# Irredundant and almost irredundant sets in $\mathbb{M}_2(\mathbb{C})$

***Original Research Article***

## Abstract

We consider irredundant and almost irredundant subsets in the *-algebra $\mathbb{M}_2(\mathbb{C})$ of all $2 \times 2$ matrices with coefficients in $\mathbb{C}$. We prove that the largest size of an irredundant subset is two, and that $\mathbb{M}_2(\mathbb{C})$ has an infinite almost irredundant subset.

*Keywords: matrix algebra; generators; irredundancy; involution.*

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# 1 Introduction

Let $\mathbb{M}_2(\mathbb{C})$ represent the algebra of all $2 \times 2$ matrices with coefficients in $\mathbb{C}$. For a given subset $S \subset \mathbb{M}_2(\mathbb{C})$, denote by $alg(S)$ the unital subalgebra of $\mathbb{M}_2(\mathbb{C})$ generated by $S$. A natural question that arises is under what conditions $alg(S) = \mathbb{M}_2(\mathbb{C})$; in other words, when can a subset $S$ generate the entire algebra $\mathbb{M}_2(\mathbb{C})$? Furthermore, an interesting problem is to determine the largest size of a set $S$ that can generate $\mathbb{M}_2(\mathbb{C})$, while ensuring that no proper subset of $S$ possesses this generating property.

These questions have been the subject of extensive research in matrix theory. Investigations into when a subset generates the full matrix algebra are detailed in (1). Additionally, T. Laffey addresses the problem of determining the maximum size of an irredundant set of generators in (5).

In this study, we also consider the algebra $\mathbb{M}_2(\mathbb{C})$ equipped with an involution $^* : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})$ defined by the conjugate transpose, $A^* = \overline{A}^t$. With this operation, $\mathbb{M}_2(\mathbb{C})$ becomes a *-algebra (or an involutive algebra). For a subset $S \subset \mathbb{M}_2(\mathbb{C})$, we denote by $alg^*(S)$ the involutive unital subalgebra of $\mathbb{M}_2(\mathbb{C})$ generated by $S$. If $alg^*(S) = \mathbb{M}_2(\mathbb{C})$, we say that $S$ is a set of *-generators for $\mathbb{M}_2(\mathbb{C})$, or that $S$ *-generates $\mathbb{M}_2(\mathbb{C})$.

Again, we can inquire about the maximum size[1] of a *-generator $S$ for $\mathbb{M}_2(\mathbb{C})$, ensuring that no proper subset of $S$ is also a set of *-generators for $\mathbb{M}_2(\mathbb{C})$. This property is referred to as 'irredundancy'.

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[1]For details on the problem of finding generating sets in matrix algebras with generic involutive maps, see (7).

**Definition 1.1.** Let $S$ be a subset of $\mathbb{M}_2(\mathbb{C})$. We say that $S$ is irredundant, if $x \notin alg(S \setminus \{x\})$ for every $x \in S$. Moreover, if $x \notin alg^*(S \setminus \{x\})$, for every $x \in S$, we say that $S$ is a *-irredundant set.

In other words, a subset $S \subset \mathbb{M}_2(\mathbb{C})$ is considered irredundant (*-irredundant) if no element of $S$ is contained in the (involutive) subalgebra generated by the other elements in $S$.

As a consequence of Laffey's result (5, Theorem 2.1), we demonstrate in Section 2 that the maximum size of a *-irredundant set of *-generators for $\mathbb{M}_2(\mathbb{C})$ is two.

The notion of *-irredundance in general infinite dimensional C*-algebras has been introduced in (4), where the question whether every C*-algebra has a large *-irredundant set was considered. In an attempt to prove the existence of large *-irredundant sets, a weaker notion of *-irredundance, termed almost irredundance, was introduced in (3). It was demonstrated that a special class of C*-algebras admits large almost irredundant sets (see (3) for details).

In this article, we establish that in the finite-dimensional context, specifically for the algebra $\mathbb{M}_2(\mathbb{C})$, the maximal size of an irredundant sets and almost irredundant sets differ significantly. In Section 3, we show that $\mathbb{M}_2(\mathbb{C})$ possesses an infinite almost irredundant set (see Proposition 3.1).

