

# On the extended $(k, t)$ -Fibonacci numbers

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## Abstract

This article studies an extension of the concept of  $k$ -Fibonacci numbers by introducing a new non-zero positive integer parameter  $t$ . In case  $t = 1$ , the numbers found are the Leonardo numbers. A homogeneous recurrence relationship is found between these new numbers, and various formulas are studied such as the Binet Identity or the generating function.

*Key words:*  $k$ -Fibonacci numbers, Binet identity, Recurrence relation, Generating function, Leonardo numbers.

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## 1 Introduction

One of the more studied sequences is the Fibonacci sequence [9], and it has been generalized in many ways [10,11,15,13]. Here, we use the following one-parameter generalization of the Fibonacci sequence.

**Definition 1** For any integer number  $k \geq 1$ , the  $k$ -Fibonacci sequence, say

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$\{F_{k,n}\}_{n \in \mathbf{N}}$  is defined recurrently by:

$$F_{k,0} = 0, F_{k,1} = 1, \text{ and } F_{k,n+1} = k F_{k,n} + F_{k,n-1} \text{ for } n \geq 1.$$

Note that for  $k = 1$  the classical Fibonacci sequence is obtained while for  $k = 2$  we obtain the Pell sequence. Some of the properties that the  $k$ -Fibonacci numbers verify and that we will need later are summarized bellow [6–8]. In particular, and since we will use it throughout this article, we indicate the Binet identity,

$$F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} \tag{1}$$

where  $\sigma_1 = \frac{k + \sqrt{k^2 + 4}}{2}$  and  $\sigma_2 = \frac{k - \sqrt{k^2 + 4}}{2}$  are the characteristic roots

of the relation of the definition. Among other properties, these roots verify  $\sigma_1 + \sigma_2 = k, \sigma_1 \cdot \sigma_2 = -1, \sigma^2 = k \sigma + 1, \sigma_1 - \sigma_2 = \sqrt{k^2 + 4}$

The sum of the first  $n$  numbers is given by the formula  $\sum_{j=0}^n F_{k,j} = \frac{F_{k,n} + F_{k,n+1} - 1}{k}$

that, for the classical Fibonacci sequence ( $k = 1$ ) is  $\sum_{j=0}^n F_j = F_{n+2} - 1$

The generating function of the  $k$ -Fibonacci numbers is  $f(k, x) = \frac{x}{1 - kx - x^2}$

and the negative  $k$ -Fibonacci numbers are  $F_{k,-n} = (-1)^{n+1} F_{k,n}$ .

### 1.1 Generating function of the sequence $\{F_{k,n+b}\}$

If  $w(k, x)$  is the generating function of the sequence  $\{F_{k,n+b}\}$ , then

$$\begin{aligned} w(k, x) &= F_{k,b} + F_{k,b+1}x + F_{k,b+2}x^2 + F_{k,b+3}x^3 \dots \\ -kxw(k, x) &= -kF_{k,b}x - kF_{k,b+1}x^2 - kF_{k,b+2}x^3 - \dots \\ -x^2w(k, x) &= -F_{k,b}x^2 - F_{k,b+1}x^3 - F_{k,b+2}x^4 - \dots \\ (1 - kx - x^2)w(k, x) &= F_{k,b} + (F_{k,b+1} - kF_{k,b})x \end{aligned}$$

So

$$w(k, x) = \frac{F_{k,b} + (F_{k,b+1} - k F_{k,b})x}{1 - kx - x^2} \tag{2}$$

### 1.2 Generating function of the sequence $\{F_{k,2n+b}\}$

From the definition of the  $k$ -Fibonacci numbers we can deduce that  $F_{k,n} = (k^2 + 2)F_{k,n-2} - F_{k,n-4}$ .

Let us suppose  $z(k, x)$  is the generating function of the sequence  $\{F_{k,2n+b}\}$ .

