On the extended (k,t)–Fibonacci numbers

Abstract

This article studies an extension of the concept of k–Fibonacci numbers by introducing a new non-zero positive integer parameter t. In case $t = 1$, the numbers found are the Leonardo numbers. A homogeneous recurrence relationship is found between these new numbers, and various formulas are studied such as the Binet Identity or the generating function.

Key words: k–Fibonacci numbers, Binet identity, Recurrence relation, Generating function, Leonardo numbers.
MSC 2020, 11B39..

1 Introduction

One of the more studied sequences is the Fibonacci sequence [9], and it has been generalized in many ways [10,12,17,14]. Here, we use the following one-parameter generalization of the Fibonacci sequence.

Definition 1 *For any integer number $k \geq 1$, the k–Fibonacci sequence, say*

$\{F_{k,n}\}_{n\in\mathbf{N}}$ *is defined recurrently by:*

$$F_{k,0}=0,\ \ F_{k,1}=1,\ \ \textit{and}\ F_{k,n+1}=k\,F_{k,n}+F_{k,n-1}\ \textit{for}\ n\geq 1.$$

Note that for $k=1$ the classical Fibonacci sequence is obtained while for $k=2$ we obtain the Pell sequence. Some of the properties that the k–Fibonacci numbers verify and that we will need later are summarized bellow [6–8]. In particular, and since we will use it throughout this article, we indicate the Binet identity,

$$F_{k,n}=\frac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2} \qquad (1)$$

where $\sigma_1=\dfrac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2=\dfrac{k-\sqrt{k^2+4}}{2}$ are the characteristic roots of the relation of the definition. Among other properties, these roots verify $\sigma_1+\sigma_2=k$, $\sigma_1\cdot\sigma_2=-1$, $\sigma^2=k\,\sigma+1$, $\sigma_1-\sigma_2=\sqrt{k^2+4}$

The sum of the first n numbers is given by the formula $\displaystyle\sum_{j=0}^{n}F_{k,j}=\frac{F_{k,n}+F_{k,n+1}-1}{k}$ that, for the classical Fibonacci sequence ($k=1$) is $\displaystyle\sum_{j=0}^{n}F_j=F_{n+2}-1$

The generating function of the k–Fibonacci numbers is $f(k,x)=\dfrac{x}{1-k\,x-x^2}$ and the negative k–Fibonacci numbers are $F_{k,-n}=(-1)^{n+1}F_{k,n}$.

1.1 The k–Lucas numbers

The k–Lucas numbers [3] are defined with the same recurrence relation that for the k–Fibonacci numbers but with initial conditions $L_{k,0}=2$ and $L_{k,1}=k$. The Binet Identity takes the form $L_{k,n}=\sigma_1^n+\sigma_2^n$, and these k–numbers are related to each other by $L_{k,n}=F_{k,n-1}+F_{k,n+1}$. Moreover, $L_{k,n-1}+L_{k,n+1}=(k^2+4)F_{k,n}$ and its generating function is $l(k,x)=\dfrac{2-k\,x}{1-k\,x-x^2}$

1.2 The Leonardo numbers

The Leonardo numbers $\{Le_n\}$ are ddefined by the recurrence relation $Le_n = Le_{n-1} + Le_{n-2} + 1$ with initial conditions $Le_0 = 1$ and $Le_1 = 1$. The first few Leonardo numbers are $\{1, 1, 3, 5, 9, 15, 25, 41, \ldots\}$ indexed as A001595 in the OEIS [13].

The study of Leonardo numbers has been expanded by introducing different extensions of them, such as the complex Leonardo numbers [11] or the Gaussian Leonardo numbers [15].

2 On the extended (k, t)–Fibonacci numbers

In this section we generalize the recurrence relation that the k–Fibonavcci numbers must satisfy by adding a complementary term that is a positive integer constant t.

