
ON THE NUMBER OF CYCLIC CODES OVER \mathbb{Z}_{31}

Abstract

Let n be a positive integer, $y^n - 1$ cyclotomic polynomial and \mathbb{Z}_q be a given finite field. In this study we determined the number of cyclic codes over \mathbb{Z}_{31} . First, we partitioned the cyclotomic polynomial $y^n - 1$ using cyclotomic cosets $31 \bmod n$ and factorized $y^n - 1$ using case to case basis. Each monic divisor obtained is a generator polynomial and generate cyclic codes. The results obtained are useful in the field of coding theory and more especially, in error correcting codes.

Keywords: Code; cyclic Codes; cyclotomic coset

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1 Introduction

The study of cyclic codes have attracted attention of many researchers since the advent of cryptography, this is because cyclic codes have direct applications in coding theory in particular error correction coding [17, 19, 8]. The problem of finding codes which are optimal, that will transmit a wide varieties of messages fast and correct many errors has been a motivating factor in the study of codes [13, 9, 15, 16]. Factorization of polynomials over a finite field is of interest in algebraic coding theory, computational number theory, computer algebra, and cryptography among other areas see [1, 4, 5, 7]. Cyclotomic cosets has been used to partition Cyclotomic polynomials $y^n - 1$ [2, 3, 18]. Cyclic codes of length n over finite fields \mathbb{Z}_q for which $q \leq 23$ has been fully characterized and a general formula for computing the number of cyclic codes given see [10, 11, 12, 13, 20, ?, 21]. In this study the number of cyclic codes over \mathbb{Z}_{31} are determined and generalizations made over \mathbb{Z}_{31} .

1.1 Definitions

- i. **Code:** Let F be a finite set with q symbols, there are q^n different sequences of length n , of these only q^k are codewords since the r check digits within any codeword are completely determined by the k message digits. The set consisting of q^k codewords of length n is called a code.
- ii. **Cyclic Code:** Let C be a linear code over a finite field $GF(q)$ of block n , C is called a cyclic code, if for every codeword $a_0, a_1, a_2, \dots, a_n$ from C , the word $a_n, a_0, a_1, a_2, \dots, a_{n-1}$ in C obtain by a cyclic right shift of component is also a codeword. This also involves the left shift. Therefore a linear code C is cyclic precisely when it is invariant under all cyclic shifts.
- iii. **Cyclotomic Coset:** Let n be relatively to q . The cyclotomic coset of $q \bmod n$ is defined by $C_i = \{i \cdot q^j \bmod n \in \mathbb{Z}_n, j = 0, 1, 2, 3, \dots\}$

2 Main Results:

2.1 Number of Cyclotomic Cosets and Factorization of $y^n - 1$ over \mathbb{Z}_{31}

- i. Let $n = 1$, Then $C_i = \{i \cdot 31^j \bmod 31 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$ We get 1 cyclotomic cosets 31 mod 1. We factorize $y - 1$. $y - 1 = (y + 30)$. The number of irreducible monic polynomial is 1 of degree 1.
- ii. Let $n = 2$, Then $C_i = \{i \cdot 31^j \bmod 31 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1\}$. We get 2 cyclotomic cosets 31 mod 2. We factorize $y^2 - 1$. $y^2 - 1 = (y - 1)(y + 1) = (y + 30)(y + 1)$. The number of irreducible monic polynomial are also 2 of degree 1.
- iii. Let $n = 3$, Then $C_i = \{i \cdot 31^j \bmod 31 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1\}$, $C_2 = \{2\}$. We get 3 cyclotomic cosets 31 mod 3. We factorize $y^3 - 1$. $y^3 - 1 = (y - 1)(y^2 + y + 1) = (y + 30)(y + 26)(y + 6)$. The number of irreducible monic polynomial are also 3 of degree 1.
- iv. Let $n = 4$, Then $C_i = \{i \cdot 31^j \bmod 31 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 3\}$, $C_2 = \{2\}$. We get 3 cyclotomic cosets 31 mod 4. We factorize $y^4 - 1$. $y^4 - 1 = (y^2 - 1)(y^2 + 1) = (y - 1)(y + 1)(y^2 + 1)$. The number of irreducible monic polynomial are also 3, 2 of degree 1 and 1 of degree 2.
- v. Let $n = 5$, Then $C_i = \{i \cdot 31^j \bmod 31 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1\}$, $C_2 = \{2\}$, $C_3 = \{3\}$, $C_4 = \{4\}$. We get 5 cyclotomic cosets 31 mod 5. We factorize $y^5 - 1$. $y^5 - 1 = (y - 1)(y^4 + y^3 + y^2 + y + 1) = (y + 30)(y + 29)(y + 27)(y + 15)(y + 23)$. The number of irreducible monic polynomial are also 5 of degree 1.
- vi. Let $n = 6$, Then $C_i = \{i \cdot 31^j \bmod 31 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1\}$, $C_2 = \{2\}$, $C_3 = \{3\}$, $C_4 = \{4\}$, $C_5 = \{5\}$. We get 6 cyclotomic cosets 31 mod 6. We factorize $y^6 - 1$. $y^6 - 1 = (y^3 - 1)(y^3 + 1) = (y + 30)(y + 26)(y + 6)(y + 1)(y + 5)(y + 25)$. The number of irreducible monic polynomial are also 6 of degree 1.
- vii. Let $n = 7$, Then $C_i = \{i \cdot 31^j \bmod 31 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 2, 3, 4, 5, 6\}$. We get 2 cyclotomic cosets 31 mod 7. We factorize $y^7 - 1$. $y^7 - 1 = (y + 30)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)$. The number of irreducible monic polynomial are also 2, 1 of degree 1 and 1 of degree 7.
- viii. Let $n = 8$, Then $C_i = \{i \cdot 31^j \bmod 31 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 7\}$, $C_2 = \{2, 6\}$, $C_3 = \{3, 5\}$, $C_4 = \{4\}$. We get 5 cyclotomic cosets 31 mod 8. We factorize $y^8 - 1$. $y^8 - 1 = (y^4 - 1)(y^4 + 1) = (y^2 - 1)(y^2 + 1)(y^4 + 1) = (y + 30)(y + 1)(y^2 + 1)(y^4 + 1) = (y + 30)(y + 1)(y^2 + 1)(y^2 + 8y + 1)(y^2 + 23y + 1)$. The number of irreducible monic polynomial are also 5, 2 of degree 1 and 3 of degree 2.

