Short Research Article

On propositions pertaining to the Riemann Hypothesis III

Abstract

In this paper, we consider further propositions concerning the range of possible distributions over the unit circle, for the Riemann zeta function as in Basu [2022], Basu [2023a] and Basu [2023b]. We also derive some new upper bounds on the sum of norms for the tail sequence corresponding to the Riemann zeta function. We discuss some hypotheses, conditional on which, properties of concentrated distributions may be obtained. Specific sub-classes are shown of distributions that occur infinitely often along the imaginary axis.

Keywords: Riemann Zeta Function; Riemann Hypothesis; Probability measures on a circle

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1 Introduction

The subject of study for this paper is the Riemann zeta function and the Riemann Hypothesis (Riemann [1859], Stein and Shakarchi [2010]). The point of view is in line with and extends prior research in Basu [2022],Basu [2023a] and Basu [2023b]. The main interest is in understanding the pattern of zeros and non-zeroes. We draw connections with associated distributions for complex functions defined by the absolutely convergent series and consider concentration phenomena in associated probability measures over the unit circle, for the given series. Such an undertaking is linked to solving for bounds on the tail of the sequence that defines the series. Previous research on this subject may be found in Mangoldt [1905], Hardy [1914], Hardy and Littlewood [1921], Conrey [2003], Lagarias [2002], Bump et al. [2000], Borwein et al. [2008], Platt and Trudgian [2021], Nicolas [2021], Johnston [2022], Maynard and Pratt [2022] and Guth and Maynard [2024].

2 Probability measures over the unit circle

We present the framework for the analysis in this paper. For the appropriate notation, we refer to Basu [2022], Basu [2023a] and Basu [2023b]. The concentration theorem is as follows. (Basu [2023a]) Let μ be a probability measure on the unit circle $\mathbb{S}^1$. Then,

$$\mathbb{E}_\mu[z] \neq 0 \text{ if there exist numbers } 0 \leq \theta' \leq \theta'' \leq 2\pi \text{ such that } \theta'' - \theta' \leq \pi \text{ and}$$
$$\mu(\{z : \theta' \leq \theta(z) \leq \theta''\}) > \frac{1}{1 + \cos\left(\frac{\theta'' - \theta'}{2}\right)}. \quad (2.1)$$

Of course, by symmetry of the circle, this also applies to arcs of the form described by $0 \leq \theta' \leq \theta'' \leq 2\pi$ and $\theta' + (2\pi - \theta'') \leq \pi$, requiring the concentration condition

$$\mu(\{z : \theta(z) \in [0, \theta'] \cup [\theta'', 2\pi]\}) > \frac{1}{1 + \cos\left(\frac{\theta' + (2\pi - \theta'')}{2}\right)}. \tag{2.2}$$

For each complex variable (Stein and Shakarchi [2010], Pierpont [1914]) given by $(\sigma, t) \in S = \{(\sigma, t) \in \mathbb{R}^2 : \sigma \in (0, 1); t \neq 0\}$, we define $Z_0(s) = (1, 0)$; $Z_n(s) = \frac{1}{(2n+1)^s} - \frac{1}{(2n)^s}$, for each $n \in \mathbb{Z}^+$. Hence, the Riemann zeta function is $\zeta(s) = (\frac{1}{1-2^{1-s}}) \times \sum_{n \geq 0} Z_n(s)$. We also define $\zeta^*(s) = \sum_{n \geq 0} Z_n(s)$. For $s = (\sigma, t) \in S$, the probability measure μ_s on $\mathbb{S}^1$ is defined as follows. For each measurable subset $A \subseteq \mathbb{S}^1$,

$$\mu_s(A) = \frac{\sum_{n \geq 0 : Z_n(s) \in A} ||Z_n(s)||}{\sum_{n \geq 0} ||Z_n(s)||}. \tag{2.3}$$

We shall say that μ_s is concentrated if either condition 2.1 or 2.2 is satisfied. Define the set of concentrated points in S to be $S^* = \{s \in S : \mu_s \text{ is concentrated}\}$.