Then

$$\begin{aligned} z(k, x) &= F_{k,b} + F_{k,b+2}x + F_{k,b+4}x^2 + F_{k,b+6}x^3 + \dots \\ -(k^2 + 2)xz(k, x) &= -(k^2 + 2)F_{k,b}x - (k^2 + 2)F_{k,b+2}x^2 - (k^2 + 2)F_{k,b+4}x^3 - \dots \\ x^2z(k, x) &= F_{k,b}x^2 + F_{k,b+2}x^3 + \dots \\ (1 - (k^2 + 2)x + x^2)z(k, x) &= F_{k,b} + (F_{k,b+2} - (k^2 + 2)F_{k,b})x \end{aligned}$$

from where

$$z(k, x) = \frac{F_{k,b} + (F_{k,b+2} - (k^2 + 2)F_{k,b})x}{1 - (k^2 + 2)x + x^2} \tag{3}$$

### 1.3 The $k$ -Lucas numbers

The  $k$ -Lucas numbers [3] are defined with the same recurrence relation that for the  $k$ -Fibonacci numbers but with initial conditions  $L_{k,0} = 2$  and  $L_{k,1} = k$ .

The Binet Identity takes the form  $L_{k,n} = \sigma_1^n + \sigma_2^n$ , and these  $k$ -numbers are related to each other by  $L_{k,n} = F_{k,n-1} + F_{k,n+1}$ . Moreover,  $L_{k,n-1} + L_{k,n+1} = (k^2 + 4)F_{k,n}$  and its generating function is  $l(k, x) = \frac{2 - kx}{1 - kx - x^2}$

## 2 On the extended $(k, t)$ -Fibonacci numbers

In this section we generalize the recurrence relation that the  $k$ -Fibonacci numbers must satisfy by adding a complementary term that is a positive integer constant  $t$ .

**Definition 1** *Let  $t$  be a positive integer number,  $t \in \mathbb{N}$ . It defines the linear non-homogeneous recurrence relation as*

$$T(k, t, n) = kT(k, t, n - 1) + T(k, t, n - 2) + t \tag{4}$$

It is necessary to indicate two initial conditions in order to determine exactly the terms of this sequence. According to this definition, this sequence takes the general form

$$\{1, 1, k + (t + 1), k^2 + (t + 1)k + (t + 1), k^3 + (t + 1)k^2 + (t + 2)k + (2t + 1), \dots\}$$

For some values of  $t$  and specific initial conditions, certain types of these numbers have already been the subject of studies.

In this article we  $t$  must be non null because if  $t = 0$

- (1) If the initial conditions are  $T(k, 0, 0) = 0$  and  $T(k, 0, 1) = 1$ , the numbers  $T(k, 0, n)$  are the  $k$ -Fibonacci numbers  $F_{k,n}$  [6,7]. In this case, if  $k = 1$ , the numbers  $F_{1,n}$  are the terms of the classical Fibonacci sequence  $F = \{F_n\}$  A000045 in the OEIS [12]. If  $k = 2$ , the Pell sequence appears:  $\{T(2, 0, n)\} = \{P_n\}$ , A000129.
- (2) If the initial conditions are  $T(k, 0, 0) = 2$  and  $T(k, 0, 1) = k$ , we have the  $k$ -Lucas numbers  $L_{k,n}$  [3]. If  $k = 1$  it is the classical Lucas sequence  $\{L_n\}$  A000032 and if  $k = 2$  it is the Pell-Lucas sequence  $\{PL_n\}$  A002203.

With the initial conditions  $T(1, t, 0) = 1$  and  $T(1, t, 1) = 1$ , only the following sequences are indexed in the OEIS.

(1) For  $t = 1$ .