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- ix. Let $n = 9$, Then $C_i = \{i \cdot 31^j \bmod 9 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 4, 7\}$, $C_2 = \{2, 6, 8\}$, $C_3 = \{3\}$, $C_5 = \{5\}$. We get 5 cyclotomic cosets 31 mod 9. We factorize $y^9 - 1 = (y-1)(y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y+30)(y+26)(y+6)(y^3+1)(y^3+26)$. The number of irreducible monic polynomial are also 5, 3 of degree 1 and 2 of degree 3.
- x. Let $n = 10$, Then $C_i = \{i \cdot 31^j \bmod 10 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1\}$, $C_2 = \{2\}$, $C_3 = \{3\}$, $C_4 = \{4\}$, $C_5 = \{5\}$, $C_6 = \{6\}$, $C_7 = \{7\}$, $C_8 = \{8\}$, $C_9 = \{9\}$. We get 10 cyclotomic cosets 31 mod 10. We factorize $y^{10} - 1 = (y^5 - 1)(y^5 + 1) = (y+30)(y+29)(y+27)(y+15)(y+23)(y^5+1) = (y+30)(y+29)(y+27)(y+15)(y+23)(y+1)(y+2)(y+4)(y+8)(y+16)$. The number of irreducible monic polynomial are also 10 of degree 1.
- xi. Let $n = 11$, Then $C_i = \{i \cdot 31^j \bmod 11 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 9, 4, 3, 5\}$, $C_2 = \{2, 6, 7, 8, 10\}$. We get 3 cyclotomic cosets 31 mod 11. We factorize $y^{11} - 1 = (y-1)(y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y+30)(y^5 + 10y^4 + 30y^3 + y^2 + 9y + 30)(y^5 + 22y^4 + 30y^3 + y^2 + 21y + 30)$. The number of irreducible monic polynomial are also 3, 1 of degree 1 and two of degree 5.
- xii. Let $n = 12$, Then $C_i = \{i \cdot 31^j \bmod 12 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 7\}$, $C_2 = \{2\}$, $C_3 = \{3, 9\}$, $C_4 = \{4\}$, $C_5 = \{5, 11\}$, $C_6 = \{6\}$, $C_8 = \{8\}$, $C_{10} = \{10\}$. We get 9 cyclotomic cosets 31 mod 12. We factorize $y^{12} - 1 = (y^6 - 1)(y^6 + 1) = (y^3 - 1)(y^3 + 1)(y^6 + 1) = (y+30)(y+26)(y+6)(y+1)(y+5)(y+25)(y^2+1)(y^2+5)(y^2+25)$. The number of irreducible monic polynomial are also 9, 6 of degree 1 and 3 of degree 2.
- xiii. Let $n = 13$, Then $C_i = \{i \cdot 31^j \bmod 13 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 5, 8, 12\}$, $C_2 = \{2, 10, 11, 13\}$, $C_4 = \{4, 7, 9, 6\}$. There are 4 cyclotomic cosets 31 mod 13. We factorize $y^{13} - 1 = (y-1)(y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y+30)(y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y+30)(y^4 + 4y^3 + 11y^2 + 4y + 1)(y^4 + 6y^3 + 29y^2 + 6y + 1)(y^4 + 22y^3 + 27y^2 + 22y + 1)$. The number of irreducible monic polynomial are also 4, 1 of degree 1 and 3 of degree 4.
- xiv. Let $n = 14$, Then $C_i = \{i \cdot 31^j \bmod 14 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 3, 9, 13, 11, 5\}$, $C_2 = \{2, 4, 6, 8, 10, 12\}$, $C_7 = \{7\}$ 4 cyclotomic cosets 31 mod 14 are obtained. We factorize $y^{14} - 1 = (y^7 - 1)(y^7 + 1) = (y+30)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)(y^7 + 1) = (y+30)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)(y+1)(y^6 + 30y^5 + y^4 + 30y^3 + y^2 + 30y + 1)$. The number of irreducible monic polynomial are also 4, 2 of degree 1 and 2 of degree 6.
- xv. Let $n = 15$, Then $C_i = \{i \cdot 31^j \bmod 15 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1\}$, $C_2 = \{2\}$, $C_3 = \{3\}$, $C_4 = \{4\}$, $C_5 = \{5\}$, $C_6 = \{6\}$, $C_7 = \{7\}$, $C_8 = \{8\}$, $C_9 = \{9\}$, $C_{10} = \{10\}$, $C_{11} = \{11\}$, $C_{12} = \{12\}$, $C_{13} = \{13\}$, $C_{14} = \{14\}$. There are 15 cyclotomic cosets 31 mod 15. We factorize $y^{15} - 1 = (y-1)(y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y+30)(y+26)(y+6)(y+3)(y+21)(y+29)(y+23)(y+27)(y+17)(y+11)(y+24)(y+22)(y+12)(y+13)(y+15)$. The number of irreducible monic polynomial are also 15 of degree 1.
- xvi. Let $n = 16$, Then $C_i = \{i \cdot 31^j \bmod 16 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 15\}$, $C_2 = \{2, 14\}$, $C_3 = \{3, 13\}$, $C_4 = \{4, 12\}$, $C_5 = \{5, 11\}$, $C_6 = \{6, 10\}$, $C_8 = \{8\}$, $C_9 = \{9, 7\}$. There are 9 cyclotomic cosets 31 mod 16. We factorize $y^{16} - 1 = (y^8 - 1)(y^8 + 1) = (y^2 - 1)(y^2 + 1)(y^4 + 1)(y^8 + 1) = (y+30)(y+1)(y^2 + 1)(y^2 + 8y + 1)(y^2 + 23y + 1)(y^8 + 1) = (y+30)(y+1)(y^2 + 1)(y^2 + 8y + 1)(y^2 + 26y + 1)(y^2 + 23y + 1)(y^2 + 5y + 1)(y^2 + 14y + 1)(y^2 + 17y + 1)$. The number of irreducible monic polynomial are also 9, 2 of degree 1 and 7 of degree 2.
- xvii. Let $n = 17$, Then $C_i = \{i \cdot 31^j \bmod 17 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ There are 2 cyclotomic cosets 31 mod 17. We factorize $y^{17} - 1 = (y-1)(y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y+30)(y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)$. The number of irreducible monic polynomial are also 2, 1 of degree 1 and 1 of degree 16.

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- xviii. Let $n = 18$, Then $C_i = \{i \cdot 31^j \bmod 18 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 7, 13\}$, $C_2 = \{2, 8, 14\}$, $C_3 = \{3\}$, $C_4 = \{4, 10, 16\}$, $C_5 = \{5, 11, 17\}$, $C_6 = \{6\}$, $C_9 = \{9\}$, $C_{12} = \{12\}$, $C_{15} = \{15\}$. There are 10 cyclotomic cosets 31 mod 18. We factorize $y^{18} - 1$. $y^{18} - 1 = (y^9 - 1)(y^9 + 1) = (y + 30)(y + 26)(y + 6)(y^3 + 6)(y^3 + 26)(y^9 + 1) = (y + 30)(y + 26)(y + 6)(y^3 + 6)(y^3 + 26)(y + 1)(y + 5)(y + 25)(y^3 + 5)(y^3 + 25)$. The number of irreducible monic polynomial are also 10, 6 of degree 1 and 4 of degree 3.
- xix. Let $n = 19$, Then $C_i = \{i \cdot 31^j \bmod 19 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 8, 10, 11, 12, 18\}$, $C_2 = \{2, 3, 5, 14, 16, 17\}$, $C_4 = \{4, 6, 7, 9, 13, 15\}$. There are 4 cyclotomic cosets 31 mod 19. We factorize $y^{19} - 1$. $y^{19} - 1 = (y - 1)(y^{18} + y^{17} + y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y + 30)(y^6 + 4y^5 + 14y^4 + 18y^3 + 14y^2 + 4y + 1)(y^6 + 5y^5 + 25y^4 + y^3 + 5y^2 + 25y + 1)(y^6 + 25y^5 + 5y^4 + y^3 + 25y^2 + 5y + 1)$. The number of irreducible monic polynomial are also 4, 1 of degree 1 and 3 of degree 6.
- xx. Let $n = 20$, Then $C_i = \{i \cdot 31^j \bmod 20 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 11\}$, $C_2 = \{2\}$, $C_3 = \{3, 13\}$, $C_4 = \{4\}$, $C_5 = \{5, 15\}$, $C_6 = \{6\}$, $C_7 = \{7, 17\}$, $C_8 = \{8\}$, $C_9 = \{9, 19\}$, $C_{10} = \{10\}$, $C_{12} = \{12\}$, $C_{14} = \{14\}$, $C_{16} = \{16\}$, $C_{18} = \{18\}$. There are 15 cyclotomic cosets 31 mod 20. We factorize $y^{20} - 1$. $y^{20} - 1 = (y^{10} - 1)(y^{10} + 1) = (y^5 - 1)(y^5 + y)(y^{10} + 1) = (y + 30)(y + 29)(y + 27)(y + 15)(y + 23)(y + 1)(y + 2)(y + 4)(y + 8)(y + 16)(y^{10} + 1) = (y + 30)(y + 29)(y + 27)(y + 15)(y + 23)(y + 1)(y + 2)(y + 4)(y + 8)(y + 16)(y^2 + 1)(y^2 + 2)(y^2 + 4)(y^2 + 8)(y^2 + 16)$. The number of irreducible monic polynomial are also 15, 10 of degree 1 and 5 of degree 2.
- xxi. Let $n = 21$, Then $C_i = \{i \cdot 31^j \bmod 21 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 4, 10, 13, 16, 19\}$, $C_2 = \{2, 5, 8, 11, 17, 20\}$, $C_3 = \{3, 6, 9, 12, 15, 18\}$, $C_7 = \{7\}$, $C_{14} = \{14\}$. There are 6 cyclotomic cosets 31 mod 21. We factorize $y^{21} - 1$. $y^{21} - 1 = (y - 1)(y^{20} + y^{19} + y^{18} + y^{17} + y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y + 30)(y + 6)(y + 26)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)(y^{12} + 30y^{11} + y^{10} + 30y^9 + y^8 + 30y^7 + y^6 + 30y^5 + y^4 + 30y^3 + y^2 + 30y + 1) = (y + 30)(y + 6)(y + 26)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)(y^6 + 5y^5 + 25y^4 + y^3 + 5y^2 + 25y + 1)(y^6 + 25y^5 + 5y^4 + y^3 + 25y^2 + 5y + 1)$ The number of irreducible monic polynomial are also 6, 3 of degree 1 and 3 of degree 6.
- xxii. Let $n = 22$, Then $C_i = \{i \cdot 31^j \bmod 22 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 3, 5, 9, 15\}$, $C_2 = \{2, 6, 8, 10, 18\}$, $C_4 = \{4, 14, 12, 16, 20\}$, $C_7 = \{7, 13, 17, 19, 21\}$, $C_{11} = \{11\}$. There are 6 cyclotomic cosets 31 mod 22. We factorize $y^{22} - 1$. $y^{22} - 1 = (y^{11} - 1)(y^{11} + 1) = (y + 30)(y^5 + 10y^4 + 30y^3 + y^2 + 9y + 30)(y^5 + 22y^4 + 30y^3 + y^2 + 21y + 30)(y + 1)(y^{10} + 30y^9 + y^8 + 30y^7 + y^6 + 30y^5 + y^4 + 30y^3 + y^2 + 30y + 1) = (y + 30)(y + 1)(y^5 + 9y^4 + 30y^3 + 30y^2 + 21y + 1)(y^5 + 10y^4 + 30y^3 + y^2 + 9y + 30)(y^5 + 21y^4 + 30y^3 + 30y^2 + 9y + 1)(y^5 + 22y^4 + 30y^3 + y^2 + 21y + 30)$. The number of irreducible monic polynomial are also 6, 2 of degree 1 and 4 of degree 5.
- xxiii. Let $n = 23$, Then $C_i = \{i \cdot 31^j \bmod 23 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18\}$, $C_5 = \{5, 10, 11, 14, 7, 15, 17, 19, 20, 21, 22\}$ There are 3 cyclotomic cosets 31 mod 23. We factorize $y^{23} - 1$. $y^{23} - 1 = (y - 1)(y^{22} + y^{21} + y^{20} + y^{19} + y^{18} + y^{17} + y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y + 30)(y^{11} + 8y^{10} + 5y^9 + 27y^8 + 20y^7 + 14y^6 + 18y^5 + 27y^4 + 4y^3 + 10y^2 + 7y + 30)(y^{11} + 24y^{10} + 21y^9 + 27y^8 + 4y^7 + 13y^6 + 17y^5 + 11y^4 + 4y^3 + 26y^2 + 23y + 30)$ The number of irreducible monic polynomial are also 3, 1 of degree 1 and 2 of degree 11.
- xxiv Let $n = 24$, Then $C_i = \{i \cdot 31^j \bmod 24 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 7\}$, $C_2 = \{2, 14\}$, $C_3 = \{3, 21\}$, $C_4 = \{4\}$, $C_5 = \{5, 11\}$, $C_6 = \{6, 18\}$, $C_8 = \{8\}$, $C_9 = \{9, 15\}$, $C_{12} = \{12\}$, $C_{13} = \{13, 19\}$, $C_{16} = \{16\}$, $C_{10} = \{10, 22\}$, $C_{17} = \{17, 23\}$, $C_{20} = \{20\}$. There are 15 cyclotomic cosets 31 mod 24. We factorize $y^{24} - 1$. $y^{24} - 1 = (y^{12} - 1)(y^{12} + 1) = (y^6 - 1)(y^6 + 1)(y^{12} + 1) = (y^3 - 1)(y^3 + 1)(y^6 + 1)(y^{12} + 1) = (y + 30)(y + 26)(y + 6)(y + 1)(y + 5)(y + 25)(y^2 + 1)(y^2 + 5)(y^2 + 25)(y^{12} + 1) = (y + 30)(y + 26)(y + 6)(y + 1)(y + 5)(y + 25)(y^2 + 1)(y^2 + 5)(y^2 + 25)(y^2 + 8y + 1)(y^2 + 23y + 1)$