We are interested in providing an upper bound for the sum of norms corresponding to the tail i.e.

$$Z^*(\sigma, t, m) := \sum_{n \geq m+1} ||Z_n(s)||. \tag{2.4}$$

Before proceeding, we introduce some notation. Suppose we are given two real-valued functions defined on a subset of $X \subseteq \mathbb{R}^d$ i.e $f, g : X \to \mathbb{R}$. We say that $f(x) = O(g(x))$ if for some $\phi > 0$ and $y > 0$, $|f(x)| \leq \phi |g(x)|$ for each $x \in X$ such that $\max_{i \in \{1, \ldots d\}} x_i \geq y$. This concerns the asymptotic order or the rate of growth of f by an upper bound that is provided by the function g.

Consider the following hypothesis.

$$\text{For each } \sigma \in [0.5, 1), \text{ there exists } \alpha \in (0, 1) \text{ and } k \in \mathbb{N} \text{ such that } Z^*(\sigma, t, m) = O((\ln(|t|))^k m^{-\alpha}). \tag{2.5}$$

The purpose of considering a bound on the tail is to be able to construct t values i.e. imaginary parts of the complex input, which position the points in the set $\{Z_n(s)\}_{1 \leq n \leq m}$ such that the points $Z_n(s)$ are all contained in an arc, which corresponds to a subset of $[0, 2\pi]$. For large m, the sum of norms $Z^*(\sigma, t, m)$ would be small, resulting in the distribution $\mu_{(\sigma,t)}$ to be concentrated on the arc. Since m is large, the t values needed for this construction would be large, which would increase the value of a bound on $Z^*(\sigma, t, m)$ that is increasing in $|t|$. For example, consider the bound from Basu [2023a] which leads to $Z^*(\sigma, t, m) \leq \left(1 + \frac{|t|}{\sigma}\right) \frac{1}{(2m+2)^\sigma}$. We now show some different bounds on $Z^*(\sigma, t, m)$. Note that

$$\begin{aligned} Z_n(s) &= \frac{1}{(2n+1)^s} - \frac{1}{(2n)^s} \\ &= \Big(\frac{1}{(2n)^s}\Big) \times \Big(\Big(\frac{2n}{2n+1}\Big)^s - 1\Big). \end{aligned} \tag{2.6}$$

Hence,

$$\begin{aligned}||Z_n(s)|| &= \Big|\Big|\frac{1}{(2n)^s}\Big|\Big| \times \Big|\Big|\Big(\frac{2n}{2n+1}\Big)^s - 1\Big|\Big| \\ &= \Big(\frac{1}{(2n)^\sigma}\Big) \times \Big|\Big|\Big(\frac{2n}{2n+1}\Big)^\sigma\Big(\cos\Big(t\ln\Big(\frac{2n}{2n+1}\Big)\Big), \sin\Big(t\ln\Big(\frac{2n}{2n+1}\Big)\Big) - (1,0)\Big|\Big| \\ &= \sqrt{((2n)^{-\sigma} - (2n+1)^{-\sigma})^2 + 2(2n)^{-\sigma}(2n+1)^{-\sigma}\Big(1 - \cos\Big(|t|\ln\Big(1+\frac{1}{2n}\Big)\Big)\Big)} \\ &\leq (2n)^{-\sigma} - (2n+1)^{-\sigma} + \sqrt{2}(2n)^{-\sigma}\Big|\sin\Big(|t|\ln\Big(1+\frac{1}{2n}\Big)\Big|\end{aligned} \tag{2.7}$$

