Finite-time synchronization of fractional-order quaternion-valued neural networks under aperiodically intermittent control: A non-separation method

Abstract

This article mainly focuses on investigating finite-time synchronization(FTS) of fractional-order quaternion-valued neural networks(FOQVNNs) with time-varying delay via aperiodically intermittent control. Firstly, a new group of fractional differential inequalities during aperiodically intermittent times is proposed, which provides a formula for calculating stable time $\tilde{t}$ related to a parameter ϖ about aperiodically intermittent control strategy, and it can make the stable time $\tilde{t}$ even shorter than that of many existing literature results. Secondly, the theoretical analysis of the entire paper is conducted in a nonseparable method, and a quaternion-valued aperiodically intermittent controller based on 1-norm is directly designed to ensure the synchronization of FOQVNNs. By constructing suitable Lyapunov function, the FTS criterion is derived. Finally, the validity of the proposed theoretical results is confirmed by numerical simulations.

Keywords: Fractional-order quaternion-valued neural networks; Finite-time synchronization; Time-varying delay; Fractional differential inequalities; Aperiodically intermittent control.

1. Introduction

Since the end of the last century, artificial neural networks have attracted more and more attention, which is an interdisciplinary research combining engineering, psychology and biology. Neural networks have a wide range of applications in the fields of system recognition, data compression, color image processing, signal processing, and secure communication[1–3]. Fractional order models have better characteristics of heritability and memory than that of traditional integer order models and can more accurately depict the dynamic behavior of complex systems, therefore, many scholars have begun to study fractional-order neural networks, and have obtained remarkable research achievements.

The concept of quaternion algebra was been proposed by W.R. Hamilton [4]. Given the lack of corresponding practical background, quaternion did not get widespread attention for a long time. In addition, multiplication operation on quaternions are not suitable for commutative law, so the difficulty and complexity of quaternion operation is much greater than that of complex numbers, which is also one of the reasons why the development of quaternions is relatively slow. Most of the research on neural networks is about real-valued neural networks(RVNNs)[5–7] or complex-valued neural networks(CVNNs)[8]. In fact, multi-dimensional data are often encountered in practical applications, while RVNNs and CVNNs cannot process these data well. so some scholars have considered introducing quaternions into neural networks, which are defined as quaternion-valued neural networks(QVNNs)[9–13]. It is precisely because the commutative law of quaternion multiplication is not satisfied that we may need to analyze the QVNNs with the help of RVNNs and CVNNs. So far, most researchers have applied three methods to the study of QVNNs: the first is changing the considered QVNNs into four equivalent RVNNs, that is real decomposition method [13]: the second is changing the considered QVNNs into two equivalent CVNNs, that is plural decomposition method [14]: and the last one is the method used in this article without decomposition [15]. Given the separation strategy of the first two methods and the non-separation strategy of the latter, it is not difficult to find that the separation method increases the difficulty of calculation and complexity of theoretical analysis. Therefore, considering the advantages of the non-separable method, our entire theoretical analysis for FTS of FOQVNNs is conducted under the inseparable method.

In the past few decades, with the rapid development of research in the field of algebra, many scholars have begun to study the synchronous behavior of FOQVNNs based on the Lyapunov function. Researchers in [16] discussed quasi-stability and quasi-synchronization control of FOQVNNs. Researchers in [17] studied FTS of FOQVNNs. However, the current research on FOQVNNs synchronization is mainly based on the discussion of the above synchronization types, there exists very few studies on FTS of time-varying delay FOQVNNs. Because there are many specific application scenarios of FTS, which is more reasonable and meaningful to study the FTS problem.

In order to complete the synchronization of FOQVNNs within a finite time, the convergence time is very important. In the literature [17] and [18], by establishing a fractional differential inequality as a criterion for FTS, the synchronization time is obtained and in [19], the authors establishes the fractional differential inequality through the period settling, such that the convergence time is more accurate. However, in practice, many dynamic behaviors exhibit aperiodic synchronization. To be closer to real life, we introduce a constant ϖ to set the control time flexibly. The constant ϖ is called the non-control rate and does not depend on each control interval and rest interval, which greatly reduces the synchronization convergence time. This is great importance for the study on FTS of QVNNs.