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- 1) $(y^2 + 17y + 5)(y^2 + 14y + 5)(y^2 + 22y + 25)(y^2 + 9y + 25)$. The number of irreducible monic polynomial are also 15, 6 of degree 1 and 9 of degree 2.
- xxv. Let $n = 25$, Then $C_i = \{i \cdot 31^j \bmod 25 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 6, 11, 16, 21\}$, $C_2 = \{2, 7, 12, 17, 22\}$, $C_3 = \{3, 18, 8, 23, 13\}$, $C_4 = \{4, 24, 19, 14, 9\}$, $C_5 = \{5\}$, $C_{10} = \{10\}$, $C_{15} = \{15\}$, $C_{20} = \{20\}$ There are 9 cyclotomic cosets 31 mod 25. We factorize $y^{25} - 1$. $y^{25} - 1 = (y - 1)(y^{24} + y^{23} + y^{22} + y^{21} + y^{20} + y^{19} + y^{18} + y^{17} + y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y + 30)(y + 29)(y + 27)(y + 23)(y + 15)(y^5 + 15)(y^5 + 23)(y^5 + 29)(y^5 + 27)$. The number of irreducible monic polynomial are also 9, 5 of degree 1 and 4 of degree 4.
- xxvi. Let $n = 26$, Then $C_i = \{i \cdot 31^j \bmod 26 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 5, 21, 25\}$, $C_2 = \{2, 10, 16, 24\}$, $C_3 = \{3, 11, 15, 23\}$, $C_4 = \{4, 20, 22, 6\}$ $C_7 = \{7, 9, 19, 17\}$, $C_8 = \{8, 14, 18, 12\}$, $C_{13} = \{13\}$. There are 8 cyclotomic cosets 31 mod 26. We factorize $y^{26} - 1$. $y^{26} - 1 = (y^{13} - 1)(y^{13} + 1) = (y - 1)(y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)(y^{13} + 1) = (y + 30)(y^4 + 4y^3 + 11y^2 + 4y + 1)(y^4 + 6y^3 + 29y^2 + 6y + 1)(y^4 + 22y^3 + 27y^2 + 22y + 1)(y^{13} + 1) = (y + 1)(y + 30)(y^4 + 4y^3 + 11y^2 + 4y + 1)(y^4 + 6y^3 + 29y^2 + 6y + 1)(y^4 + 22y^3 + 27y^2 + 22y + 1)(y^4 + 9y^3 + 27y^2 + 9y + 1)(y^4 + 27y^3 + 11y^2 + 27y + 1)(y^4 + 25y^3 + 29y^2 + 25y + 1)$. The number of irreducible monic polynomial are also 8, 2 of degree 1 and 6 of degree 4.
- xxvii. Let $n = 27$, Then $C_i = \{i \cdot 31^j \bmod 27 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 4, 16, 10, 13, 25, 19, 22, 7\}$, $C_2 = \{2, 8, 5, 20, 26, 23, 11, 17, 14\}$, $C_3 = \{3, 12, 21\}$, $C_6 = \{6, 24, 15\}$, $C_9 = \{9\}$, $C_{18} = \{18\}$. There are 7 cyclotomic cosets 31 mod 27. We factorize $y^{27} - 1$. $y^{27} - 1 = (y - 1)(y^{26} + y^{25} + y^{24} + y^{23} + y^{22} + y^{21} + y^{20} + y^{19} + y^{18} + y^{17} + y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y + 30)(y + 6)(y + 26)(y^3 + 6)(y^3 + 26)(y^9 + 6)(y^9 + 16)$. The number of irreducible monic polynomial are also 7, 3 of degree 1, 2 of degree 3 and 2 of degree 9.
- xxviii. Let $n = 28$, Then $C_i = \{i \cdot 31^j \bmod 28 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 3, 9, 27, 25, 19\}$, $C_2 = \{2, 6, 18, 26, 22, 10\}$, $C_4 = \{4, 12, 8, 24, 16, 20\}$, $C_5 = \{5, 15, 17, 23, 13, 11\}$, $C_7 = \{7, 21\}$, $C_{14} = \{14\}$ There are 7 cyclotomic cosets 31 mod 28. We factorize $y^{28} - 1$. $y^{28} - 1 = (y^{14} - 1)(y^{14} + 1) = (y^7 - 1)(y^7 + 1)(y^{14} + 1) = (y + 30)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)(y^7 + 1)(y^{14} + 1) = (y + 30)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)(y + 1)(y^6 + 30y^5 + y^4 + 30y^3 + y^2 + 30y + 1)(y^{14} + 1) = (y + 30)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)(y + 1)(y^6 + 30y^5 + y^4 + 30y^3 + y^2 + 30y + 1)(y^2 + 1)(y^{12} + 30y^{10} + y^8 + 30y^6 + y^4 + 30y^2 + 1) = (y + 30)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)(y + 1)(y^6 + 30y^5 + y^4 + 30y^3 + y^2 + 30y + 1)(y^2 + 1)(y^6 + 10y^5 + 3y^4 + 10y^3 + 3y^2 + 10y + 1)(y^6 + 21y^5 + 3y^4 + 21y^3 + 3y^2 + 21y + 1)$. The number of irreducible monic polynomial are also 7, 2 of degree 1 and 1 of degree 2, 4 of degree 6.
- xxix. Let $n = 29$, Then $C_i = \{i \cdot 31^j \bmod 29 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\}$. There are 2 cyclotomic cosets 31 mod 29. We factorize $y^{29} - 1$. $y^{29} - 1 = (y - 1)(y^{28} + y^{27} + y^{26} + y^{25} + y^{24} + y^{23} + y^{22} + y^{21} + y^{20} + y^{19} + y^{18} + y^{17} + y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1) = (y + 30)(y^{28} + y^{27} + y^{26} + y^{25} + y^{24} + y^{23} + y^{22} + y^{21} + y^{20} + y^{19} + y^{18} + y^{17} + y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)$. The number of irreducible monic polynomial are also 2, 1 of degree 1, 1 of degree 28.
- xxx. Let $n = 30$, Then $C_i = \{i \cdot 31^j \bmod 15 \in \mathbb{Z}_{31} : j = 0, 1, 2, \dots\}$, $C_0 = \{0\}$, $C_1 = \{1\}$, $C_2 = \{2\}$, $C_3 = \{3\}$, $C_4 = \{4\}$, $C_5 = \{5\}$, $C_6 = \{6\}$, $C_7 = \{7\}$, $C_8 = \{8\}$, $C_9 = \{9\}$, $C_{10} = \{10\}$, $C_{11} = \{11\}$, $C_{12} = \{12\}$, $C_{13} = \{13\}$, $C_{14} = \{14\}$, $C_{15} = \{15\}$, $C_{16} = \{16\}$, $C_{17} = \{17\}$, $C_{18} = \{18\}$, $C_{19} = \{19\}$, $C_{20} = \{20\}$, $C_{21} = \{21\}$, $C_{22} = \{22\}$, $C_{23} = \{23\}$, $C_{24} = \{24\}$, $C_{25} = \{25\}$, $C_{26} = \{26\}$, $C_{27} = \{27\}$, $C_{28} = \{28\}$, $C_{29} = \{29\}$. There are 30 cyclotomic cosets 31 mod 15. We factorize $y^{30} - 1$. $y^{30} - 1 = (y^{15} - 1)(y^{15} + 1) = (y + 30)(y + 26)(y + 6)(y + 3)(y + 21)(y + 29)(y + 23)(y + 27)(y + 17)(y + 11)(y + 24)(y + 22)(y + 12)(y + 13)(y + 15)(y^{15} + 1) = (y + 30)(y + 26)(y + 6)(y + 3)(y + 21)(y + 29)(y + 23)(y + 27)(y + 17)(y + 11)(y + 24)(y + 22)(y + 12)(y + 13)(y + 15)(y + 4)(y + 5)(y + 7)(y + 8)(y + 9)(y + 10)(y + 14)(y + 16)(y +$

18)($y + 19$)($y + 20$)($y + 25$)($y + 28$)($y + 1$)($y + 2$)
 The number of irreducible monic polynomial are also 30 of degree 1.