(a) If  $k = 1$ , the extended  $(1, 1)$ -Fibonacci number  $T(1, 1, n)$  is called the Leonardo number  $Le_n$  [2,1], and the sequence of the Leonardo numbers is  $\{1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, \dots\}$ : A001595

(b) If  $k = 2$  it is  $\{T(2, 1, n)\} = \{1, 1, 4, 10, 25, 61, 148, 358, 865, \dots\}$ : A033539

(2) For  $t = 2$

(a) If  $k = 1$  it is  $\{T(1, 2, n)\} = \{1, 1, 4, 7, 13, 22, 37, 61, 100, 163, 265, \dots\}$ : A111314

(b) If  $k = 2$ , then  $\{T(2, 2, n)\} = \{1, 1, 5, 13, 33, 81, 197, 477, 1153, 2785, \dots\}$ : 100227

Since the first two addends of the recurrence equation above (Equation (4)) are exactly the same ones that define the recurrence relation of the  $k$ -Fibonacci numbers, we will call these numbers the extended  $(k, t)$ -Fibonacci numbers. However, and without taking into account the complementary term  $t$ , there is a difference between both that resides in the initial conditions since while in the  $k$ -Fibonacci numbers are  $F_{k,0} = 0$  and  $F_{k,1} = 1$  in the extended  $(k, t)$ -Fibonacci numbers are  $T(k, t, 0) = 1$  and  $T(k, t, 1) = 1$ . Therefore, even for  $t = 0$ , there is a difference between both values since it is  $T(k, 0, n) = F_{k,n-1}$ , which indicates that there is a shift in both sequences. If  $t \neq 0$ , the difference is total between them.

In fact,  $T(k, t, n)$  is the sum of a polynomial  $P(k) = F_{k,n-1} + F_{k,n}$  of degree  $n$  plus the product of  $t$  times another polynomial  $Q(k) = \frac{F_{k,n-1} + F_{k,n} - 1}{k}$  of

degree  $n - 1$ , as we will see in the following theorem.

In order to simplify the writing and as long as there is no room for confusion, we will represent the elements  $T(k, t, n)$  as  $T_n$ .

**Theorem 1** *If  $T_n$  verifies the recurrence relation (4), then*

$$T_n = \frac{(k + t)(F_{k,n} + F_{k,n-1}) - t}{k} \tag{5}$$

We will prove this theorem by induction, first proving it for  $n = 0, 1, 2$  and later extending it to  $n$ . Remember that  $F_{k,-n} = (-1)^{n+1}F_{k,n}$  so  $F_{k,-1} = F_{k,1} = 1$ .

$$\begin{aligned} T_0 &= \frac{(k + t)(F_{k,0} + F_{k,-1}) - t}{k} = \frac{(k + t) - t}{k} = 1 \\ T_1 &= \frac{(k + t)(F_{k,1} + F_{k,0}) - t}{k} = \frac{(k + t) - t}{k} = 1 \\ T_2 &= \frac{(k + t)(F_{k,2} + F_{k,1}) - t}{k} = \frac{(k + t)(k + 1) - t}{k} = \frac{k^2 + tk + k + t - t}{k} \\ &= k + t + 1 \end{aligned}$$

Suppose the formula (5) is true up to  $T_{n-1}$ . Then

$$\begin{aligned} T_{n-1} &= \frac{(k + t)(F_{k,n-1} + F_{k,n-2}) - t}{k} \\ T_{n-2} &= \frac{(k + t)(F_{k,n-1} + F_{k,n-2}) - t}{k} \\ k T_{n-1} + T_{n-2} + t &= \frac{1}{k} [(k + t)(k F_{k,n-1} + k F_{k,n-2} + F_{k,n-2} + F_{k,n-3} - kt - t)] + t \\ &= \frac{1}{k} ((k + t)(F_{k,n} + F_{k,n+1}) - t) = T_n \end{aligned}$$

