	34.49054	-65.6198	15.08474	7466.641	-53.8322	290.8466	37.01957
$\hat{\bar{Y}}_{81}^{ST}$	-230.8916	5331.789	16.45162	-70.82577	501.6972	14.37192	149.24	6315.38	16.9143	-127.334	1621.555	38.61101	-200.685	4028.102	15.74475	-103.197	1064.784	15.38036	-96.154	924.8662	121.7823	-77.6898	603.7272	17.83424
$\hat{\bar{Y}}_{82}^{ST}$	-864.5172	77899.26	9.688552	268.2003	7338.302	9.769847	928.79	89546.7	9.951 94	1778.15	22212.07	10.26594	-447.405	61819.45	9.277389	-386.284	15209.5	9.942604	-942.81	100630	2.562048	-288.023	8884.211	9.507056
$\hat{\bar{Y}}_{91}^{ST}$	-2970.82	962718.9	0.090983	-921.1678	88418.37	0.081904	-3169.9	1075599	0.09937	-1578.68	262861.6	0.105303	-2658.93	785624.7	0.080827	-1318.48	182220.1	0.090015	-4405.4	1967194	0.057223	-998.371	116474.2	0.092878
$\hat{\bar{Y}}_{PCRDOSE}^{*,ST}$	136.2099	1860.394	405.6971	44.15938	196.0911	365.9888	159.37	2546.6	349.942	-77.1917	597.1727	382.0837	119.624	1435.403	399.5553	61.6236	380.1873	398.0152	-15.726	142.9112	1804.749	49.7726	249.4568	338.6991
$\hat{\bar{Y}}_{PCRDOCE}^{*,ST}$	92.21787	850.4137	887.5168	26.28987	69.11576	1038.361	102.14	1043.28	854.194	-51.8998	269.3592	847.0845	78.3861	614.438	933.4108	39.6993	157.6003	960.1525	-105.84	1120.291	230.2249	32.026	102.5666	823.765

Table 6 show the results of the Biases, MSEs and PREs of the proposed and related existing estimators considered in this study obtained from simulated data under eight different distributions respectively. The results revealed that proposed estimators (62) and (63) have the minimum Bias, MSE and higher PRE compared to the review estimators of (1), (2), (3), (10), (11), (12), (13), (22), (26), (27), (32), (33), (34), (35), (44), (45), (50), (53), (54) and (59) related estimators reviews in the study.

4.3 Results and Findings

Twenty ratio and product or type estimators were review from different authors and two ratio-product-type estimators is proposed for separate and combined in stratified random sampling under two auxiliaries variables.The estimator were formulated using ideas and version of [39, 24, 37] and adopted strategy given by [16, 27, 40, 23] in terms of two auxiliaries variables in simple random sampling. The bias, MSE, and PRE of the proposed estimators were derived up to fist order of approximation by Taylor series expansion. The efficiency comparison were perform theoretical and empirical. It was observed that proposed estimator are better than existing estimator with less minimum mean square error and maximum percentage relative efficiency indicate in tables 3, 4 and 6.

5 Conclusion

The results of the empirical study revealed that proposed estimators do better than other mentioned ratio estimators having less Bias, mean square error (MSE) and the maximum Percentage Relative Efficiency (PRE) which proved that its more efficient table.

In general, proposed estimator(On The Efficiency of Modified Exponential Dual to Ratio-Product-Cum Type Estimator) are better than existing estimators.

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