Table 1: The Number of Cyclic Code of Length $n = 1, 2, 3, \dots, 30$ over \mathbb{Z}_{31}

n	$y^n - 1$	Number of q Cyclotomic Cosets equal h	Number of Cyclic code equal 2^h
1	$y - 1$	1	$2^1 = 2$
2	$y^2 - 1$	2	$2^2 = 4$
3	$y^3 - 1$	3	$2^3 = 8$
4	$y^4 - 1$	3	$2^3 = 8$
5	$y^5 - 1$	5	$2^5 = 32$
6	$y^6 - 1$	6	$2^6 = 64$
7	$y^7 - 1$	2	$2^2 = 4$
8	$y^8 - 1$	5	$2^5 = 32$
9	$y^9 - 1$	5	$2^5 = 32$
10	$y^{10} - 1$	10	$2^{10} = 1,024$
11	$y^{11} - 1$	3	$2^3 = 8$
12	$y^{12} - 1$	9	$2^9 = 512$
13	$y^{13} - 1$	4	$2^4 = 16$
14	$y^{14} - 1$	4	$2^4 = 16$
15	$y^{15} - 1$	15	$2^{15} = 32,768$
16	$y^{16} - 1$	9	$2^9 = 512$
17	$y^{17} - 1$	2	$2^2 = 4$
18	$y^{18} - 1$	10	$2^{10} = 1,024$
19	$y^{19} - 1$	4	$2^4 = 16$
20	$y^{20} - 1$	15	$2^{15} = 32,768$
21	$y^{21} - 1$	6	$2^6 = 64$
22	$y^{22} - 1$	6	$2^6 = 64$
23	$y^{23} - 1$	3	$2^3 = 8$
24	$y^{24} - 1$	15	$2^{15} = 32,768$
25	$y^{25} - 1$	9	$2^9 = 512$
26	$y^{26} - 1$	8	$2^8 = 256$
27	$y^{27} - 1$	7	$2^7 = 128$
28	$y^{28} - 1$	7	$2^7 = 128$
29	$y^{29} - 1$	2	$2^2 = 4$
30	$y^{30} - 1$	30	$2^{30} = 1,073,741,824$

Conjecture 2.1. Suppose h is the number of cyclotomic coset 31 mode n , then the number of cyclic codes over N over \mathbb{Z}_{31} is given by $N = 2^h$

2.2 Number of Cyclic Codes when $n = 31w$, $1 \leq w \leq 15$

$$y^n - 1 = (y - 1)^n = ((y + 30)^n) = (y + 30)^{31w}$$

- i. Let $w = 1$, then $n = 31$ and $y^{31} - 1 = (y - 1)^{31} = (y + 30)^{31}$. Number of cyclic codes are, $(31 + 1) = 32$.
- ii. Let $w = 2$, then $n = 62$ and $y^{62} - 1 = (y^2 - 1)^{31} = (y - 1)^{31}(y + 1)^{31} = (y + 30)^{31}(y + 1)^{31}$. Number of cyclic codes are, $(31 + 1)^2 = 32^2 = 1,024$.
- iii. Let $w = 3$, then $n = 93$ and $y^{93} - 1 = (y^3 - 1)^{31} = (y + 30)(y + 26)(y + 6)$
- iv. Let $w = 4$, then $n = 124$ and $y^{124} - 1 = (y^4 - 1)^{31} = (y^2 - 1)(y^2 + 1) = (y + 30)^{31}(y + 1)^{31}(y^2 + 1)^{31}$ Number of cyclic codes are, $(31 + 1)^3 = 32^3 = 32,768$.
- v. Let $w = 5$, then $n = 155$ and $y^{155} - 1 = (y^5 - 1)^{31} = ((y - 1)(y^4 + y^3 + y^2 + y + 1))^{31} = ((y + 30)(y + 29)(y + 27)(y + 15)(y + 23))^{31}$ Number of cyclic codes are, $(31 + 1)^5 = 32^5 = 33,554,432$.
- vi. Let $w = 6$, then $n = 186$ and $y^{186} - 1 = (y^6 - 1)^{31} = ((y^3 - 1)(y^3 + 1))^{31} = ((y + 30)(y + 26)(y + 6)(y + 1)(y + 5)(y + 25))^{31}$ Number of cyclic codes are, $(31 + 1)^6 = 32^6 = 887,503,681$.
- vii. Let $w = 7$, then $n = 217$ and $y^{217} - 1 = (y^7 - 1)^{31} = ((y + 30)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1))^{31}$ Number of cyclic codes are, $(31 + 1)^7 = 32^7 = 1,024$.
- viii. Let $w = 8$, then $n = 248$ and $y^{248} - 1 = (y^8 - 1)^{31} = ((y^4 - 1)(y^4 + 1))^{31} = ((y + 30)^{31}(y + 1)^{31}(y^2 + 1)^{31}(y^2 + 8y + 1)(y^2 + 23y + 1))^{31}$ Number of cyclic codes are, $(31 + 1)^8 = 32^8 = 28,629,151$.
- ix. Let $w = 9$, then $n = 279$ and $y^{279} - 1 = (y^9 - 1)^{31} = ((y + 30)(y + 26)(y + 6)(y^3 + 6)(y^3 + 26))^{31}$ Number of cyclic codes are, $(31 + 1)^9 = 32^9 = 28,629,151$.
- x. Let $w = 10$, then $n = 310$ and $y^{310} - 1 = (y^{10} - 1)^{31} = ((y^5 - 1)(y^5 + 1))^{31} = ((y + 30)(y + 29)(y + 27)(y + 15)(y + 23)(y + 1)(y + 2)(y + 4)(y + 8)(y + 16))^{31}$ Number of cyclic codes are, $(31 + 1)^{10} = 32^{10} = 819,628,286,980,801$.
- xi. Let $w = 11$, then $n = 341$ and $y^{341} - 1 = (y^{11} - 1)^{31} = ((y - 1)(y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1))^{31} = ((y + 30)(y^5 + 10y^4 + 30y^3 + y^2 + 9y + 30)(y^5 + 22y^4 + 30y^3 + y^2 + 21y + 30))^{31}$ Number of cyclic codes are, $(31 + 1)^{11} = 32^{11} = 29,791$.
- xii. Let $w = 12$, then $n = 372$ and $y^{372} - 1 = (y^{12} - 1)^{31} = ((y^6 - 1)(y^6 + 1))^{31} = ((y + 30)(y + 26)(y + 6)(y + 1)(y + 5)(y + 25)(y^2 + 1)(y^2 + 5)(y^2 + 25))^{31}$ Number of cyclic codes are, $(31 + 1)^{12} = 32^{12} = 26,439,622,160,671$.
- xiii. Let $w = 13$, then $n = 403$ and $y^{403} - 1 = (y^{13} - 1)^{31} = ((y + 30)(y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1))^{31}$ Number of cyclic codes are, $(31 + 1)^{13} = 32^{13} = 923,521$.
- xiv. Let $w = 14$, then $n = 434$ and $y^{434} - 1 = (y^{14} - 1)^{31} = ((y^7 - 1)(y^7 + 1))^{31} = ((y + 30)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)(y^7 + 1))^{31} = ((y + 30)(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)(y + 1)(y^6 + 30y^5 + y^4 + 30y^3 + y^2 + 30y + 1))^{31}$ Number of cyclic codes are, $(31 + 1)^{14} = 32^{14} = 923,521$.
- xv. Let $w = 15$, then $n = 465$ and $y^{465} - 1 = (y^{15} - 1)^{31} = ((y - 1)(y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1))^{31} = ((y + 30)(y + 26)(y + 6)(y + 3)(y + 21)(y + 29)(y + 23)(y + 27)(y + 17)(y + 11)(y + 24)(y + 22)(y + 12)(y + 13)(y + 15))^{31}$ Number of cyclic codes are, $(31 + 1)^{15} = 32^{15} = 23,465,261,991,844,685,929,951$.