Encouraged by the above discussion, the main work of this paper is to explore the FTS problem of FOQVNNs under aperiodic intermittent control. The main contributions are listed as follow:

- At first, inspired by literature[18–20], a group of fractional differential inequality is established under non-periodic time conditions, which provide a shorter synchronization time, and time is more flexible and more in line with real life.
- Secondly, as a general form of symbolic functions for real and complex numbers, the quaternion symbolic function is introduced, and some properties of the quaternion symbolic function are established, which can prevent QVNNs from being decomposed into CVNNs or RVNNs and reduce computational complexity.
- Thirdly, a quaternion numerical state controller composed of symbolic functions and norms are flexibly designed, and based on the application of the Lyapunov function method in nonlinear dynamics, Lyapunov functions related to 1-norm are constructed.
- Finally, some new synchronization criteria are obtained by establishing fractional-order differential inequalities in aperiodic periods and fractional-order Razumikhin theorems.

Notation: In this paper, $\mathbb{N}, \mathbb{R}, \mathbb{Q}, \mathbb{Q}^n$ denote the set of natural numbers, the set of real numbers, the set of quaternion numbers, and n-dimensional quaternion space, respectively. For any quaternion $x = x^R + ix^I + jx^J + kx^K \in \mathbb{Q}$, where $x^R, x^I, x^J, x^K \in \mathbb{R}$, i, j, k are standard units, and obey Hamilton relus: $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. $\overline{x} = x^R - ix^I - jx^J - kx^K$ denote the conjugate of x. For $x \in \mathbb{Q}$, $|x|_1 = |x^R| + |x^I| + |x^J| + |x^K|$. For $x \in \mathbb{Q}^n$, $\|x\|_1 = \sum\limits_{r=1}^{n} |x_r|_1$.

2. Preliminaries

In this section, we introduce some essential basics of fractional-order calculus by recalling some definitions, lemmas, and establish two new lemmas.

Definition 1.([21]) The Caputo fractional-order derivative of the function $F(t)$ is given as

$$ {}_{t_0}^{c}D_t^{\wp}F(t) = \frac{1}{\Gamma(1-\wp)} \int_{t_0}^{t} (t-s)^{-\wp} F'(s) \mathrm{d}s, $$

in which $0 < \wp < 1$, $\Gamma(\wp)$ is the Gamma function defined by $\Gamma(\wp) = \int_0^{+\infty} e^{-t} t^{\wp-1} \mathrm{d}t$. For simplicity, use $D^{\wp}F(t)$ in place of ${}_{t_0}^{c}D_t^{\wp}F(t)$.

The time sequences $\{t_m\}$ and $\{s_m\}(m \in \mathbb{N})$ satisfy $0 = t_0 < s_0 < t_1 < s_1 \cdots < t_m < s_m < \cdots$, and $\lim_{m\to+\infty} t_m = \lim_{m\to+\infty} s_m = +\infty$. The control interval is $[t_m, s_m)$ and the rest interval is $[s_m, t_{m+1})$.

Definition 2.([22]) For the aperiodically intermittent control, we denote

$$ \varpi = \lim_{m\to\infty} \sup_{m\in\mathbb{N}} \frac{t_{m+1} - s_m}{t_{m+1} - t_m}, $$

where $0 \le \varpi < 1$, when $\varpi = 0$, the aperiodically intermittent control becomes continuous control.

Definition 3.([23]) For $x \in \mathbb{Q}$, the sign function of x is denoted

$$ [x] = sign(x^R) + isign(x^I) + jsign(x^J) + ksign(x^K). $$

Lemma 1.([24]) For any $m = 0, 1, 2, \cdots$, we denote

$$ \varpi(t) = \frac{t - s_m}{t - t_m}, \quad t \in [s_m, t_{m+1}). \tag{1} $$

Obviously, $\varpi(t)$ is a strictly increasing function and $\varpi(t) \le \frac{t_{m+1}-s_m}{t_{m+1}-t_m}$.

Lemma 2.([15]) Suppose function $X(t) \in \mathbb{Q}$ is a differentiable function, then

$$ D^{\wp}(\bar{X}(t)X(t)) \le D^{\wp}\bar{X}(t))X(t) + \bar{X}(t)D^{\wp}X(t), $$

where $0 < \wp < 1$, $t \ge 0$.