# 2 Irredundant sets in $\mathbb{M}_2(\mathbb{C})$

By definition, *-irredundancy implies irredundancy, and every irredundant set is, in particular, a linearly independent set. Consequently, the size of a *-irredundant set in $\mathbb{M}_2(\mathbb{C})$ is bounded above by four.

Consider a subset $S$ of $\mathbb{M}_2(\mathbb{C})$. Suppose $|S| = 1$. Then, $alg(S)$ is a commutative algebra, implying $alg(S) \neq \mathbb{M}_2(\mathbb{C})$. This demonstrates the absence of any $S \subset \mathbb{M}_2(\mathbb{C})$ of size 1 capable of generating the entire algebra $\mathbb{M}_2(\mathbb{C})$.

Now, let's suppose $|S| = 2$. According to Burnside's theorem[2], a subset $S = \{A_1, A_2\}$ generates $\mathbb{M}_2(\mathbb{C})$ if $A_1$ and $A_2$ do not share a common eigenvector. Particularly, if $E_{i,j}$ denotes the $2 \times 2$ matrix with a one in the $(i, j)$-position and zero elsewhere, then $S = \{E_{1,2}, E_{2,1}\}$ constitutes an irredundant set of generators of size 2 for the algebra $\mathbb{M}_2(\mathbb{C})$. Thus, the smallest possible size of a set of irredundant generators for $\mathbb{M}_2(\mathbb{C})$ as an algebra is 2.

If we consider an involution, then $A = E_{1,2}$ satisfies

$$\{A, A^*, AA^*, A^*A\} = \{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}\}.$$

This demonstrates that $S = \{A\}$ is a *-irredundant set (due to its sole member), and $alg^*(S) = \mathbb{M}_2(\mathbb{C})$. Particularly, when an involution is incorporated in our operations, the lower bound of an irredundant set of generators for the algebra $\mathbb{M}_2(\mathbb{C})$, which is two, reduces to one.

According to (5, Theorem 2.1), the maximum size of an irredundant set of generators for $\mathbb{M}_2(\mathbb{C})$ is three. We establish that upon integrating an involution into our operations, the upper bound is similarly reduced by one unit. In other words, we demonstrate that the maximum size of a *-irredundant set that *-generates $\mathbb{M}_2(\mathbb{C})$ is two. Before delving into the proof, we introduce some auxiliary lemmas.

Recall that a matrix $A \in \mathbb{M}_2(\mathbb{C})$ is self-adjoint if $A^* = A$, and unitary if $AA^* = A^*A = Id$. Each matrix can be expressed as a linear combination of two self-adjoint elements. For every $A \in \mathbb{M}_2(\mathbb{C})$, we denote $A = B + iC$, where $B = \frac{1}{2}(A + A^*)$ and $C = \frac{1}{2i}(A - A^*)$ are self-adjoint matrices.

The following lemma[3] states that elements in a *-irredundant set can be replaced with self-adjoint elements, resulting in another *-irredundant set.

**Lemma 2.1.** *Let $F$ be a *-irredundant set of *-generators for $\mathbb{M}_2(\mathbb{C})$ of size $n$, where $n$ represents the largest size of such a set. Then, there exists a *-irredundant set of *-generators $F'$ of size $n$ composed entirely of self-adjoint elements.*

[2] We recommend (6) for a concise proof of Burnside's theorem.

[3] For a version of this lemma applicable to all C*-algebras, refer to (4, Proposition 3.2).

*Proof.* Let $F = \{A_1, A_2, \cdots, A_n\}$ be a *-irredundant set. Write $A_1 = B_1 + iC_1$, where $B_1 = \frac{1}{2}(A_1 + A_1^*)$ and $C_1 = \frac{1}{2i}(A_1 - A_1^*)$ are self-adjoint elements. If $B_1, C_1 \in alg^*(\{A_2, A_3, \cdots, A_n\})$, then $A_1 = B_1 + iC_1 \in alg^*(\{A_2, A_3, \cdots, A_n\})$, which contradicts the fact that $\{A_1, A_2, \cdots, A_n\}$ is a *-irredundant set.