The table below gives summary of the number of cyclic code

Table 2: The Number of Cyclic Code for $n = 31w$ over \mathbb{Z}_{31}

w	$n = 31w$	Number of Factors	Number of Cyclic code
1	31	1	$32^1 = 32$
2	62	2	$32^2 = 1,024$
3	93	3	$32^3 = 32,768$
4	124	3	$32^3 = 32,768$
5	155	5	$32^5 = 33,554,432$
6	186	6	$32^6 = 1,073,741,824$
7	217	2	$32^2 = 1,024$
8	248	5	$32^5 = 33,554,432$
9	279	5	$32^5 = 33,554,432$
10	310	10	$32^{10} = 1,125,899,906,842,624$
11	341	3	$32^3 = 32,768$
12	372	9	$32^9 = 35,184,372,088,832$
13	403	4	$32^4 = 1,048,576$
14	434	4	$32^4 = 1,048,576$
15	465	15	$32^{15} = 32,768$

Conjecture 2.2. Suppose $n = 31w, w \in \mathbb{Z}^+$ and $y^n - 1$ factorizes into f linear factors then the number of cyclic code N over \mathbb{Z}_{31} is given by; $N = 32^f$

2.3 Number of Cyclic Codes when $n = 31^w, 1 \leq w \leq 10$

- i. Let $w = 1$, then $n = 31$ and $y^{31} - 1 = (y - 1)^{31} = (y + 30)^{31}$. Number of cyclic codes are, $31^1 + 1 = 31 + 1 = 32$
- ii. Let $w = 2$, then $n = 31^2$ and $y^{31^2} - 1 = (y - 1)^{31^2} = (y + 30)^{31^2}$ Number of cyclic codes are, $31^2 + 1 = 961 + 1 = 962$
- iii. Let $w = 3$, then $n = 31^3$ and $y^{31^3} - 1 = (y - 1)^{31^3} = (y + 30)^{31^3}$ Number of cyclic codes are, $31^3 + 1 = 29,791 + 1 = 29,792$
- iv. Let $w = 4$, then $n = 31^4$ and $y^{31^4} - 1 = (y - 1)^{31^4} = (y + 30)^{31^4}$. Number of cyclic codes are, $31^4 + 1 = 923,521 + 1 = 923,522$
- v. Let $w = 5$, then $n = 31^5$ and $y^{31^5} - 1 = (y - 1)^{31^5} = (y + 30)^{31^5}$ Number of cyclic codes are, $31^5 + 1 = 31 + 1 = 28,629,152$
- vi. Let $w = 6$, then $n = 31^6$ and $y^{31^6} - 1 = (y - 1)^{31^6} = (y + 30)^{31^6}$ Number of cyclic codes are, $31^6 + 1 = 887,503,681 + 1 = 887,503,682$
- vii. Let $w = 7$, then $n = 31^7$ and $y^{31^7} - 1 = (y - 1)^{31^7} = (y + 30)^{31^7}$ Number of cyclic codes are, $31^7 + 1 = 27,512,614,111 + 1 = 27,512,614,112$
- viii. Let $w = 8$, then $n = 31^8$ and $y^{31^8} - 1 = (y - 1)^{31^8} = (y + 30)^{31^8}$ Number of cyclic codes are, $31^8 + 1 = 852,891,037,441 + 1 = 852,891,037,442$
- ix. Let $w = 9$, then $n = 31^9$ and $y^{31^9} - 1 = (y - 1)^{31^9} = (y + 30)^{31^9}$ Number of cyclic codes are, $31^9 + 1 = 26,439,622,160,671 + 1 = 26,439,622,160,672$

- x. Let $w = 10$, then $n = 31^{10}$ and $y^{31^{10}} - 1 = (y - 1)^{31^{10}} = (y + 30)^{31^{10}}$ Number of cyclic codes are, $31^{10} + 1 = 819, 628, 286, 980, 801 + 1 = 819, 628, 286, 980, 802$

The table below gives summary of the number of cyclic code.

Table 3: The Number of Cyclic Code for $n = 31^w$ over \mathbb{Z}_{31}

w	$n = 31^w$	Factors	Number of Cyclic code
1	31^1	$(y - 1)^{31^1}$	$31^1 + 1$
2	31^2	$(y - 1)^{31^2}$	$31^2 + 1$
3	31^3	$(y - 1)^{31^3}$	$31^3 + 1$
4	31^4	$(y - 1)^{31^4}$	$31^4 + 1$
5	31^5	$(y - 1)^{31^5}$	$31^5 + 1$
6	31^6	$(y - 1)^{31^6}$	$31^6 + 1$
7	31^7	$(y - 1)^{31^7}$	$31^7 + 1$
8	31^8	$(y - 1)^{31^8}$	$31^8 + 1$
9	31^9	$(y - 1)^{31^9}$	$31^9 + 1$
10	31^{10}	$(y - 1)^{31^{10}}$	$31^{10} + 1$

In general one can easily infer that for any positive w then, $y^{31^w} - 1 = (y - 1)^{31^w} = (y + 30)^{31^w}$. and the number of cyclic codes is given by $31^w + 1$.

Conjecture 2.3. Suppose $n = 31^w, w \in \mathbb{Z}^+$ and $y^n - 1$ factorizes into f linear factors such that $y^n - 1 = (y - 1)^n = (y - 1)^{31^w}$ then the number of cyclic code N over \mathbb{Z}_{31} is given by $N = 31^w + 1$

2.4 Number of Cyclic Codes when $n = q \cdot 31^r$ where $r = 0, 1, 2, 3, \dots$, and q is prime.

- i. Let $q = 2$ then $n = 2(31)^r$ for $r = 0, 1, 2, \dots$ and $y^{2(31)^r} - 1 = (y^2 - 1)^{31^r} = (y + 30)^{31^r} (y + 1)^{31^r}$ when $r = 0$, $(y^2 - 1)^{31^0} = y^2 - 1 = (y + 30)(y + 1)$. The number of factors f equals $(31^0 + 1) = 2$ and the number cyclic codes = $f^2 = 4$
 When $r = 1$, $(y^2 - 1)^{31^1} = (y^2 - 1)^{31} = ((y + 30)(y + 1))^{31} = (y + 30)^{31} (y + 1)^{31}$, number of factors f equals $31^1 + 1 = 32$ number of cyclic codes = $(31 + 1)^2 = 1024$
 when $r = 2$, $y^n - 1 = (y + 30)^{31^2} (y + 1)^{31^2}$, number of factors f equals $31^2 + 1 = 62$ number of cyclic codes = $(31^2 + 1)^2 = 925, 444$
 In general, $y^{2(31)^r} - 1 = [(y - 1)^{31^r}]^2$. number of factors f equals = $31^r + 1$ and number of cyclic codes = $(31^r + 1)^2$
- ii. Let $q = 3$ then $n = 3(31)^r$ for $r = 0, 1, 2, \dots$ and $y^n - 1 = y^{3(31)^r} - 1 = (y^3 - 1)^{31^r} = (y + 30)^{31^r} (y + 26)^{31^r} (y + 6)^{31^r}$
 When $r = 0$, $y^n - 1 = (y + 30)^{31^0} (y + 26)^{31^0} (y + 6)^{31^0}$. Number of cyclic codes = $(31^0 + 1)^3 = 8$.
 When $r = 1$, $y^n - 1 = (y + 30)^{31^1} (y + 26)^{31^1} (y + 6)^{31^1}$ Number of cyclic codes = $(31^1 + 1)^3 = 32, 768$.
 When $r = 2$, $y^n - 1 = (y + 30)^{31^2} (y + 26)^{31^2} (y + 6)^{31^2}$. Number of cyclic codes = $(31^2 + 1)^3$.
 In general, number of cyclic codes = $(31^r + 1)^3$.
- iii. Let $q = 5$ then $n = 5(31)^r, r = 0, 1, 2, \dots$ and $y^n - 1 = y^{5(31)^r} - 1 = (y^5 - 1)^{31^r} = (y + 30)^{31^r} (y + 29)^{31^r} (y + 27)^{31^r} (y + 15)^{31^r} (y + 23)^{31^r}$