Lemma 3.([25]) For any $\sigma \in \mathbb{Q}$, the following equation holds

$$ [\bar{\sigma}]\sigma + [\sigma]\bar{\sigma} = 2|\sigma|_1. \tag{2} $$

Lemma 4.([19]) Suppose $\alpha \in (0, 1)$, $q \in \mathbb{R}$, the Caputo fractional-order derivative for function $F(t)$

has the following properties

$$D^{\alpha}F^{q}(t)=\frac{\Gamma(1+q)}{\Gamma(1+q-\alpha)}F^{q-\alpha}(t)D^{\alpha}F(t).$$

Lemma 5.([26]) Let $f(t)$ be a differentiable function, then one has

$$D^{\wp}|f(t)|\leq sign(f(t))D^{\wp}(f(t)),$$

where $\wp\in(0,1)$.

Lemma 6.([26]) For any $o\in\mathbb{Q}$, we can obtain

$$o+\bar{o}=2Re(o)\leq 2|o|_1. \tag{3}$$

Lemma 7.([27]) Let $0<\wp<1$, there exists function $m(t)\in\mathbb{Q}$, such that

$$D^{\wp}|m(t)|_1\leq\frac{1}{2}([\bar{m}(t)]D^{\wp}m(t)+[m(t)]D^{\wp}\bar{m}(t)).$$

Lemma 8.([28]) The following inequality holds

$$(\sum_{i=1}^{n}|z_i|)^{\wp}\leq\sum_{i=1}^{n}|z_i|^{\wp},$$

where $z_i\in\mathbb{Q}$, $\wp\in(0,1]$.

Lemma 9. Suppose that a continuous and non-negative function $\chi(t)$ satisfies the following conditions

$$\begin{cases} D^{\wp}\chi(t)\leq-\mu\chi^{\rho}(t), & t_m\leq t<s_m,\\ D^{\wp}\chi(t)\leq 0, & s_m\leq t<t_{m+1}, \end{cases} \tag{4}$$

where $m=0,1,2,\cdots$, $\mu>0$, $0<\rho<\wp$. If there are a constant $\varpi\in(0,1)$, and ϖ is introduced in Definition 2, then we can obtain inequality as below

$$\begin{cases} \chi^{\wp-\rho}(t)\leq\chi^{\wp-\rho}(t_0)-\frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(1-\varpi^{\wp})(t-t_0)^{\wp}, & t<\tilde{t},\\ \chi(t)=0, & t\geq\tilde{t}, \end{cases} \tag{5}$$

where $\tilde{t} = t_0 + \left(\frac{\chi^{\wp-\rho}(t_0)\Gamma(1-\rho)\Gamma(1+\wp)}{\mu\Gamma(1+\wp-\rho)\Gamma(1-\varpi^{\wp})}\right)^{\frac{1}{\wp}}$.

Proof. Take $H_0 = \chi^{\wp-\rho}(t_0)$ and $\aleph(t) = \chi^{\wp-\rho}(t) + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(t-t_0)^{\wp}$, $t \geq t_0 = 0$. Set $\Re(t) = \aleph(t) - H_0$. Obviously, it gets

$$\Re(t) = 0, \text{ for } t = 0. \tag{6}$$

Next, $\Re(t) \leq 0$, for $t \in [0, s_0)$ will be proved.

When $t \in [0, s_0)$, it has

$$\begin{aligned}
D^{\wp}\Re(t) &= D^{\wp}[\chi^{\wp-\rho}(t) + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(t-t_0)^{\wp} - H_0] \\
&= \frac{\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)}\chi^{-\rho}(t)D^{\wp}\chi(t) + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}D^{\wp}[(t-t_0)^{\wp}] \\
&\leq -\mu\frac{\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)} + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\frac{\Gamma(1+\wp)}{\Gamma(1)} \\
&= 0.
\end{aligned} \tag{7}$$

It is east to see that $\Re(t)$ is non-increasing in $[0, s_0)$. Hence, $\Re(t) \leq \Re(0) = 0$, for $t \in [0, s_0)$.

Assume $Q_1(t) = \aleph(t) - H_0 - \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(t-s_0)^{\wp}$. In the following steps, $t \in [s_0, t_1)$, $Q_1(t) \leq 0$ will be proved.

When $\forall\ t \in [s_0, t_1)$, it yields

$$\begin{aligned}
D^{\wp}Q_1(t) &= D^{\wp}[\aleph(t) - H_0 - \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(t-s_0)^{\wp}] \\
&= \frac{\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)}\chi^{-\rho}(t)D^{\wp}(t) + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}D^{\wp}[(t-t_0)^{\wp}] \\
&\quad - \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}D^{\wp}[(t-s_0)^{\wp}] \\
&\leq \mu\frac{\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)} - \mu\frac{\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\frac{\Gamma(1+\wp)}{\Gamma(1)} \\
&= 0.
\end{aligned} \tag{8}$$

That is, the function $Q(t)$ is non-increasing over the interval $[s_0, t_1)$. Hence, $Q_1(t) \leq Q_1(s_0) = \Re(s_0) \leq 0$.