Definition 1 *Let t be a positive integer number, $t \in N$. It defines the linear non–homogeneus recurrence relation as*

$$T(k, t, n) = k\,T(k, t, n-1) + T(k, t, n-2) + t \tag{2}$$

It is necessary to indicate two initial conditions in order to determine exactly the terms of this sequence. According to this definition, this sequence takes the general form

$\{1, 1, k + (t+1), k^2 + (t+1)k + (t+1), k^3 + (t+1)k^2 + (t+2)k + (2t+1), \ldots\}$

For some values of t and specific initial conditions, certain types of these numbers have already been the subject of studies.

In this article we t must be non null because if $t = 0$

(1) If the initial conditions are $T(k, 0, 0) = 0$ and $T(k, 0; 1) = 1$, the numbers $T(k, 0, n)$ are the k–Fibonacci numbers $F_{k,n}$ [6,7]. In this case, if $k = 1$, the numbers $F_{1,n}$ are the terms of the classical Fibonacci sequence $F = \{F_n\}$ A000045 in the OEIS [13]. If $k = 2$, the Pell sequence appears: $\{T(2, 0, n)\} = \{P_n\}$, A000129.

(2) If the initial conditions are $T(k, 0, 0) = 2$ and $T(k, 0, 1) = k$, we have the k–Lucas numbers $L_{k,n}$ [3]. If $k = 1$ it is the classical Lucas sequence $\{L_n\}$ A000032 and if $k = 2$ it is the Pell–Lucas sequence $\{PL_n\}$ A002203.

If $k = 1$, the extended $(1, t)$–Fibonacci numbers are a generalization of the classical Leonardo numbers and we will represent them as $Le_n(t)$.

With the initial conditions $T(1, t, 0) = 1$ and $T(1, t, 1) = 1$, only the following sequences are indexed in the OEIS.

(1) For $t = 1$.
   (a) If $k = 1$, the extended $(1, 1)$–Fibonacci number $T(1, 1, n)$ is called the Leonardo number Le_n [2,1], and the sequence of the Leonardo numbers is $\{1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, \ldots\}$: A001595
   (b) If $k = 2$ it is $\{T(2, 1, n)\} = \{1, 1, 4, 10, 25, 61, 148, 358, 865, \ldots\}$: A033539

(2) For $t = 2$
   (a) If $k = 1$ it is $\{T(1, 2, n)\} = \{1, 1, 4, 7, 13, 22, 37, 61, 100, 163, 265, \ldots\}$: A111314
   (b) If $k = 2$, then $\{T(2, 2, n)\} = \{1, 1, 5, 13, 33, 81, 197, 477, 1153, 2785, \ldots\}$: 100227

Since the first two addends of the recurrence equation above (Equation (2))are exactly the same ones that define the recurrence relation of the k–Fibonacci numbers, we will call these numbers the extended (k, t)–Fibonacci numbers. However, and without taking into account the complementary term t, there is a difference between both that resides in the initial conditions since while in the k–Fibonacci numbers are $F_{k,0} = 0$ and $F_{k,1} = 1$ in the extended (k, t)–Fibonacci numbers are $T(k, t, 0) = 1$ and $T(k, t, 1) = 1$. Therefore, even for $t = 0$, there is a difference between both values since it is $T(k, 0, n) = F_{k,n-1}$, which indicates that there is a shift in both sequences . If $t \neq 0$, the difference is total between them.

In fact, $T(k, t, n)$ is the sum of a polynomial $P(k) = F_{k,n-1} + F_{k,n}$ of degree n plus the product of t times another polynomial $Q(k) = \frac{F_{k,n-1} + F_{k,n} - 1}{k}$ of degree $n - 1$, as we wiil see in the following theorem.

In order to simplify the writing and as long as there is no room for confusion, we will represent the elements $T(k, t, n)$ as T_n.