The last inequality follows from the fact that for any two real numbers $x, y \geq 0$, it follows that $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ and also from the equality $|\sin(x)| = \sqrt{1-\cos(x)}$. Further,

$$\begin{aligned}(2n)^{-\sigma} - (2n+1)^{-\sigma} + \sqrt{2}(2n)^{-\sigma}\Big|\sin\Big(|t|\ln\Big(1+\frac{1}{2n}\Big)\Big)\Big| &\leq (2n)^{-\sigma} - (2n+1)^{-\sigma} + 2\sqrt{2}(2n)^{-\sigma}\ln\Big(1+|t|\ln\Big(1+\frac{1}{2n}\Big)\Big) \\ &\leq (2n)^{-\sigma} - (2n+1)^{-\sigma} + 2\sqrt{2}(2n)^{-\sigma}\ln\Big(1+\frac{|t|}{2n}\Big).\end{aligned}$$

In the first inequality above, we apply the fact that for any $x \geq 0$, we have that $|\sin(x)| \leq 2\ln(1+x)$. In the second inequality, we apply the fact that for any $x \geq 0$, we have $\ln(1+x) \leq x$, and that $\ln(x)$ is an increasing function. Hence, it follows that

$$\sum_{n\geq m+1} ||Z_n(s)|| \leq \sum_{n\geq m+1} (2n)^{-\sigma} - (2n+1)^{-\sigma} + 2\sqrt{2}(2n)^{-\sigma}\ln\Big(1+\frac{|t|}{2n}\Big).$$

Since $\ln(x)$ is increasing in x and n^{-1} is decreasing in n, the latter term, which depends on t, in the above sum may be bounded above by a definite integral as

$$\sum_{n\geq m+1} 2\sqrt{2}(2n)^{-\sigma}\ln\Big(1+\frac{|t|}{2n}\Big) \leq 2\sqrt{2}\int_{m+1}^{\infty} x^{-\sigma}\ln(1+x^{-1}|t|)dx \tag{2.8}$$

$$= 2\sqrt{2}\int_{(\frac{m}{m+1},1)} (1-\omega')^{\sigma-2}\ln(1+(1-\omega')|t|)d\nu(\omega') \tag{2.9}$$

$$= 2\sqrt{2}\int_{(0,\frac{1}{m+1})} \omega^{\sigma-2}\ln(1+\omega|t|)d\nu(\omega), \tag{2.10}$$

where ν is the uniform probability measure on $(0,1)$. The above holds due to the change of variables substitution $(1-\omega')^{-1} = x$. Another change of variables $\omega = 1-\omega'$ is applied, however, the last equality holds due to the symmetry of the density of the uniform probability measure around 0.5. We obtain the expectation of an unbounded random variable in ω over the measurable set $(0, \frac{1}{m+1})$.

One may also derive the polar form of $-Z_n(s)$. This is useful since, by varying t, one would position $-Z_n(s)$ in a given arc where the distribution is to be concentrated. Note that if $Z_n(s)$ leads to concentration on an arc, then so does $-Z_n(s)$ on the arc that is polar opposite on the unit circle.

Since $-Z_n(s) = \Big(\frac{1}{(2n)^s}\Big) \times \Big(1 - \Big(\frac{2n}{2n+1}\Big)^s\Big)$, one need only derive the angle of $1 - \Big(\frac{2n}{2n+1}\Big)^s$, then add it to the angle of $\frac{1}{(2n)^s}$ which is $-t\ln(2n)$. Note that as $\Big(\frac{2n}{2n+1}\Big)^s = \Big(\frac{2n}{2n+1}\Big)^\sigma\Big(\cos\Big(t\ln\Big(\frac{2n}{2n+1}\Big)\Big), \sin\Big(t\ln\Big(\frac{2n}{2n+1}\Big)\Big)$, by geometry, we may show that the angle $\hat{\theta}_n(\sigma,t)$ for $1 - \Big(\frac{2n}{2n+1}\Big)^s$ is

$$\hat{\theta}_n(\sigma,t) = \tan^{-1}\left(\frac{\sin(t\ln(\frac{2n}{2n+1}))}{\Big(\frac{2n+1}{2n}\Big)^\sigma - \cos(t\ln(\frac{2n}{2n+1}))}\right). \tag{2.11}$$