If  $k = 1$ , the  $k$ -Fibonacci numbers are the classical Fibonacci fibonacci numbers  $\{0, 1, 1, 2, 3, 5, 8, 13, 21, \dots\}$  and formula (5) becomes

$$T_n = \frac{(1 + t)(F_n + F_{n-1}) - t}{1} = (1 + t)F_{n+1} - t. \text{ If } t = 0, \text{ the formula (4) defines}$$

the Fibonacci numbers, so  $T(1, 0, n) = F_n + F_{n-1} = F_{n+1}$

But if  $t = 1$ , the relation (4) defines the Leonardo numbers [1,2] and then

$$Le_n = \frac{2(F_n + F_{n-1}) - 1}{1} = 2F_{n+1} - 1$$

**Proposition 1** *The non-homogeneous recurrence relation (4) can be transformed in a homogeneous recurrence relation by the formula*

$$T_n = (k + 1)T_{n-1} - (k - 1)T_{n-2} - T_{n-3} \tag{6}$$

for  $n \geq 3$  and initial conditions  $T_0 = 1, T_1 = 1, T_2 = k + t + 1$

*Proof.* From Equation (5)

$$\begin{cases} (k + 1)T_{n-1} = \frac{k + 1}{k}((k + t)(F_{k,n-1} + F_{k,n-2}) - t) \\ -((k - 1)T_{n-2} = \frac{-(k - 1)}{k}((k + t)(F_{k,n-2} + F_{k,n-3}) - t) \\ -T_{n-3} = \frac{-1}{k}((k + t)(F_{k,n-3} + F_{k,n-4}) - t) \end{cases}$$

from where

$$\begin{aligned} &(k + 1)T_{n-1} - (k - 1)T_{n-2} - T_{n-3} = \\ &\frac{k + t}{k}((k + 1)F_{k,n-1} - (k - 1)F_{k,n-2} - F_{k,n-3}) \\ &+ \frac{k + t}{k}((k + 1)F_{k,n-2} - (k - 1)F_{k,n-3} - F_{k,n-4}) \\ &- \frac{t}{k}((k + 1) - (k - 1) - 1) \\ &= \frac{k + t}{k}(k F_{k,n-1} + F_{k,n-1} - k F_{k,n-2} + F_{k,n-3} - F_{k,n-3}) \\ &+ \frac{k + t}{k}(k F_{k,n-2} + F_{k,n-2} - k F_{k,n-3} + F_{k,n-3} - F_{k,n-4}) - \frac{t}{k} \\ &\frac{k + t}{k}((k F_{k,n-1} + F_{k,n-2}) + (k F_{k,n-2} + F_{k,n-3})) - \frac{t}{k} \\ &= \frac{1}{k}((k + t)(F_{k,n} + F_{k,n-1}) - t) = T_n \end{aligned}$$

**Theorem 2** *The formula (5) can also be proven from this equation, as we demonstrate below.*

*Proof.* The characteristic equation of the relation (6) is  $r^3 - (k + 1)r^2 + (k - 1)r + 1 = 0$  and its solutions are  $r_1 = 1$ ,  $r_2 = \sigma_1 = \frac{k + \sqrt{k^2 + 4}}{2}$  and  $r_3 = \sigma_2 = \frac{k - \sqrt{k^2 + 4}}{2}$ . Therefore, the general term will have the form  $T_n = C_1 + C_2\sigma_1^n + C_3\sigma_2^n$  with the conditions  $T_0 = 1$ ,  $T_1 = 1$  and  $T_2 = k + (t + 1)$ .

So, we have the linear system

$$\begin{aligned} T_0 &= C_1 + C_2 + C_3 = 1 \\ T_1 &= C_1 + C_2 \sigma_1 + C_3 \sigma_2 = 1 \\ T_2 &= C_1 + C_2 \sigma_1^2 + C_3 \sigma_2^2 = k + (t + 1) \end{aligned}$$

Solving this system of equations we find the values  $C_0 = -\frac{t}{k}$ ,  $C_1 = \frac{1}{k} \frac{1 - \sigma_2}{\sigma_1 - \sigma_2} (k + t)$  and  $C_2 = \frac{1 - 1 + \sigma_1}{k \sigma_1 - \sigma_2} (k + t)$ . And taking into account that  $(1 - \sigma_2)\sigma_1^n = \sigma_1^n + \sigma_1^{n-1}$ ,  $(-1 + \sigma_1)\sigma_2^n = -\sigma_2^n - \sigma_2^{n-1}$  and the Binet Identity (1)  $F_{k,r} = \frac{\sigma_1^r - \sigma_2^r}{\sigma_1 - \sigma_2}$ , finally the formula (5)  $T_n = \frac{1}{k}(-t + (k + t)(F_{k,n} + F_{k,n-1}))$  results.