Theorem 1 *If T_n verifies the recurrence relation (2), then*

$$T_n = \frac{(k + t)(F_{k,n} + F_{k,n-1}) - t}{k} \tag{3}$$

We will prove this theorem by induction, first proving it for $n = 0, 1, 2$ and later extending it to n. Remember that $F_{k,-n} = (-1)^{n+1}F_{k,n}$ so $F_{k,-1} = Fk, 1 = 1$.

$$T_0 = \frac{(k + t)(F_{k,0} + F_{k,-1}) - t}{k} = \frac{(k + t) - t}{k} = 1$$
$$T_1 = \frac{(k + t)(F_{k,1} + F_{k,0}) - t}{k} = \frac{(k + t) - t}{k} = 1$$
$$T_2 = \frac{(k + t)(F_{k,2} + F_{k,1}) - t}{k} = \frac{(k + t)(k + 1) - t}{k} = \frac{k^2 + t\,k + k + t - t}{k}$$
$$= k + t + 1$$

Suppose the formula (3) is true up to T_{n-1}. Then

$$
\begin{aligned}
&T_{n-1} = \frac{(k+t)(F_{k,n-1}+F_{k,n-2})-t}{k} \\
&T_{n-2} = \frac{(k+t)(F_{k,n-1}+F_{k,n-2})-t}{k} \\
&k\,T_{n-1}+T_{n-2}+t \\
&= \frac{1}{k}\left[(k+t)(k\,F_{k,n-1}+k\,F_{k,n-2}+F_{k,n-2}+F_{k,n-3}-k\,t-t\right]+t \\
&= \frac{1}{k}\left((k+t)(F_{k,n}+F_{k,n+1})-t\right) = T_n
\end{aligned}
$$

If $k=1$, the k–Fibonacci numbers are the classical Fibonacci fibonacci numbers $\{0,1,1,2,3,5,8,13,21,\ldots\}$ and formula (3) becomes the formula for the generalized t–Leonardo numbers

$T_n = Le_n(t) = \dfrac{(1+t)(F_n+F_{n-1})-t}{1} = (1+t)F_{n+1}-t.$

If $t=0$, the formula (2) defines the Fibonacci numbers, so $T(1,0,n) = F_n + F_{n-1} = F_{n+1}$

But if $t=1$, the relation (2) defines the Leonardo numbers [1,2] and then $Le_n = \dfrac{2(F_n+F_{n-1})-1}{1} = 2F_{n+1}-1$

Proposition 1 *The non-homogeneous recurrence relation (2) can be transformed in a homogeneous recurrence relation by the formula*

$$T_n = (k+1)T_{n-1} - (k-1)T_{n-2} - T_{n-3} \tag{4}$$

for $n \geq 3$ and initial conditions $T_0 = 1$, $T_1 = 1$, $T_2 = k+t+1$

Proof. From Equation (3)

$$\begin{cases} (k+1)T_{n-1} = \dfrac{k+1}{k}((k+t)(F_{k,n-1}+F_{k,n-2})-t) \\ -((k-1)T_{n-2} = \dfrac{-(k-1)}{k}((k+t)(F_{k,n-2}+F_{k,n-2})-t) \\ -T_{n-3} = \dfrac{-1}{k}((k+t)(F_{k,n-3}+F_{k,n-4})-t) \end{cases}$$

from where

$$\begin{aligned}
&(k+1)T_{n-1}-(k-1)T_{n-2}-T_{n-3} = \\
&\frac{k+t}{k}((k+1)F_{k,n-1}-(k-1)F_{k,n-2}-F_{k,n-3}) \\
&+\frac{k+t}{k}((k+1)F_{k,n-2}-(k-1)F_{k,n-3}-F_{k,n-4}) \\
&-\frac{t}{k}((k+1)-(k-1)-1) \\
&= \frac{k+t}{k}(k\,F_{k,n-1}+F_{k,n-1}-k\,F_{k,n-2}+F_{k,n-3}-F_{k,n-3}) \\
&+\frac{k+t}{k}(k\,F_{k,n-2}+F_{k,n-2}-k\,F_{k,n-3}+F_{k,n-3}-F_{k,n-4})-\frac{t}{k} \\
&\frac{k+t}{k}((k\,F_{k,n-11}+F_{k,n-2})+(k\,F_{k,n-2}+F_{k,n-3}))-\frac{t}{k} \\
&= \frac{1}{k}((k+t)(F_{k,n}+F_{k,n-1})-t) = T_n
\end{aligned}$$