Hence, the polar form of $-Z_n(s)$ may be derived as

$$-Z_n(s) = ||Z_n(s)||(\cos(-t\ln(2n) - \hat{\theta}_n(\sigma,t)), \sin(-t\ln(2n) - \hat{\theta}_n(\sigma,t))). \tag{2.12}$$

Now, we shall define an optimization problem based on the above polar form.

$$\min_{(t,(q_n)_{n=2}^m)\in\mathbb{R}\times\mathbb{Z}^{m-1}} t$$

$$\text{subject to : } \Big| -t\Big(\frac{\ln(2n)}{2\pi}\Big) - \frac{\theta_n(\sigma,t)}{2\pi} - q_n\Big| \leq \frac{r}{2\pi} \text{ for all } 2 \leq n \leq m. \tag{2.13}$$

$$q_n \leq 0 \text{ for all } 2 \leq n \leq m. \tag{2.14}$$

$$\sum_{n=2}^{m} q_n \leq -1. \tag{2.15}$$

$$t \geq 0. \tag{2.16}$$

Note that the optimization problem has non-linear constraints (Schrijver [1998],Boyd and Vandenberghe [2004],Conforti et al. [2014]). Denote as $t^*(m,r)$, the optimal value of the above optimization problem. It exists since the set of feasible t values is non-empty and closed in $\mathbb{R}_+$. Non-emptiness would follow from simultaneous Diophantine approximation (Schmidt [1996],Kuipers and Niederreiter [2012]) of the real numbers $(\frac{-\ln(2n)}{2\pi}, \ln(\frac{2n}{2n+1}))_{1\leq n\leq m}$, applying continuity of $\theta_n(\sigma,t)$ in t. The continuity also yields closedness of the feasible t region. Note also that $t^*(m,r)$ would be non-decreasing in m, given that more constraints are added with larger m.

Consider the following hypotheses

$$\text{For each } r \in (0,1),\ \lim_{m\to+\infty} t^*(m,r) = +\infty \tag{2.17}$$

and

$$\text{For each } r \in (0,1),\ \text{there exists } Q \in \mathbb{N} \text{ such that } t^*(m,r) = O(m^Q). \tag{2.18}$$

Now, we prove the following proposition. Suppose that the hypotheses in 2.5, 2.17 and 2.18 are satisfied. Suppose that $\sigma \in [0.5,1)$. Then, there exist countably many pairwise disjoint intervals $\{[\underline{t}_k,\bar{t}_k]\}_{k\in\mathbb{Z}^+}$, such that $\lim_{k\to\infty}\underline{t}_k \to +\infty$ and for each k and $t \in [\underline{t}_k,\bar{t}_k]$, we have that $(\sigma,t) \in S_+^*$ i.e. $\mu_{(\sigma,t)}$ is concentrated.

Proof. Consider the following function of m,

$$Z^*(\sigma, t^*(m,r), m). \tag{2.19}$$

Note that by the given hypothesis, we have that $Z^*(\sigma,t^*(m,r),m) = O((Q\ln(m))^k m^{-\alpha})$, which converges to zero as $m \to +\infty$. Hence, for small r, by choosing $0 \leq t \leq t^*(m,r)$, we may simultaneously diminish the norm of the tail $Z^*(\sigma,t,m)$ and also position the points $\{Z_n(s)\}_{2\leq n\leq m}$ inside an arc around $(1,0)$, on which the probability measure $\mu_{(\sigma,t)}$ would be concentrated. □

Prime zeta function We next study some more properties of concentrated measures corresponding to series. We particularly consider the prime zeta function (Glaisher [1891],Fröberg [1968]). We recall a definition from Basu [2023b]. Suppose $f : [0,\pi] \to [0,1]$ is increasing. Then, a distribution μ on $\mathbb{S}^1$ is said to be f-equitable, if for each $0 \leq \theta' \leq \theta'' \leq 2\pi$ such that $\theta'' - \theta' \leq \pi$, we have that

$$\mu(\{z : \theta' \leq \theta(z) \leq \theta''\}) \leq f(\theta'' - \theta') \tag{2.20}$$

and for $0 \leq \theta' \leq \theta'' \leq 2\pi$ such that $\theta' + (2\pi - \theta'') \leq \pi$,

$$\mu(\{z : \theta(z) \in [0,\theta'] \cup [\theta'',2\pi]\}) \leq f(\theta' + (2\pi - \theta'')). \tag{2.21}$$