For now, we therefore have three different ways to find the numbers  $T_n$ :

- Through the recurrence relation (4):  $T_n = kT_{n-1} + T_{n-2} + t$
- Using the formula (5):  $T_n = \frac{(k + t)(F_{k,n} + F_{k,n-1}) - t}{k}$
- By the second recurrence relation (6):  $T_n = (k + 1)T_{n-1} - (k - 1)T_{n-2} - T_{n-3}$

The first recurrence relation needs the two initial conditions  $T_0 = 1$ ,  $T_1 = 1$  and the second recurrence relations needs a third condition  $T_2 = k + (t + 1)$

For the second formula, it is necessary to use the Binet Identity (1).

You can also use a combinatorial formula like  $F_{k,n} = \sum_{j=0}^{n-1} \binom{n-1-j}{j} k^{n-1-2j}$  [7].

For the sequence A033539 ( $k = 2$ ,  $t = 1$ ), it is  $T(2, 1; n) = 3T(2, 1; n - 1) - T(2, 1; n - 2) - T(2, 1; n - 3)$ .

A case that we will study in more depth is that of Leonardo numbers.

The recurrence relation 6 for  $k = 1$  and  $t = 1$   $Le_n = 2Le_{n-1} - Le_{n-3}$ , so the characteristic equation is  $r^3 - 2r^2 + 1 = 0$  whose solutions are  $\{1, \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}\}$ , being  $\frac{1 + \sqrt{5}}{2}$  the golden ratio  $\phi$  and  $\frac{1 - \sqrt{5}}{2} = 1 + \psi = -\frac{1}{\phi}$ . So, the general term  $Le_n$  will be  $Le_n = C_0 + C_1\phi^n + C_2\psi^n$ . Solving the system  $\{Le_0, Le_1, Le_2\} = \{1, 1, 3\}$ , we obtain the coefficients  $C_0 = -1, C_1 = \frac{1 + \sqrt{5}}{\sqrt{5}}, -\frac{1 - \sqrt{5}}{\sqrt{5}}$ . Finally the Binet formula for the Leonardo numbers results:  $Le_n = -1 + \frac{2}{\sqrt{5}}(\phi^{n+1} + \psi^{n+1})$ . That is  $Le_n = 2F_{n+1} - 1$

**Lemma 1** *Just as the formula (5) indicates that  $T_n$  can be expressed in terms of the  $F_{k,n}$ , it is also possible to express  $F_{k,n}$  in terms of the  $T_n$  as indicated in the following formula.*

$$F_{k,n} = \frac{T_{n+1} - T_n}{k + t} \tag{7}$$

*Proof.*

$$\begin{aligned} T_n &= \frac{(k + t)F_{k,n} + F_{k,n-1}}{k} \\ T_{n+1} &= \frac{(k + t)F_{k,n+1} + F_{k,n}}{k} \rightarrow F_{k,n} = \frac{k T_{k,n+1} + t}{k + t} - F_{k,n+1} \\ T_n &= \frac{(k + t) \left( \frac{k T_{k,n+1} + t}{k + t} - F_{k,n+1} + F_{k,n-1} \right) - t}{k} \\ &= \frac{k T_{n+1} + t - (k + t) k F_{k,n} - t}{k} = T_{n+1} - (k + t)F_{k,n} \\ &\rightarrow F_{k,n} = \frac{T_{n+1} - T_n}{k + t} \end{aligned}$$

It is interesting to note that although the parameter  $t$  is included in the second member, in reality the formula is independent of its value since  $F_{k,n}$  does not depend on  $t$ .