Theorem 2 *The formula (3) can also be proven from this equation, as we demonstrate below.*

Proof. The characteristic equation of the relation (4) is $r^3-(k+1)r^2+(k-1)r+1=0$ and its solutions are $r_1=1$, $r_2=\sigma_1=\dfrac{k+\sqrt{k^2+4}}{2}$ and $r_3=\sigma_2=\dfrac{k-\sqrt{k^2+4}}{2}$. Therefore, the general term will have the form $T_n=C_1+C_2\sigma_1^n+C_3\sigma_2^n$ with the conditions $T_0=1$, $T_1=1$ and $T_2=k+(t+1)$.

So, we have the linear system

$$\begin{aligned}
T_0 &= C_1+C_2+C_3=1 \\
T_1 &= C_1+C_2\,\sigma_1+C_3\,\sigma_2=1 \\
T_2 &= C_1+C_2\,\sigma_1^2+C_3\,\sigma_2^2=k+(t+1)
\end{aligned}$$

Solving this system of equations we find the values $C_0=-\dfrac{t}{k}$, $C_1=\dfrac{1}{k}\dfrac{1-\sigma_2}{\sigma_1-\sigma_2}(k+$

$t)$ and $C_2 = \dfrac{1}{k}\dfrac{-1+\sigma_1}{\sigma_1 - \sigma_2}(k+t)$. And taking into account that $(1-\sigma_2)\sigma_1^n = \sigma_1^n + \sigma_1^{n-1}$, $(-1+\sigma_1)\sigma_2^n = -\sigma_2^n - \sigma_2^{n-1}$ and the Binet Identity (1) $F_{k,r} = \dfrac{\sigma_1^r - \sigma_2^r}{\sigma_1 - \sigma_2}$, finally the formula (3) $T_n = \dfrac{1}{k}(-t + (k+t)(F_{k,n} + F_{k,n-1}))$ results.

For the sequence A033539 $(k = 2,\ t = 1)$, it is $T(2,1;n) = 3T(2,1;n-1) - T(2,1;n-2) - T(2,1;n-3)$.

A case that we will study in more depth is that of Leonardo numbers.

The recurrence relation 4 for $k = 1$ and $t = 1$ $Le_n = 2Le_{n-1} - Le_{n-3}$, so the characteristic equation is $r^3 - 2r^2 + 1 = 0$ whose solutions are $\{1, \dfrac{1+\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}\}$, being $\dfrac{1+\sqrt{5}}{2}$ the golden ratio ϕ and $\dfrac{1-\sqrt{5}}{2} = 1 + \psi = -\dfrac{1}{\phi}$. So, the general term Le_n will be $Le_n = C_0 + C_1\phi^n + C_2\psi^n$. Solving the system $\{Le_0, Le_1, Le_2\} = \{1, 1, 3\}$, we obtain the coefficients $C_0 = -1,\ C_1 = \dfrac{1+\sqrt{5}}{\sqrt{5}}, -\dfrac{1-\sqrt{5}}{\sqrt{5}}$. Finally the Binet formula for the Leonardo numbers results: $Le_n = -1 + \dfrac{2}{\sqrt{5}}\left(\phi^{n+1} + \psi^{n+1}\right)$. That is $Le_n = 2F_{n+1} - 1$

Lemma 1 *Just as the formula (3) indicates that T_n can be expressed in terms of the $F_{k,n}$, it is also possible to express $F_{k,n}$ in terms of the T_n as indicated in the following formula.*

$$F_{k,n} = \frac{T_{n+1} - T_n}{k+t} \tag{5}$$

Proof.