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- When $r = 0$, $y^n - 1 = (y + 30)(y + 29)(y + 27)(y + 15)(y + 23)$. Number of cyclic codes $= (31^0 + 1)^5 = 32$.
- When $r = 1$, $y^n - 1 = (y + 30)^{31}(y + 29)^{31}(y + 27^{31})(y + 15)^{31}(y + 23)^{31}$. Number of cyclic codes $= (31^1 + 1)^5$
- When $r = 2$, $y^n - 1 = (y + 30)^{31^2}(y + 29)^{31^2}(y + 27^{31^2})(y + 15)^{31^2}(y + 23)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^5$.
- In general, number of cyclic codes $= (31^r + 1)^5$.
- iv. Let $q = 7$ then $n = 7(31)^r$, $r = 0, 1, 2, \dots$ and $y^n - 1 = y^{7(31)^r} - 1 = (y^7 - 1)^{31^r} = (y + 30)^{31^r}(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)^{31^r}$.
- When $r = 0$, $y^n - 1 = (y + 30)^{31^0}(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)^{31^0}$. Number of cyclic codes $= (31^0 + 1)^2$
- When $r = 1$, $y^n - 1 = (y + 30)^{31^1}(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)^{31^1}$ Number of cyclic codes $= (31^1 + 1)^2$.
- When $r = 2$, $y^n - 1 = (y + 30)^{31^2}(y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)^{31^2}$ Number of cyclic codes $= (31^2 + 1)^2$. In general, number of cyclic codes $= (31^r + 1)^2$.
- v. Let $q = 11$ then $n = 11(31)^r$, $r = 0, 1, 2, \dots$ and $y^n - 1 = y^{11(31)^r} - 1 = (y^{11} - 1)^{31^r} = (y + 30)^{31^r}(y^5 + 10y^4 + 30y^3 + y^2 + 9y + 30)^{31^r}(y^5 + 22y^4 + 30y^3 + y^2 + 21y + 30)^{31^r}$.
- When $r = 0$, $y^n - 1 = (y + 30)^{31^0}(y^5 + 10y^4 + 30y^3 + y^2 + 9y + 30)^{31^0}(y^5 + 22y^4 + 30y^3 + y^2 + 21y + 30)^{31^0}$ Number of cyclic codes $= (31^0 + 1)^3$.
- When $r = 1$, $y^n - 1 = (y + 30)^{31^1}(y^5 + 10y^4 + 30y^3 + y^2 + 9y + 30)^{31^1}(y^5 + 22y^4 + 30y^3 + y^2 + 21y + 30)^{31^1}$ Number of cyclic codes $= (31^1 + 1)^3$.
- When $r = 2$, $y^n - 1 = (y + 30)^{31^2}(y^5 + 10y^4 + 30y^3 + y^2 + 9y + 30)^{31^2}(y^5 + 22y^4 + 30y^3 + y^2 + 21y + 30)^{31^2}$ Number of cyclic codes $= (31^2 + 1)^3$.
- vi. Let $q = 13$ then $n = 13(31)^r$, $r = 0, 1, 2, \dots$ and $y^n - 1 = y^{13(31)^r} - 1 = (y^{13} - 1)^{31^r} = (y + 30)^{31^r}(y^4 + 4y^3 + 11y^2 + 4y + 1)^{31^r}(y^4 + 6y^3 + 29y^2 + 6y + 1)^{31^r}(y^4 + 22y^3 + 28y^2 + 22y + 1)^{31^r}$
- When $r = 0$, $y^n - 1 = (y + 30)^{31^0}(y^4 + 4y^3 + 11y^2 + 4y + 1)^{31^0}(y^4 + 6y^3 + 29y^2 + 6y + 1)^{31^0}(y^4 + 22y^3 + 28y^2 + 22y + 1)^{31^0}$. Number of cyclic codes $= (31^0 + 1)^4$.
- When $r = 1$, $y^n - 1 = (y + 30)^{31^1}(y^4 + 4y^3 + 11y^2 + 4y + 1)^{31^1}(y^4 + 6y^3 + 29y^2 + 6y + 1)^{31^1}(y^4 + 22y^3 + 28y^2 + 22y + 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^4$.
- When $r = 2$, $y^n - 1 = (y + 30)^{31^2}(y^4 + 4y^3 + 11y^2 + 4y + 1)^{31^2}(y^4 + 6y^3 + 29y^2 + 6y + 1)^{31^2}(y^4 + 22y^3 + 28y^2 + 22y + 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^4$
- vii. Let $q = 17$ then $n = 17(31)^r$, $r = 0, 1, 2, \dots$ and $y^n - 1 = y^{17(31)^r} - 1 = (y^{17} - 1)^{31^r} = (y + 30)^{31^r}(y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)^{31^r}$.
- When $r = 0$, $y^n - 1 = (y + 30)^{31^0}(y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)^{31^0}$. Number of cyclic codes $= (31^0 + 1)^2$.
- When $r = 1$, $y^n - 1 = (y + 30)^{31^1}(y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^2$. When $r = 2$, $y^n - 1 = (y + 30)^{31^2}(y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^2$.
- viii. Let $q = 19$ then $n = 19(31)^r$, $r = 0, 1, 2, \dots$ and $y^n - 1 = y^{19(31)^r} - 1 = (y^{19} - 1)^{31^r} = (y + 30)^{31^r}(y^6 + 4y^5 + 14y^4 + 18y^3 + 14y^2 + 4y + 1)^{31^r}(y^6 + 5y^5 + 25y^4 + y^3 + 5y^2 + 25y + 1)^{31^r}(y^6 + 25y^5 + 5y^4 + y^3 + 25y^3 + 5y + 1)^{31^r}$.
- when $r = 0$, $y^n - 1 = (y + 30)^{31^0}(y^6 + 4y^5 + 14y^4 + 18y^3 + 14y^2 + 4y + 1)^{31^0}(y^6 + 5y^5 + 25y^4 + y^3 + 5y^2 + 25y + 1)^{31^0}(y^6 + 25y^5 + 5y^4 + y^3 + 25y^3 + 5y + 1)^{31^0}$. Number of cyclic codes $= (31^0 + 1)^4$.

When $r = 1$, $y^n - 1 = (y + 30)^{31^1} (y^6 + 4y^5 + 14y^4 + 18y^3 + 14y^2 + 4y + 1)^{31^1}$
 $(y^6 + 5y^5 + 25y^4 + y^3 + 5y^2 + 25y + 1)^{31^1} (y^6 + 25y^5 + 5y^4 + y^3 + 25y^3 + 5y + 1)^{31^1}$.
Number of cyclic codes = $(31^0 + 1)^4$.
When $r = 2$, $y^n - 1 = (y + 30)^{31^2} (y^6 + 4y^5 + 14y^4 + 18y^3 + 14y^2 + 4y + 1)^{31^2}$
 $(y^6 + 5y^5 + 25y^4 + y^3 + 5y^2 + 25y + 1)^{31^2} (y^6 + 25y^5 + 5y^4 + y^3 + 25y^3 + 5y + 1)^{31^2}$. Number
of cyclic codes = $(31^2 + 1)^4$.

- ix. Let $q = 23$ then $n = 23(31)^r$, $r = 0, 1, 2, \dots$ and $y^n - 1 = y^{23(31)^r} - 1 = (y^{23} - 1)^{31^r} =$
 $(y + 30)^{31^r} (y^{11} + 8y^{10} + 5y^9 + 27y^8 + 20y^7 + 14y^6 + 18y^5 + 27y^4 + 4y^3 + 10y^2 + 7y + 30)^{31^r} (y^{11} +$
 $24y^{10} + 21y^9 + 27y^8 + 4y^7 + 13y^6 + 17y^5 + 11y^4 + 4y^3 + 26y^2 + 23y + 30)^{31^r}$.
When $r = 0$, $y^n - 1 = (y + 30)^{31^0} (y^{11} + 8y^{10} + 5y^9 + 27y^8 + 20y^7 + 14y^6 + 18y^5 + 27y^4 + 4y^3 +$
 $10y^2 + 7y + 30)^{31^0} (y^{11} + 24y^{10} + 21y^9 + 27y^8 + 4y^7 + 13y^6 + 17y^5 + 11y^4 + 4y^3 + 26y^2 + 23y + 30)^{31^0}$.
Number of cyclic codes = $(31^0 + 1)^3$.
When $r = 1$, $y^n - 1 = (y + 30)^{31^1} (y^{11} + 8y^{10} + 5y^9 + 27y^8 + 20y^7 + 14y^6 + 18y^5 + 27y^4 + 4y^3 +$
 $10y^2 + 7y + 30)^{31^1} (y^{11} + 24y^{10} + 21y^9 + 27y^8 + 4y^7 + 13y^6 + 17y^5 + 11y^4 + 4y^3 + 26y^2 + 23y + 30)^{31^1}$.
Number of cyclic codes = $(31^1 + 1)^3$.
When $r = 2$, $y^n - 1 = (y + 30)^{31^2} (y^{11} + 8y^{10} + 5y^9 + 27y^8 + 20y^7 + 14y^6 + 18y^5 + 27y^4 + 4y^3 +$
 $10y^2 + 7y + 30)^{31^2} (y^{11} + 24y^{10} + 21y^9 + 27y^8 + 4y^7 + 13y^6 + 17y^5 + 11y^4 + 4y^3 + 26y^2 + 23y + 30)^{31^2}$.
Number of cyclic codes = $(31^2 + 1)^3$.
- x. Let $q = 29$ then $n = 29(31)^r$, $r = 0, 1, 2, \dots$ and $y^n - 1 = y^{29(31)^r} - 1 = (y^{29} - 1)^{31^r} (y^{29} - 1)^{31^r} =$
 $((y + 30)(y^{28} + y^{27} + y^{26} + y^{25} + y^{24} + y^{23} + y^{22} + y^{21} + y^{20} + y^{19} + y^{18} + y^{17} + y^{16} + y^{15} +$
 $y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1))^{31^r}$.
When $r = 0$, $(y^{29} - 1) = ((y + 30)(y^{28} + y^{27} + y^{26} + y^{25} + y^{24} + y^{23} + y^{22} + y^{21} + y^{20} + y^{19} +$
 $y^{18} + y^{17} + y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1))$.
Number of cyclic codes $(31^0 + 1)^2$.
When $r = 1$, $(y^{29} - 1)^{31} = ((y + 30)(y^{28} + y^{27} + y^{26} + y^{25} + y^{24} + y^{23} + y^{22} + y^{21} + y^{20} + y^{19} +$
 $y^{18} + y^{17} + y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} + y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1))^{31}$.
Number of cyclic codes $(31^1 + 1)^2$ When $r = 2$, $(y^{29} - 1)^{31^2} = ((y + 30)(y^{28} + y^{27} + y^{26} +$
 $y^{25} + y^{24} + y^{23} + y^{22} + y^{21} + y^{20} + y^{19} + y^{18} + y^{17} + y^{16} + y^{15} + y^{14} + y^{13} + y^{12} + y^{11} + y^{10} +$
 $y^9 + y^8 + y^7 + y^6 + y^5 + y^4 + y^3 + y^2 + y + 1))^{31^2}$.
Number of cyclic codes $(31^2 + 1)^2$

The table below gives summary of the number of cyclic code.