Define $f_0(\theta) = \frac{\theta}{2\pi}$ and $f_1(\theta) = \frac{1}{1+\cos(\frac{\theta}{2})}$. The uniform measure on the unit circle is the only f_0-equitable measure. Note that μ is f_1-equitable if and only if μ is not concentrated. We will now study f_1-equitable distributions for points in the domain of the prime zeta function.

Suppose that $\mathbb{P} \subseteq \mathbb{N}$ is the set of all prime numbers. For each $s = (\sigma, t)$, define the function

$$P(s) = \sum_{p\in\mathbb{P}} \frac{1}{p^s}. \tag{2.22}$$

The function P is the prime zeta function. Now, define μ_s^P to be the associated probability measure on the unit circle $\mathbb{S}^1$ as

$$\mu_s^P(A) := \frac{\sum_{p\in\mathbb{P}:p^{-s}\in A} p^{-\sigma}}{\sum_{p\in\mathbb{P}} p^{-\sigma}}. \tag{2.23}$$

We will first prove the following proposition in planar geometry that will be useful. Suppose that $\varphi_1, \varphi_2, \varphi_3 \geq 0$ such that $\varphi_1 + \varphi_2 + \varphi_3 = 1$. Then, $\max\{\varphi_1, \varphi_2, \varphi_3\} \leq 0.5$ if and only if there exist $z_1, z_2, z_3 \in \mathbb{S}^1$ such that

$$\varphi_1 z_1 + \varphi_2 z_2 + \varphi_3 z_3 = 0. \tag{2.24}$$

Proof. Note that in the plane there exists $z_1, z_2, z_3 \in \mathbb{S}^1$ such that $\varphi_1 z_1 + \varphi_2 z_2 + \varphi_3 z_3 = 0$ if and only if φ_1, φ_2 and φ_3 are the lengths of three sides of a triangle. This, in turn is equivalent to the condition $\max\{\varphi_1, \varphi_2, \varphi_3\} \leq 0.5$. □

Now, we shall prove the following proposition. There exists an interval $[\sigma_*, \sigma^*] \subseteq [1, \frac{7}{4}]$ such that for each $\sigma \in [\sigma_*, \sigma^*]$, there exist countably many pairwise disjoint intervals $\{[\underline{t}_k, \bar{t}_k]\}_{k\in\mathbb{Z}^+}$, such that $\lim_{k\to\infty} \underline{t}_k \to +\infty$ and for each k and $t \in [\underline{t}_k, \bar{t}_k]$, we have that $\mu^P_{(\sigma,t)}$ is f_1-equitable.

Proof. Let $\sigma^* > 1$ such that

$$\sum_{p\in\mathbb{P}:p\geq 7} \frac{1}{p^{\sigma^*}} = 0.5. \tag{2.25}$$

This follows from the continuity of the prime zeta function. We will now exhibit some necessary calculations. Note that, by an integral comparison,

$$\sum_{p\in\mathbb{P}:p\geq 7} \frac{1}{p^{\frac{7}{4}}} \leq \frac{4}{3(6)^{3/4}} < 0.5 \tag{2.26}$$

Hence, by the monotonicity of the prime zeta function in σ on the real line, $\sigma^* < \frac{7}{4}$. It follows that

$$\frac{1}{2^{\sigma^*}} + \frac{1}{3^{\sigma^*}} + \frac{1}{5^{\sigma^*}} > \frac{1}{2^{\frac{7}{4}}} + \frac{1}{3^{\frac{7}{4}}} + \frac{1}{5^{\frac{7}{4}}} > 0.5. \tag{2.27}$$