$$\begin{aligned}
T_n &= \frac{(k+t)F_{k,n} + F_{k,n-1}) - t}{k} \\
T_{n+1} &= \frac{(k+t)F_{k,n+1} + F_{k,n}) - t}{k} \to F_{k,n} = \frac{k\,T_{k,n+1} + t}{k+t} - F_{k,n+1} \\
T_n &= \frac{(k+t)\left(\dfrac{k\,T_{k,n+1} + t}{k+t} - F_{k,n+1} + F_{k,n-1}\right) - t}{k}
\end{aligned}$$

$$= \frac{k\,T_{n+1} + t - (k+t)\,k\,F_{k,n} - t}{k} = T_{n+1} - (k+t)F_{k,n}$$

$$\rightarrow F_{k,n} = \frac{T_{n+1} - T_n}{k+t}$$

It is interesting to note that although the parameter t is included in the second member, in reality the formula is independent of its value since $F_{k,n}$ does not depend on t.

2.1 Generating function

The generating function of the sequence of the extended (k,t)–Fibonacci numbers T_n is

$$f(k,t,x) = \frac{1 - k\,x - (1-k-t)x^2}{(1-x)(1-k\,x-x^2)} \tag{6}$$

Wee will use the homgeneous recurrence relation (4).

$$\begin{aligned}
f(k,t,x) &= T_0 + T_1x + T_2x^2 + T_3x^3 + \cdots \\
&= T_0 + T_1x + T_2x^2 \\
&\quad +((k+1)T_2 - (k-1)T_1 - T_0))x^3 \\
&\quad +((k+1)T_3 - (k-1)T_2 - T_1))x^4 \\
&\quad +((k+1)T_4 - (k-1)T_3 - T_2))x^5 + \cdots \\
&= T_0 + T_1x + T_2x^2 \\
&\quad +(k+1)(T_2x^2 + T_3x^3 + T_4x^4 + \cdots)x \\
&\quad -(k-1)(T_1x + T_2x^2 + T_3x^3 + \cdots)x^2 \\
&\quad -(T_0 + T_1x + T_2x^2 + \cdots)x^3 \\
&= T_0 + T_1x + T_2x^2 + (k+1)(f(k,t,x) - T_1x - T_0)x \\
&\quad -(k-1)(f(k,t,x) - T_0)x^2 - f(k,t,x)x^3 \\
&= T_0 + T_1x + T_2x^2 - (T_1x + T_0)(k+1)x + T_0(k-1)x^2 \\
&\quad +f(x)\left((k+1)x - (k-1)x^2 - x^3\right)
\end{aligned}$$

because all the remaining addends are null. So

$$
\begin{aligned}
&\left(1-(k+1)x+(k-1)x^2+x^3\right) f(k,t,x)\\
&=1+x+(k+t+1)x^2-(x+1)(k+1)x+(k-1)x^2\\
&=1-k\,x+(t+k-1)x^2\\
f(k,t,x)=&\frac{1-k\,x+(t+k-1)x^2}{(1-x)(1-k\,x-x^2)}
\end{aligned}
$$

Remark. There is a direct relationship between the equations (6) and (4) because the denominator $1-(k+1)x+(k-1)x^2+x^3$ of the generating function determines the recurrence relation indicated in Formula (4).

Corollary 1 *(1) For $k=1$*

(a) *If $t=0$, it results the generating function of the classical Fibonacci numbers (without F_0) $f(1,0,x)=\dfrac{1}{1-x-x^2}$.*

(b) *If $t=1$, $f(1,1,x)=\dfrac{1-x+x^2}{(1-x)(1-x-x^2)}$ is the generating function of the Leonardo sequence A001595.*

(c) *If $t=2$, $f(1,2,x)=\dfrac{1-x+2x^2}{(1-x)(1-x-x^2)}$ is the generating function of the sequence A111314.*

(2) For $k=2$:

(a) *If $t=1$, $f(2,1,x)=\dfrac{1-2x+2x^2}{1-2x-x^2}$ is the generating function of the sequence A0033539.*

(b) *If $t=2$, $f(2,2,x)=\dfrac{1-2x+3x^2}{(1-x)(1-2x-x^2)}$ generates the sequence A1002275.*

(3) For $k=3$ if $t=1$, $f(3,1,x)=\dfrac{1-3x+3x^2}{(1-x)(1-3x-x^2)}$ is the generating function of the sequence A033538