**Claim 1.** It is always possible to choose $D \in \{B_1, C_1\}$ such that $\{D, A_2, A_3, \cdots, A_n\}$ *-generates $\mathbb{M}_2(\mathbb{C})$ and $D \notin alg^*(\{A_2, A_3, \cdots, A_n\})$.

In fact, since $\{A_1, A_2, A_3, \cdots, A_n\}$ *-generates $\mathbb{M}_2(\mathbb{C})$, it is sufficient to show that we can choose $D$ such that $D \notin alg^*(\{A_2, A_3, \cdots, A_n\})$ and $A_1 \in alg^*(\{D, A_2, A_3, \cdots, A_n\})$.

If $B_1 \in alg^*(\{C_1, A_2, A_3, \cdots, A_n\})$, choose $D = C_1$. Observe that $D = C_1 \notin alg^*(\{A_2, A_3, \cdots, A_n\})$, otherwise we would have $B_1, C_1 \in alg^*(\{A_2, A_3, \cdots, A_n\})$ and therefore,

$$A_1 = B_1 + iC_1 \in alg^*(\{A_2, A_3, \cdots, A_n\})$$

which is a contradiction with the irredundancy of $F$.

Suppose now that $B_1 \notin alg^*(\{C_1, A_2, A_3, \cdots, A_n\})$ and define $D = B_1$. We claim that $C_1 \in alg^*(\{D, A_2, A_3, \cdots, A_n\})$; In fact, since

$$alg^*(\{B_1, C_1, A_2, \cdots, A_{j-1}, A_{j+1}, \cdots, A_n\}) = alg^*(\{A_1, A_2, \cdots, A_{j-1}, A_{j+1}, \cdots, A_n\})$$

for every $2 \leq j \leq n$, we would have that $\{B_1, C_1, A_2, \cdots, A_n\}$ is a *-irredundant set which *-generates $\mathbb{M}_2(\mathbb{C})$ and contains $n + 1$ elements, which contradicts the choice of $n$.

Fix $D \in \{B_1, C_1\}$ as in Claim 1. Then $\{D, A_2, A_3, \cdots, A_n\}$ *-generates $\mathbb{M}_2(\mathbb{C})$. Let us proove that $\{D, A_2, A_3, \cdots, A_n\}$ is a *-irredundant set. Since $D \notin alg^*(\{A_2, A_3, \cdots, A_n\})$ it sufficies to show that there is no $2 \leq j \leq n$ such that

$$A_j \in alg^*(\{D, A_2, \cdots, A_{j-1}, A_{j+1}, \cdots, A_n\})$$

Suppose there exists such $j$ and let us derive a contradiction. Since $D \in alg^*(A_1)$ it follows that $alg^*(D, A_2, \cdots, A_{j-1}, A_{j+1}, \cdots, A_n)$ is a subset of $alg^*(\mathcal{F} \setminus \{A_j\})$ and therefore, $A_j \in alg^*(\mathcal{F} \setminus \{A_j\})$, contradicting the *-irredundancy of $\mathcal{F}$. In particular, $\{D, A_2, A_3, \cdots, A_n\}$ is an *-irredundant set of *-generators comprising self-adjoint elements. ☐

Now, using the fact that we are only working in the field of complex numbers, we can rewrite (5, Theorem 2.1) as follows:

**Proposition 2.1** ((5, Theorem 2.1))**.** *Let $S \subset \mathbb{M}_2(\mathbb{C})$ be an irredundant set of self-adjoint elements such that $alg(S) = \mathbb{M}_2(\mathbb{C})$. Then $|S| \leq 3$. Moreover, if $|S| = 3$, then there is a unitary $U \in \mathbb{M}_2(\mathbb{C})$ such that $U^*SU = \{A, B, C\}$, where $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \alpha I_2$, $B = \begin{pmatrix} b & c \\ 0 & 0 \end{pmatrix} + \beta I_2$ and $C = \begin{pmatrix} d & 0 \\ e & 0 \end{pmatrix} + \gamma I_2$, where $a, b, c, d, \alpha, \beta, \gamma \in \mathbb{C}$, with $a \neq 0$ and $bd + ce = 0$.*