From the above derivations in 2.25,2.26 and 2.27 we have that

$$\max\left\{\frac{2^{-\sigma^*}}{P(\sigma^*)}, \frac{3^{-\sigma^*} + 5^{-\sigma^*}}{P(\sigma^*)}, \frac{\sum_{p\in\mathbb{P}:p\geq 7} p^{-\sigma^*}}{P(\sigma^*)}\right\} < 0.5. \tag{2.28}$$

In the above maximum, the three numbers add up to one. Hence, from Proposition 2, there exist points $z_A, z_B, z_C \in \mathbb{S}^1$ such that

$$\frac{2^{-\sigma^*}}{P(\sigma^*)} z_A + \frac{3^{-\sigma^*} + 5^{-\sigma^*}}{P(\sigma^*)} z_B + \frac{\sum_{p\in\mathbb{P}:p\geq 7} p^{-\sigma^*}}{P(\sigma^*)} z_C = 0. \tag{2.29}$$

Suppose, for large m, we now consider three disjoint sets $A = \{2^{\sigma^*}(2)^{-(\sigma^*,t)}\}$, $B = \{3^{\sigma^*}(3)^{-(\sigma^*,t)}, 5^{\sigma^*}(5)^{-(\sigma^*,t)}\}$ and $C = \{p^{\sigma^*}(p)^{-(\sigma^*,t)}\}_{p\in\mathbb{P}:7\le p\le m}$. Note that $A, B, C \subseteq \mathbb{S}^1$. Since for $m \ge 3$, the collection of numbers $\{\frac{\ln(p)}{2\pi}\}_{p\in\mathbb{P}:p\le m}$ is linearly independent over $\mathbb{Q}$, we apply the multidimensional Weyl criterion (Kuipers and Niederreiter [2012]) to the sequence $\left\{\{t\frac{\ln(p)}{2\pi}\}_{p\in\mathbb{P}:p\le m}\right\}_{t\in\mathbb{N}}$ to position A arbitrarily close to z_A, B arbitrarily close to z_B, and C arbitrarily close to z_C, by varying t. Since m is large, the norm of the tail would be negligible, so that we would approximate the zero mean distribution on the three points z_A, z_B, z_C on the unit circle with probability weights as in 2.28. Lastly, we choose $\sigma_* < \sigma^*$ such that σ_* is close to σ^*, so that the appropriate strict inequalities from above would be satisfied. □

We also have the following proposition. There exists an interval $[\sigma_*, \sigma^*] \subseteq [1, \frac{7}{4}]$ such that for each $\sigma \in [\sigma_*, \sigma^*]$, there exists an infinite subset of primes $T \subseteq \mathbb{P}$ such that for each $t \in T$, we have that $\mu^P_{(\sigma,t)}$ is f_1-equitable.

Proof. See Vinogradov [1948], Kuipers and Niederreiter [2012] and Harman [1991]. □

One may remark that in the above propositions, we present uncountably many f_1-equitable points. By appealing to the concentration phenomena in associated measures over the unit circle, we are able to concentrate the measure on any possible arc with span less than π, by varying t for the prime zeta function. This would also allow us to apply the argument principle on triangular contours, where the ends points would be concentrated on polar opposite arcs, ensuring a positive argument/angular change. Such an approach would generate countably many zeros and hence, countably many f_1-equitable points. In contrast, the above proposition constructs approximate zeros as we approximate a zero mean distribution on three points on the circle.

The concentration on any arc (not necessarily around $(1, 0)$) is possible because of the linear independence of $\{\frac{\ln(p)}{2\pi}\}_{p\in\mathbb{P}:p\le m}$ over $\mathbb{Q}$. Hence, equidistribution follows allowing inhomogenous Diophantine approximation with intercepts. This is in contrast with the Riemann zeta function $\zeta(s) = \sum_{n\ge1} n^{-s}$ on $\sigma > 1$, for which the associated collection $\{\frac{\ln(n)}{2\pi}\}_{n\in\mathbb{N}:2\le n\le m}$ is linearly dependent over $\mathbb{Q}$. However, it is possible to consider intercepts which are proportional to the real numbers to be simultaneously approximated.