Table 4: The Number of Cyclic Code for $n = q.31^r$ over \mathbb{Z}_{31}

q	$n = q.31^r$	Factors	number of factors in $y^q - 1$	Number of Cyclic code
2	2.31^r	$(y^2 - 1)^{31^r}$	2	$(31^r + 1)^2$
3	3.31^r	$(y^3 - 1)^{31^r}$	3	$(31^r + 1)^3$
5	5.31^r	$(y^5 - 1)^{31^r}$	5	$(31^r + 1)^5$
7	7.31^r	$(y^7 - 1)^{31^r}$	2	$(31^r + 1)^2$
11	11.31^r	$(y^{11} - 1)^{31^r}$	3	$(31^r + 1)^3$
13	13.31^r	$(y^{13} - 1)^{31^r}$	4	$(31^r + 1)^4$
17	17.31^r	$(y^{17} - 1)^{31^r}$	2	$(31^r + 1)^2$
19	19.31^r	$(y^{19} - 1)^{31^r}$	4	$(31^r + 1)^4$
23	23.31^r	$(y^{23} - 1)^{31^r}$	3	$(31^r + 1)^3$
29	29.31^r	$(y^{29} - 1)^{31^r}$	2	$(31^r + 1)^2$

Conjecture 2.4. Suppose $n = q \cdot 31^r$, $r = 0, 1, 2, \dots$, q prime and $y^n - 1 = (y - 1)^n = (y - 1)^{q \cdot 31^r}$ factorizes into $31^r + 1$ linear factors then the number of cyclic code N over \mathbb{Z}_{31} is given by $N = (31^r + 1)^q$

2.5 Number of Cyclic Codes when $n = q^r \cdot 31^r$ where $r = 1, 2, 3, \dots$ and q is prime.

- i. Let $q = 2$ then $n = 2^r(31)^r$ for $r = 1, 2, \dots$ and $y^{2^r(31)^r} - 1 = (y^{2^r} - 1)^{31^r}$.
 When $r = 1$, $(y^{2^1} - 1)^{31^1} = (y^2 - 1)^{31} = ((y + 30)(y + 1))^{31} = (y + 30)^{31}(y + 1)^{31}$,
 then the number of cyclic codes $= (31 + 1)^2 = 1024$.
 When $r = 2$, $y^n - 1 = (y^{2^2} - 1)^{31^2}$, then the number of cyclic codes $= (31^2 + 1)^{2^2}$.
 In general, number of cyclic codes $= (31^r + 1)^{2^r}$.
- ii. Let $q = 3$ then $n = 3^r(31)^r$ for $r = 1, 2, \dots$ and $y^n - 1 = y^{3^r(31)^r} - 1 = (y^{3^r} - 1)^{31^r}$.
 When $r = 1$, $y^n - 1 = y^{3^1(31^1)} - 1 = (y^{3^1} - 1)^{31^1}$,
 then the number of cyclic codes $= (31^1 + 1)^{3^1} = 32, 768$.
 When $r = 2$, $y^n - 1 = (y^{3^2} - 1)^{31^2}$, then the number of cyclic codes $= (31^2 + 1)^{3^2}$.
 In general, number of cyclic codes $= (31^r + 1)^{3^r}$.
- iii. Let $q = 5$ then $n = 5^r(31)^r$, $r = 1, 2, \dots$ and $y^n - 1 = y^{5^r(31)^r} - 1 = (y^{5^r} - 1)^{31^r}$.
 When $r = 1$, $y^n - 1 = (y^{5^1} - 1)^{31^1}$, then the number of cyclic codes $= (31^1 + 1)^{5^1}$.
 When $r = 2$, $y^n - 1 = (y^{5^2} - 1)^{31^2}$, then the number of cyclic codes $= (31^2 + 1)^{5^2}$.
 In general, number of cyclic codes $= (31^r + 1)^{5^r}$.
- iv. Let $q = 7$ then $n = 7^r(31)^r$, $r = 1, 2, \dots$ and $y^n - 1 = y^{7^r(31)^r} - 1 = (y^{7^r} - 1)^{31^r}$.
 When $r = 1$, $y^n - 1 = (y^{7^1} - 1)^{31^1}$, then the number of cyclic codes $= (31^1 + 1)^{7^1}$.
 When $r = 2$, $y^n - 1 = (y^{7^2} - 1)^{31^2}$, then the number of cyclic codes $= (31^2 + 1)^{7^2}$.
 In general, number of cyclic codes $= (31^r + 1)^{7^r}$.
- v. Let $q = 11$ then $n = 11^r(31)^r$, $r = 1, 2, \dots$ and $y^n - 1 = y^{11^r(31)^r} - 1 = (y^{11^r} - 1)^{31^r}$.
 When $r = 1$, $y^n - 1 = (y^{11^1} - 1)^{31^1}$, then the number of cyclic codes $= (31^1 + 1)^{11^1}$.
 When $r = 2$, $y^n - 1 = (y^{11^2} - 1)^{31^2}$, then the number of cyclic codes $= (31^2 + 1)^{11^2}$.
 In general, number of cyclic codes $= (31^r + 1)^{11^r}$.
- vi. Let $q = 13$ then $n = 13^r(31)^r$, $r = 1, 2, \dots$ and $y^n - 1 = y^{13^r(31)^r} - 1 = (y^{13^r} - 1)^{31^r}$.
 When $r = 1$, $y^n - 1 = (y^{13^1} - 1)^{31^1}$, then the number of cyclic codes $= (31^1 + 1)^{13^1}$.
 When $r = 2$, $y^n - 1 = (y^{13^2} - 1)^{31^2}$, then the number of cyclic codes $= (31^2 + 1)^{13^2}$.
 In general, number of cyclic codes $= (31^r + 1)^{13^r}$.
- vii. Let $q = 17$ then $n = 17^r(31)^r$, $r = 1, 2, \dots$ and $y^n - 1 = y^{17^r(31)^r} - 1 = (y^{17^r} - 1)^{31^r}$.
 When $r = 1$, $y^n - 1 = (y^{17^1} - 1)^{31^1}$, then the number of cyclic codes $= (31^1 + 1)^{17^1}$.
 When $r = 2$, $y^n - 1 = (y^{17^2} - 1)^{31^2}$, then the number of cyclic codes $= (31^2 + 1)^{17^2}$.
 In general, number of cyclic codes $= (31^r + 1)^{17^r}$.
- viii. Let $q = 19$ then $n = 19^r(31)^r$, $r = 1, 2, \dots$ and $y^n - 1 = y^{19^r(31)^r} - 1 = (y^{19^r} - 1)^{31^r}$.
 When $r = 1$, $y^n - 1 = (y^{19^1} - 1)^{31^1}$. then the number of cyclic codes $= (31^1 + 1)^{19^1}$.
 When $r = 2$, $y^n - 1 = (y^{19^2} - 1)^{31^2}$. then the number of cyclic codes $= (31^2 + 1)^{19^2}$.
 In general, number of cyclic codes $= (31^r + 1)^{19^r}$.
- ix. Let $q = 23$ then $n = 23^r(31)^r$, $r = 1, 2, \dots$ and $y^n - 1 = y^{23^r(31)^r} - 1 = (y^{23^r} - 1)^{31^r}$.
 When $r = 1$, $y^n - 1 = (y^{23^1} - 1)^{31^1}$, then the number of cyclic codes $= (31^1 + 1)^{23^1}$.
 When $r = 2$, $y^n - 1 = (y^{23^2} - 1)^{31^2}$, then the number of cyclic codes $= (31^2 + 1)^{23^2}$.
 In general, number of cyclic codes $= (31^r + 1)^{23^r}$.

- x. Let $q = 29$ then $n = 29^r(31)^r$, $r = 1, 2, \dots$ and $y^n - 1 = y^{29^r(31)^r} - 1 = (y^{29^r} - 1)^{31^r}$
 When $r = 1$, $y^n - 1 = (y^{29^1} - 1)^{31^1}$, then the number of cyclic codes $= (31^1 + 1)^{2^1}$.
 When $r = 2$, $y^n - 1 = (y^{29^2} - 1)^{31^2}$. then the number of cyclic codes $= (31^2 + 1)^{2^2}$,
 In general, number of cyclic codes $= (31^r + 1)^{2^r}$.

The table below gives summary of the number of cyclic code.

Table 5: The Number of Cyclic Code for $n = q^r \cdot 31^r$ over \mathbb{Z}_{31}

q	$n = q^r \cdot 31^r$	Factors	number of factors in $y^q - 1$	Number of Cyclic code
2	$2^r \cdot 31^r$	$(y^{2^r} - 1)^{31^r}$	2	$(31^r + 1)^{2^r}$
3	$3^r \cdot 31^r$	$(y^{3^r} - 1)^{31^r}$	3	$(31^r + 1)^{3^r}$
5	$5^r \cdot 31^r$	$(y^{5^r} - 1)^{31^r}$	5	$(31^r + 1)^{5^r}$
7	$7^r \cdot 31^r$	$(y^{7^r} - 1)^{31^r}$	2	$(31^r + 1)^{2^r}$
11	$11^r \cdot 31^r$	$(y^{11^r} - 1)^{31^r}$	3	$(31^r + 1)^{3^r}$
13	$13^r \cdot 31^r$	$(y^{13^r} - 1)^{31^r}$	4	$(31^r + 1)^{4^r}$
17	$17^r \cdot 31^r$	$(y^{17^r} - 1)^{31^r}$	2	$(31^r + 1)^{2^r}$
19	$19^r \cdot 31^r$	$(y^{19^r} - 1)^{31^r}$	4	$(31^r + 1)^{4^r}$
23	$23^r \cdot 31^r$	$(y^{23^r} - 1)^{31^r}$	3	$(31^r + 1)^{3^r}$
29	$29^r \cdot 31^r$	$(y^{29^r} - 1)^{31^r}$	2	$(31^r + 1)^{2^r}$

Conjecture 2.5. Suppose $n = q^r \cdot 31^r$, $r = 1, 2, \dots$, q prime and $y^n - 1 = (y - 1)^n = (y - 1)^{q^r \cdot 31^r}$ is such that the number of factors in $y^q - 1$ is f then the number of cyclic code N over \mathbb{Z}_{31} is given by $N = (31^r + 1)^{f^r}$