**Proposition 2.2.** *Let $n$ be the largest possible size of a *-irredundant set of *-generators for $\mathbb{M}_2(\mathbb{C})$. Then $n \leq 2$.*

*Proof.* Suppose $S \subset \mathbb{M}_2(\mathbb{C})$ is an *-irredundant set which *-generates $\mathbb{M}_2(\mathbb{C})$ as involutive algebra with the largest possible size. From Lemma 2.1, we can assume that all the elements in $S$ are self-adjoint elements. As $S$ is formed by self-adjoint elements, $alg^*(S) = alg(S)$. Then, $S$ is a *-irredundant set (and therefore irredundant) which generates $\mathbb{M}_2(\mathbb{C})$ as an algebra. In particular, from Proposition 2.1, we have $|S| \leq 3$. Assume that $S = \{A_1, A_2, A_3\}$ and lets get a contradiction. By Proposition 2.1, there is an unitary $U$ such that $U^*SU = \{A, B, C\}$, where

$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \alpha I_2$, $B = \begin{pmatrix} b & c \\ 0 & 0 \end{pmatrix} + \beta I_2$ and $C = \begin{pmatrix} d & 0 \\ e & 0 \end{pmatrix} + \gamma I_2$ for some $a, b, c, d, \alpha, \beta, \gamma \in \mathbb{C}$.

Since $S$ if formed by self-adjoint elements and $U$ is unitary, the matrices $A, B, C$ are all diagonal matrices. In particular, $\{A, B, C\}$ are linearly dependent. Since the map $a \to U^*aU$ defines a bijective involutive morphism, it follows that $\{A_1, A_2, A_3\}$, should be linearly dependent, which contradicts the fact that $S = \{A_1, A_2, A_3\}$ is a *-irredundant set. ☐

The following remark shows that the upper bound for *-irredundant set is attained:

*Remark* 2.1. Consider $\mathcal{F} = \{A, B\}$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Observe that $A$ and $B$ have no commom-invariant subspaces. Then, by Burside's theorem, $\{A, B\}$ generates $\mathbb{M}_2(\mathbb{C})$ (as an algebra and, in particular, as an involutive algebra). In conclusion, we observe that $A \notin alg^*(B)$ and $B \notin alg^*(A)$, which shows that $\{A, B\}$ is a *-irredundant set.

# 3 Almost irredundant sets in $\mathbb{M}_2(\mathbb{C})$

We focus now on a weaker notion of *-irredundance introduced in (3). Let $S \subset \mathbb{M}_2(\mathbb{C})$ be a self-adjoint subset of $\mathbb{M}_2(\mathbb{C})$. Then, $S$ is *-irredundant if and only if for every $a \in S$, the element $a$ does not belong to the involute subalgebra generated by $S \setminus \{a\}$. That is, $a$ cannot be written as $\sum_{i=1}^{n} \lambda_i \prod_{j=1}^{n_i} a_{i,j}$, where $a_{i,j} \in S \setminus \{a\}$ and $\lambda_i \in \mathbb{C}$.

Let us restrict the coefficients $\lambda$'s and define the following weak notion of *-irredundance:

**Definition 3.1.** Let $S$ be a subset of $\mathbb{M}_2(\mathbb{C})$. Then, $S$ is almost irredundant if and only if, for every $a \in S$, the element $a$ cannot be written as $\sum_{i=1}^{n} \lambda_i \prod_{j=1}^{n_i} a_{i,j}$, where $a_{i,j} \in S \setminus \{a\}$ and $\sum_{i=1}^{n} |\lambda_i| \leq 1$.

Observe that the main difference in the definition of *-irredundant sets and almost irredundant sets is that in the first, we allow any linear combinations, whereas in the second, we allow only convex linear combinations. In particular, any *-irredundant set is an almost irredundant set. However, we will see that these two notions behave differently when we consider the maximal size of such sets. We will prove that $\mathbb{M}_2(\mathbb{C})$ has an infinite almost irredundant set.

First, some lemmas are required. Recall that a self-adjoint matrix $A \in \mathbb{M}_2(\mathbb{C})$ is positive if its spectrum is positive, and that a linear map $\tau : \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$ is positive if $\tau(A) \geq 0$ whenever $A \in \mathbb{M}_2(\mathbb{C})$ is positive. We say that a matrix $A \in \mathbb{M}_2(\mathbb{C})$ is a projection if $A$ is self-adjoint and $A^2 = A$.