2.2 The Binet Identity for the extended (k,t)–Fibonacci numbers

The roots of the denominator $(1-x)(1-k\,x+x^2)$ of the generating function are $\{1,\sigma_1,\sigma_2\}$ so the general term T_n must be of the form $T_n = C_0 + C_1\sigma_1^n + C_2\sigma_2^n$ with initial conditions $\{1,1,k+(1+t)\}$. The solution of the system $\{T_0,T_1,T_2\} = \{1,1,k+(1+t)\}$ is $C_0 = -\frac{t}{k}$, $C_1 = \frac{(1-\sigma_2)(k+t)}{k\sqrt{k^2+4}}$, and $C_2 = -\frac{(1-\sigma_1)(k+t)}{k\sqrt{k^2+4}}$. Consequently the Binet identity for the extended (k,t)–Fibonaccinumbers takes the form

$$T_n = -\frac{t}{k} + \frac{(1-\sigma_2)(k+t)}{k\sqrt{k^2+4}}\sigma_1^n - \frac{(1-\sigma_1)(k+t)}{k\sqrt{k^2+4}}\sigma_2^n.$$

This formula can also be obtained in a simpler way as indicated in the following paragraph.

2.3 Use of the Binet Identity of the k–Fibonacci numbers for the extended (k,t)–Fibonacci numbers

If the Binet Identity for the k–Fibonacci numbers (1) is substituted in Equation (3), $T(k,t,n) = \frac{(k+t)(F_{k,n}+F_{k,n-1})-t}{k}$, the Binet formula for the extended (k,t)–Fibonacci numbers results:

$$T(k,t,n) = \frac{k+t}{k}\left(\frac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2} + \frac{\sigma_1^{n-1}-\sigma_2^{n-1}}{\sigma_1-\sigma_2}\right) - \frac{t}{k} \tag{7}$$

You can also use the combinatorial formula indicated in the introduction for the k–Fibonacci numbers and then

$$T(k,t,n) = \frac{k+t}{k}\left(\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-j}{j}k^{n-1-2j} + \sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-2-j}{j}k^{n-2-2j}\right) - \frac{t}{k}$$

For $k=t=1$, the Leonardo numbers results and then

$$Le_n = 2\left(\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-j}{j} + \sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-2-j}{j}\right) - 1$$

So, we therefore have five different ways to find the numbers T_n:

(1) Through the recurrence relation (2): $T_n = k\,T_{n-1} + T_{n-2} + t$

(2) Using the formula (3): $T_n = \dfrac{(k+t)(F_{k,n} + F_{k,n-1}) - t}{k}$

(3) By the second recurrence relation (4): $T_n = (k+1)T_{n-1} - (k-1)T_{n-2} - T_{n-3}$

(4) Throught the Binet Identity $T_n = \dfrac{k+t}{k}\left(\dfrac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} + \dfrac{\sigma_1^{n-1} - \sigma_2^{n-1}}{\sigma_1 - \sigma_2}\right) - \dfrac{t}{k}$

(5) Using the binomial formula

$$T_n = \frac{k+t}{k}\left(\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-j}{j}k^{n-1-2j} + \sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-2-j}{j}k^{n-2-2j}\right) - \frac{t}{k}$$

The first recurrence relation needs the two initial conditions $T_0 = 1$, $T_1 = 1$ and the second recurrence relations needs a third condition $T_2 = k + (t+1)$ For the classical Leonardo numbers, $k = t = 1$, and the first three formulas are, respectively,

- $Le_n = Le_{n-1} + Le_{n-2} + 1$
- $Le_n = 2F_{n+1} - 1$
- $Le_n = 2Le_{n-1} - Le_{n-3}$

with initial conditions $Le_0 = 1, Le_1 = 1, Le_2 = 3$

Conclusions

We have presented a new type of generalization of the Fibonacci numbers that relates them to the classical numbers of Fibonacci and those of Leonardo. This new contribution opens a very extensive field of research that should be taken into account in future contributions.

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