### 2.1 Generating function

The generating function of the sequence of the extended  $(k, t)$ -Fibonacci numbers  $T_n$  is

$$f(k, t, x) = \frac{1 - kx - (1 - k - t)x^2}{(1 - x)(1 - kx - x^2)} \tag{8}$$

We will use the homogeneous recurrence relation (6).

$$\begin{aligned} f(k, t, x) &= T_0 + T_1x + T_2x^2 + T_3x^3 + \dots \\ &= T_0 + T_1x + T_2x^2 \\ &\quad + ((k + 1)T_2 - (k - 1)T_1 - T_0)x^3 \\ &\quad + ((k + 1)T_3 - (k - 1)T_2 - T_1)x^4 \\ &\quad + ((k + 1)T_4 - (k - 1)T_3 - T_2)x^5 + \dots \\ &= T_0 + T_1x + T_2x^2 \\ &\quad + (k + 1)(T_2x^2 + T_3x^3 + T_4x^4 + \dots)x \\ &\quad - (k - 1)(T_1x + T_2x^2 + T_3x^3 + \dots)x^2 \\ &\quad - (T_0 + T_1x + T_2x^2 + \dots)x^3 \\ &= T_0 + T_1x + T_2x^2 + (k + 1)(f(k, t, x) - T_1x - T_0)x \\ &\quad - (k - 1)(f(k, t, x) - T_0)x^2 - f(k, t, x)x^3 \\ &= T_0 + T_1x + T_2x^2 - (T_1x + T_0)(k + 1)x + T_0(k - 1)x^2 \\ &\quad + f(k, t, x) \left( (k + 1)x - (k - 1)x^2 - x^3 \right) \end{aligned}$$

because all the remaining addends are null. So

$$\begin{aligned} & \left( 1 - (k + 1)x + (k - 1)x^2 + x^3 \right) f(k, t, x) \\ &= 1 + x + (k + t + 1)x^2 - (k + 1)x + (k - 1)x^2 \\ &= 1 - kx + (t + k - 1)x^2 \\ f(k, t, x) &= \frac{1 - kx + (t + k - 1)x^2}{(1 - x)(1 - kx - x^2)} \end{aligned}$$

**Remark.** There is a direct relationship between the equations (8) and (6) because the denominator  $1 - (k + 1)x + (k - 1)x^2 + x^3$  of the generating function determines the recurrence relation indicated in Formula (6).

**Corollary 1** (1) For  $k = 1$

(a) If  $t = 0$ , it results the generating function of the classical Fibonacci numbers (without  $F_0$ )  $f(1, 0, x) = \frac{1}{1 - x - x^2}$ .

(b) If  $t = 1$ ,  $f(1, 1, x) = \frac{1 - x + x^2}{(1 - x)(1 - x - x^2)}$  is the generating function of the Leonardo sequence A001595.

(c) If  $t = 2$ ,  $f(1, 2, x) = \frac{1 - x + 2x^2}{(1 - x)(1 - x - x^2)}$  is the generating function of the sequence A111314.

(2) For  $k = 2$ :

(a) If  $t = 1$ ,  $f(2, 1, x) = \frac{1 - 2x + 2x^2}{1 - 2x - x^2}$  is the generating function of the sequence A0033539.

(b) If  $t = 2$ ,  $f(2, 2, x) = \frac{1 - 2x + 3x^2}{(1 - x)(1 - 2x - x^2)}$  generates the sequence A1002275.