**Lemma 3.1.** *Let $P \in \mathbb{M}_2(\mathbb{C})$ be a projection, and let $\tau : \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$ be the map defined by $\tau(A) = trace(PA)$ for every $A \in \mathbb{M}_2(\mathbb{C})$, where $trace : \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$ is the canonical trace of a matrix. Let $A_1, \cdots, A_n$ be projections such that $\tau(A_i) < 1$ for every $1 \leq i \leq n$. Then*

$$\tau(A_1 A_2 \cdots A_n) < 1.$$

*Proof.* First, we observe that the linear map $\tau$ is positive, therefore, the map $(A, B) \to \tau(B^*A)$ defines a positive sesquilinear form on $\mathbb{M}_2(\mathbb{C})$. In particular, we apply the Cauchy–Schwarz inequality to show that $|\tau(B^*A)|^2 \leq \tau(B^*B)\tau(A^*A)$ holds for every $A, B \in \mathbb{M}_2(\mathbb{C})$.

The proof proceeds via an induction on $n$. Suppose that we have proven for the products of $n-1$ elements. Now, since $trace(AB) = trace(BA)$, we have that

$$\begin{aligned}
|\tau(A_1 A_2 \cdots A_n)|^2 &= |\tau(A_1 A_2 \cdots A_n)|^2 \\
&= |\tau((A_1)(A_2 \cdots A_n))|^2 \\
&\leq \tau(A_1^* A_1)\tau((A_2 \cdots A_n)^*(A_2 \cdots A_n)) \\
&\leq \tau(A_1)\tau(A_n^* \cdots A_2^* A_2 \cdots A_n) \\
&< trace(P A_n^* \cdots A_2^* A_2 \cdots A_n) \\
&< trace(P A_2 A_3 \cdots A_n) \\
&< \tau(A_2 A_3 \cdots A_n) \\
&< 1
\end{aligned}$$

□

**Lemma 3.2.** *Consider the function $f:[0,1]\times[0,1]\to\mathbb{R}$ given by*

$$f(x,y)=(xy+\sqrt{1-x^2}\sqrt{1-y^2})^2.$$

*Then, there exists an infinite family of distinct points $(x_i)_{i\in\mathbb{N}}$ such that:*

1. *$f(x_i,x_i)=1$ for every $i\in\mathbb{N}$;*
2. *$f(x_i,x_j)<1$ for every $i\neq j\in\mathbb{N}$.*

*Proof.* Consider a sequence of distinct points $(\theta_n)_{n\in\mathbb{N}}$ in $[0,\pi/2]$ and define $x_n=\cos(\theta_n)$ for each $n\in\mathbb{N}$. We claim that the family of points $(x_n)_{n\in\mathbb{N}}$ has the desirable properties: one has:

$$\begin{aligned} F(x_i,x_j) &= \cos(\theta_i)\cos(\theta_j)+\sin(\theta_i)\sin(\theta_j) \\ &= \cos(\theta_i-\theta_j) \end{aligned}$$

It follows that $f(x_i,x_j)<1$ when $i\neq j$ and $f(x_i,x_i)=1$ for each $i,j\in\mathbb{N}$ as required. □

**Proposition 3.1.** *$\mathbb{M}_2(\mathbb{C})$ has an infinite almost irredundant set (of projections).*

*Proof.* Fix $(x_i)_{i\in\mathbb{N}}$ given by Lemma 3.2. For each $i\in\mathbb{N}$, define $y_i=\sqrt{1-x_i^2}$ and the orthogonal projection onto the vector $v_{x_i}=(x_i,y_i)$ given by the matrix $A_i=\begin{pmatrix} x_i^2 & x_iy_i \\ x_iy_i & y_i^2\end{pmatrix}$. Let $\tau_i:\mathbb{M}_2(\mathbb{C})\to\mathbb{C}$ be a linear map defined as $\tau_i(A)=trace(A_iA)$. Given $i,j\in\mathbb{N}$ we have that