2.6 Number of Cyclic Codes when $n = q^k \cdot 31^r$ where $r = 1, 2, 3, \dots$ and $k = 1, 2, \dots$ q is prime .

- i. Let $q = 2$ the $n = 2^k(31)^r$ for $r = 1, 2, \dots$ and $y^{2^k(31)^r} - 1 = (y^{2^k} - 1)^{31^r}$.
 Let $k = 1$ be fixed then $y^n - 1 = y^{2^1(31)^r} - 1 = y^{2(31)^r} - 1$. When $r = 1$, $(y^2 - 1)^{31^1} = (y^2 - 1)^{31} = ((y + 30)(y + 1))^{31} = (y + 30)^{31}(y + 1)^{31}$. Number of cyclic codes $= (31 + 1)^2 = 1024$. When $r = 2$, $y^n - 1 = (y^2 - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^2$. In general, $y^{2^r(31)^r} - 1 = [(y - 1)^{31^r}]^2$ and Number of cyclic codes $= (31^r + 1)^2$
- ii. Let $q = 3$ then $n = 3^k(31)^r$ for $r = 1, 2, \dots$ and $y^n - 1 = y^{3^k(31)^r} - 1 = (y^{3^k} - 1)^{31^r}$.
 Let $k = 1$ be fixed then $y^n - 1 = y^{3^1(31)^r} - 1 = y^{3(31)^r} - 1$. When $r = 1$, $y^n - 1 = y^{3(31^1)} - 1 = (y^3 - 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^3 = 32,768$. When $r = 2$, $y^n - 1 = (y^3 - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^3$. In general, number of cyclic codes $= (31^r + 1)^3$
- iii. Let $q = 5$ then $n = 5^r(31)^r$, $r = 1, 2, \dots$ and $y^n - 1 = y^{5^k(31)^r} - 1 = (y^{5^k} - 1)^{31^r}$.
 Let $k = 1$ be fixed then $y^n - 1 = y^{5^1(31)^r} - 1 = y^{5(31)^r} - 1$. When $r = 1$, $y^n - 1 = (y^5 - 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^5$. When $r = 2$, $y^n - 1 = (y^5 - 1)^{31^2}$. number of cyclic codes $= (31^2 + 1)^5$. In general, number of cyclic codes $= (31^r + 1)^5$.
- iv. Let $q = 7$ then $n = 7^k(31)^r$, $r = 1, 2, \dots$ and $y^n - 1 = y^{7^k(31)^r} - 1 = (y^{7^k} - 1)^{31^r}$.
 Let $k = 1$ be fixed then $y^n - 1 = y^{7^1(31)^r} - 1 = y^{7(31)^r} - 1$. When $r = 1$, $y^n - 1 = (y^7 - 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^2$. When $r = 2$, $y^n - 1 = (y^7 - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^2$. In general, number of cyclic codes $= (31^r + 1)^2$.

- v. Let $q = 11$ then $n = 11^k(31)^r, r = 1, 2, \dots$ and $y^n - 1 = y^{11^k(31)^r} - 1 = (y^{11^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{11(31)^r} - 1 = y^{11(31)^r} - 1$. When $r = 1, y^n - 1 = (y^{11} - 1)^{31^1}$.
Number of cyclic codes $= (31^1 + 1)^3$. When $r = 2, y^n - 1 = (y^{11} - 1)^{31^2}$. Number of cyclic codes $= (31 + 1)^{3^2}$. In general, number of cyclic codes $= (31^r + 1)^{3^r}$.
- vi. Let $q = 13$ then $n = 13^k(31)^r, r = 1, 2, \dots$ and $y^n - 1 = y^{13^k(31)^r} - 1 = (y^{13^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{13(31)^r} - 1 = y^{13(31)^r} - 1$. When $r = 1, y^n - 1 = (y^{13} - 1)^{31^1}$.
Number of cyclic codes $= (31^1 + 1)^4$. When $r = 2, y^n - 1 = (y^{13} - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^4$. In general, number of cyclic codes $= (31^r + 1)^4$.
- vii. Let $q = 17$ then $n = 17^k(31)^r, r = 1, 2, \dots$ and $y^n - 1 = y^{17^k(31)^r} - 1 = (y^{17^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{17(31)^r} - 1 = y^{17(31)^r} - 1$. When $r = 1, y^n - 1 = (y^{17} - 1)^{31^1}$.
Number of cyclic codes $= (31^1 + 1)^2$. When $r = 2, y^n - 1 = (y^{17} - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^2$. In general, number of cyclic codes $= (31^r + 1)^2$.
- viii. Let $q = 19$ then $n = 19^k(31)^r, r = 1, 2, \dots$ and $y^n - 1 = y^{19^k(31)^r} - 1 = (y^{19^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{19(31)^r} - 1 = y^{19(31)^r} - 1$. When $r = 1, y^n - 1 = (y^{19} - 1)^{31^1}$.
Number of cyclic codes $= (31^1 + 1)^4$. When $r = 2, y^n - 1 = (y^{19} - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^4$. In general, number of cyclic codes $= (31^r + 1)^4$.
- ix. Let $q = 23$ then $n = 23^k(31)^r, r = 1, 2, \dots$ and $y^n - 1 = y^{23^k(31)^r} - 1 = (y^{23^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{23(31)^r} - 1 = y^{23(31)^r} - 1$.
When $r = 1, y^n - 1 = (y^{23} - 1)^{31^1}$. Number of cyclic codes $= (31^1 + 1)^3$. When $r = 2, y^n - 1 = (y^{23} - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^3$. In general, number of cyclic codes $= (31^r + 1)^3$.
- x. Let $q = 29$ then $n = 29^k(31)^r, r = 1, 2, \dots$ and $y^n - 1 = y^{29^k(31)^r} - 1 = (y^{29^k} - 1)^{31^r}$.
Let $k = 1$ be fixed then $y^n - 1 = y^{29(31)^r} - 1 = y^{29(31)^r} - 1$. When $r = 1, y^n - 1 = (y^{29} - 1)^{31^1}$.
Number of cyclic codes $= (31 + 1)^2$. When $r = 2, y^n - 1 = (y^{29} - 1)^{31^2}$. Number of cyclic codes $= (31^2 + 1)^2$. In general, number of cyclic codes $= (31^r + 1)^2$.

The table below gives summary of the number of cyclic code.

Table 6: The Number of Cyclic Code for $n = q^k \cdot 31^r$ over \mathbb{Z}_{31}

q	$n = q^k \cdot 31^r$	Factors	number of factors in $y^q - 1$	Number of Cyclic code
2	$2^k \cdot 31^r$	$(y^{2^k} - 1)^{31^r}$	2	$(31^r + 1)^{2^k}$
3	$3^k \cdot 31^r$	$(y^{3^k} - 1)^{31^r}$	3	$(31^r + 1)^{3^k}$
5	$5^k \cdot 31^r$	$(y^{5^k} - 1)^{31^r}$	5	$(31^r + 1)^{5^k}$
7	$7^k \cdot 31^r$	$(y^{7^k} - 1)^{31^r}$	2	$(31^r + 1)^{2^k}$
11	$11^k \cdot 31^r$	$(y^{11^k} - 1)^{31^r}$	3	$(31^r + 1)^{3^k}$
13	$13^k \cdot 31^r$	$(y^{13^k} - 1)^{31^r}$	4	$(31^r + 1)^{4^k}$
17	$17^k \cdot 31^r$	$(y^{17^k} - 1)^{31^r}$	2	$(31^r + 1)^{2^k}$
19	$19^k \cdot 31^r$	$(y^{19^k} - 1)^{31^r}$	4	$(31^r + 1)^{4^k}$
23	$23^k \cdot 31^r$	$(y^{23^k} - 1)^{31^r}$	3	$(31^r + 1)^{3^k}$
29	$29^k \cdot 31^r$	$(y^{29^k} - 1)^{31^r}$	2	$(31^r + 1)^{2^k}$

Conjecture 2.6. Suppose $n = q^k \cdot 31^r$, $r, k \in \mathbb{Z}^+$, q prime and $y^n - 1 = (y - 1)^n = (y - 1)^{q^k \cdot 31^r}$ is such that the number of factors in $y^q - 1$ is f then for a fixed k the number of cyclic code N over \mathbb{Z}_{31} is given by $N = (31^r + 1)^{f^k}$

3 Conclusion

In this study, we partitioned cyclotomic polynomial $y^n - 1$ using cyclotomic cosets $31 \bmod n$ and factorize $y^n - 1$ using case to case basis. We then investigated the number of cyclic codes for each case. The following conclusions were made.

1. The number of cyclotomic cosets equals to the number of irreducible monic factors in $y^n - 1$
2. Let h be the number of cyclotomic cosets, then the number of cyclic codes equal to 2^h
3. Let \mathbb{Z}_{31} be a given finite field, $y^n - 1$ be cyclotomic polynomial and f be the number of irreducible factors in $y^n - 1$. Then the number of cyclic codes N over \mathbb{Z}_{31} is given by;

$$N = \begin{cases} (31 + 1)^f & : n = 31w, w \in \mathbb{Z}^+ \\ 31^w + 1 & : n = 31^w, w \in \mathbb{Z}^+ \\ (31^r + 1)^f & : n = q \cdot 31^r, r = 0, 1, 2, \dots, q \text{ is prime} \\ (31^r + 1)^{f^r} & : n = q^r \cdot 31^r, r \in \mathbb{Z}^+ \\ (31^r + 1)^{f^k} & : n = q^k \cdot 31^r, k, r \in \mathbb{Z}^+ \end{cases}$$

Competing Interests

Authors have declared no competing interest.

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