(3) For  $k = 3$  if  $t = 1$ ,  $f(3, 1, x) = \frac{1 - 3x + 3x^2}{(1 - x)(1 - 3x - x^2)}$  is the generating function of the sequence A033538

## 2.2 Use of the Binet Identity of the $k$ -Fibonacci numbers for the extended $(k, t)$ -Fibonacci numbers

If the Binet Identity for the  $k$ -Fibonacci numbers (1) is substituted in Equation (5),  $T(k, t, n) = \frac{(k + t)(F_{k,n} + F_{k,n-1}) - t}{k}$ , the Binet formula for the extended  $(k, t)$ -Fibonacci numbers results:

$$T(k, t, n) = \frac{k + t}{k} \left( \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} + \frac{\sigma_1^{n-1} - \sigma_2^{n-1}}{\sigma_1 - \sigma_2} \right) - \frac{t}{k} \tag{9}$$

You can also use the combinatorial formula indicated in the introduction for the  $k$ -Fibonacci numbers and then

$$T(k, t, n) = \frac{k+t}{k} \left( \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} k^{n-1-2j} + \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-j}{j} k^{n-2-2j} \right) - \frac{t}{k}$$

For  $k = t = 1$ , the Leonardo numbers results and then

$$Le_n = 2 \left( \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} + \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-j}{j} \right) - 1$$

In the following sections we will study different sums of extended  $(k, t)$ -Fibonacci numbers, finding not only the formulas that allow to find these sums but also their recurrence relations as well as their generating functions and the respective Binet formulas.

## References

- [1] Alp Y. and Gökçen Koçer, *Some Properties of Leonardo Numbers*, Konuralp Journal of Mathematics 9 (1) (2021), 183 – 189
- [2] Catarino P. and Borges, A. *On Leonardo numbers* Acta Math. Univ. Comenianae, LXXXIX 1 (2020), 75 – 86
- [3] S. Falcon *On the  $k$ -Lucas numbers*, International Journal of Contemporary Mathematical Sciences, 6 (21) (2011), 1039 –1050.
- [4] Falcon, S. *On the powers of the  $k$ -Fibonacci numbers*, Ars Combinatoria. Vol. CXXVII July (2016), 329–338.
- [5] S. Falcon, *Generating Function of Some  $k$ -Fibonacci and  $k$ -Lucas Sequences*, International Journal of Innovation in Sciences and Technology, 7(2) (2019), 109–116.

- [6] S. Falcon, A. Plaza, *On the Fibonacci  $k$ -numbers*, Chaos, Solit. & Fract. 32(5) (2007), 1615–624.
- [7] S. Falcon, A. Plaza, *The  $k$ -Fibonacci sequence and the Pascal 2-triangle*, Chaos, Solit. & Fract. 33(1) (2007), 38–49.
- [8] Falcon, S. and Plaza, A., *On the  $k$ -Fibonacci numbers of arithmetic indexes*. Applied Mathematics and Computation, Elsevier, 208 (1) (2009), 180 – 185.  
<http://dx.doi.org/10.1016/j.amc.2008.11.031>
- [9] V.E. Hoggat, *Fibonacci and Lucas numbers*. Palo Alto, CA: Houghton-Mifflin; 1969.
- [10] A.F. Horadam, *A generalized Fibonacci sequence*, Math. Mag. 68 (1961). 455–459.
- [11] E. Kilic, *The Binet formula, sums and representations of generalized Fibonacci  $p$ -numbers*, Eur. J. Combin., 29(3) (2008), 701–711.
- [12] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences* (2006), [www.research.att.com/~njas/sequences/](http://www.research.att.com/~njas/sequences/).
- [13] A.A. Ocal, N. Tuglu, E. Altinisik, *On the representation of  $k$ -generalized Fibonacci and Lucas numbers*, Appl. Math. Comput., 170(1) (2005), 584–596.
- [14] H.S.Wilf, *generatingfunctionology*, available in <https://www2.math.upenn.edu/~wilf/gfology2.pdf>
- [15] S. Yang, *On the  $k$ -generalized Fibonacci numbers and high-order linear recurrence relations*, Appl. Math. Comput., 196(2) (2008). 850–857.