3 Conclusion

We have shown some new results, which extends prior research. We have obtained propositions concerning distributional properties that occur infinitely often along the imaginary axis. This presents new insights into the problem studied in the paper.

References

Pathikrit Basu. On the riemann hypothesis. *Asian Journal of Probability and Statistics*, 20(4):51–56, 2022.

Pathikrit Basu. On propositions pertaining to the riemann hypothesis. *Asian Journal of Probability and Statistics*, 21(2):16–21, 2023a.

Pathikrit Basu. On propositions pertaining to the riemann hypothesis ii. *Asian Journal of Probability and Statistics*, 25(2):63–74, 2023b.

Peter Borwein, Stephen Choi, Brendan Rooney, and Andrea Weirathmueller. *The Riemann hypothesis: a resource for the afficionado and virtuoso alike*. Springer, 2008.

Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Daniel Bump, Kwok-Kwong Choi, Pär Kurlberg, and Jeffrey Vaaler. A local riemann hypothesis, i. *Mathematische Zeitschrift*, 233(1):1–18, 2000.

Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. *Integer programming models*. Springer, 2014.

J Brian Conrey. The riemann hypothesis. *Notices of the AMS*, 50(3):341–353, 2003.

Carl-Erik Fröberg. On the prime zeta function. *BIT Numerical Mathematics*, 8:187–202, 1968.

JWL Glaisher. On the sums of inverse powers of the prime numbers. *Quart. J. Math*, 25:347–362, 1891.

Larry Guth and James Maynard. New large value estimates for dirichlet polynomials. *arXiv preprint arXiv:2405.20552*, 2024.

Godfrey H Hardy and John E Littlewood. The zeros of riemann's zeta-function on the critical line. *Mathematische Zeitschrift*, 10(3):283–317, 1921.

Godfrey Harold Hardy. Sur les zéros de la fonction ζ (s) de riemann. *CR Acad. Sci. Paris*, 158(1914): 1012, 1914.

Glyn Harman. Diophantine approximation by prime numbers. *Journal of the London Mathematical Society*, 2(2):218–226, 1991.

Daniel R Johnston. Improving bounds on prime counting functions by partial verification of the riemann hypothesis. *The Ramanujan Journal*, 59(4):1307–1321, 2022.

Lauwerens Kuipers and Harald Niederreiter. *Uniform distribution of sequences*. Courier Corporation, 2012.

Jeffrey C Lagarias. An elementary problem equivalent to the riemann hypothesis. *The American Mathematical Monthly*, 109(6):534–543, 2002.

H von Mangoldt. Zur verteilung der nullstellen der riemannschen funktion........ *Mathematische Annalen*, 61:1–19, 1905.

James Maynard and Kyle Pratt. Half-isolated zeros and zero-density estimates. *arXiv preprint arXiv:2206.11729*, 2022.

Jean-Louis Nicolas. The sum of divisors function and the riemann hypothesis. *The Ramanujan Journal*, pages 1–45, 2021.

James Pierpont. *Functions of a complex variable*. Ginn, 1914.

Dave Platt and Tim Trudgian. The riemann hypothesis is true up to 3· 10 12. *Bulletin of the London Mathematical Society*, 53(3):792–797, 2021.

Bernhard Riemann. Ueber die anzahl der primzahlen unter einer gegebenen grosse. *Ges. Math. Werke und Wissenschaftlicher Nachlaß*, 2:145–155, 1859.

Wolfgang M Schmidt. *Diophantine approximation*. Springer Science & Business Media, 1996.

Alexander Schrijver. *Theory of linear and integer programming*. John Wiley & Sons, 1998.

Elias M Stein and Rami Shakarchi. *Complex analysis*, volume 2. Princeton University Press, 2010.

Ivan Matveevich Vinogradov. On an estimate of trigonometric sums with prime numbers. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 12(3):225–248, 1948.