$$\begin{aligned} \tau_i(A_j) &= trace\left(\begin{pmatrix} x_i^2 & x_iy_i \\ x_iy_i & y_i^2\end{pmatrix}\begin{pmatrix} x_j^2 & x_jy_j \\ x_jy_j & y_j^2\end{pmatrix}\right) \\ &= trace\left(\begin{pmatrix} x_i^2x_j^2+x_iy_ix_jy_j & \cdots \\ \cdots & y_i^2y_j^2+x_iy_ix_jy_j\end{pmatrix}\right) \\ &= x_i^2x_j^2+y_i^2y_j^2+2x_iy_ix_jy_j \\ &= (x_ix_j+y_iy_j)^2 \\ &= (x_ix_j+\sqrt{1-x_i^2}\sqrt{1-x_j^2})^2 \\ &= f(x_i,x_j) \end{aligned}$$

where $f:[0,1]\times[0,1]\to\mathbb{R}$ is the function from Lemma 3.2. In particular, by Lemma 3.2 we have that

1. $\tau_i(A_i)=1$ and
2. $\tau_i(A_j)<1$ if $i\neq j$.

Let us prove that $(A_i)_{i\in\mathbb{N}}$ is an almost irredundant set. Suppose by contradiction that $(A_i)_{i\in\mathbb{N}}$ is not an almost irredundant set. Without loss of generality, suppose that we can write $A_1=\sum_{i=1}^m\lambda_i\prod_{j=1}^{n_i}a_{i,j}$ where $a_{i,j}\neq A_1$ and $\sum_{i=1}^m|\lambda_i|\leq 1$. By Lemma 3.1 we conclude that

$$\begin{aligned} 1 &= |\tau_1(A_1)| \\ &= |\tau_1(\sum_{i=1}^m\lambda_i\prod_{j=1}^{n_i}a_{i,j})| \\ &\leq \sum_{i=1}^m|\lambda_i||\tau_1(\prod_{j=1}^{n_i}a_{i,j})| \\ &< \sum_{i=1}^m|\lambda_i| \\ &\leq 1 \end{aligned}$$

which is a contradiction. □

## 4 CONCLUSIONS

The notion of *-irredundance in general infinite dimensional C*-algebras has been introduced in (4) and it is defined in an analogous manner as for matrix algebras. Because every C*-algebra is in particular a Banach space, every infinite-dimensional C*-algebra has an uncountable linear dimension; therefore, some other cardinals are more appropriate to tell something about the “size” of the algebra. For instance, the topological density of the algebra. Then, we can ask whether every large C*-algebra (in the sense of big density) has a large *-irredundant set. Some answers to this question have some set-theoretic flavours in the sense that we need to add some extra set-theoretic axioms to the standard ZFC axioms. One of the fundamental results is the example of a commutative C*-algebra with a larger density without uncountable *-irredundant sets, which is obtained as a C*-algebra of the form $C(K)$, where $K$ is the Kunen space obtained under the Continuum Hypothesis (see (8)). The question of whether such an example exists in ZFC remains open. An important partial result in this direction is the result of Todorcevic (see (9; 10)). We refer the reader to (4) for further details on *-irredundant sets in C*-algebras.

The notion of an almost irredundant set was introduced in (3) in an attempt to answer questions on *-irredundant sets. In particular, we mention (3, Theorem 1.3), where the author proved that it is consistent with the ZFC that large C*-algebras of some special class of C*-algebras admit an uncountable, almost irredundant set. Also, we refer the reader to (2) for some cardinal inequalities for almost irredundant sets.

In this article, we have proved that the maximal size of a *-irredundant set in $\mathbb{M}_2(\mathbb{C})$ is 2, while $\mathbb{M}_2(\mathbb{C})$ has an infinite almost irredundant set. In the infinite dimensional case, every infinite-dimensional C*-algebra has an infinite *-irredundant set (see (4, Proposition 3.12)). It is an open question whether the maximal size of a *-irredundant set is equal to the size of an almost irredundant set for an infinite dimensional C*-algebra. In particular, it is open if there can be a nonseparable C*-algebra, with an uncountable almost irredundant set, and with no uncountable *-irredundant set.

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