Combining $Q_1(t) \leq 0$, for $t \in [s_0, t_1)$, it has

$$\begin{aligned}
\aleph(t) &\leq H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(t-s_0)^{\wp} \\
&\leq H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(t_1-s_0)^{\wp}.
\end{aligned} \tag{9}$$

Then, we have

$$\aleph(t) \leq H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(t_1 - s_0)^{\wp}, \text{ for all } t \in [0, t_1).$$

Similar to the (7), for $t \in [t_1, s_1)$, we can prove that

$$\aleph(t) \leq H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(t_1 - s_0)^{\wp}.$$

And for any $t \in [s_1, t_2)$, similar to the proof of (8), taking $Q_2(t) = \aleph(t) - H_0 - \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(t_1 - s_0)^{\wp} - \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(t - s_1)^{\wp}$, we can prove $Q_2(t) \leq Q_2(s_1) \leq 0$. Denote $\tilde{Q}_2(t) = \aleph(t) - H_0 - \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}[(t - s_0) + (t - s_1)]^{\wp}$. According to the Lemma 8, it has $Q_2(t) \leq \tilde{Q}_2(t)$, which has similar proof process as (8), for any $t \in [s_1, t_2)$, $\tilde{Q}_2(t) \leq 0$ holds that is

$$\aleph(t) \leq H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}[(t_1 - s_0) + (t - s_1)]^{\wp}.$$

By induction, we can derive the following estimation of $\aleph(t)$ for any integer m.

When $t_m \leq t < s_m$, it has

$$\aleph(t) \leq H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\Big[\sum_{k=1}^{m}(t_k - s_{k-1})\Big]^{\wp}, \tag{10}$$

and for $s_m \leq t < t_{m+1}$, we obtain

$$\aleph(t) \leq H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\Big\{\sum_{k=1}^{m}[(t_k - s_{k-1}) + (t - s_m)]\Big\}^{\wp}. \tag{11}$$

Therefore, for any $t \geq 0$, there are a natural number l, such that $t_l \leq t < s_l$, the following estimates of $\aleph(t)$ for any t by (10) and (11), for $t_l \le t < s_l$, we can get

$$\begin{aligned}
\aleph(t) &\le H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\Big[\sum_{k=1}^{l}(t_k - s_{k-1})\Big]^{\wp} \\
&= H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\Big[\sum_{k=1}^{l}\frac{t_k - s_{k-1}}{t_k - t_{k-1}}(t_k - t_{k-1})\Big]^{\wp} \\
&\le H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\Big[\varpi\sum_{k=1}^{l}(t_k - t_{k-1})\Big]^{\wp} \\
&= H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(\varpi t_l)^{\wp} \\
&\le H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\varpi^{\wp} t^{\wp}.
\end{aligned} \tag{12}$$

And for $s_l \le t \le t_{l+1}$, it has

$$\begin{aligned}
\aleph(t) &\le H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\Big[\sum_{k=1}^{l}(t_k - s_{k-1}) + (t - s_l)\Big]^{\wp} \\
&= H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\Big[\sum_{k=1}^{l}\frac{t_k - s_{k-1}}{t_k - t_{k-1}}(t_k - t_{k-1}) + \frac{t - s_l}{t - t_l}(t - t_l)\Big]^{\wp} \\
&\le H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\Big[\sum_{k=1}^{l}\frac{t_k - s_{k-1}}{t_k - t_{k-1}}(t_k - t_{k-1}) + \frac{t_{l+1} - s_l}{t_{l+1} - t_l}(t - t_l)\Big]^{\wp} \\
&\le H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\Big\{\varpi[\sum_{k=1}^{l}(t_k - t_{k-1}) + (t - t_l)]\Big\}^{\wp} \\
&= H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(\varpi t)^{\wp} \\
&= H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\varpi^{\wp} t^{\wp}.
\end{aligned} \tag{13}$$

Thus, we can get

$$\begin{aligned}
\chi^{\wp-\rho}(t) &\le H_0 + \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}\varpi^{\wp} t^{\wp} - \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(t - t_0)^{\wp} \\
&\le \chi^{\wp-\rho}(t_0) - \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(1 - \varpi^{\wp})(t - t_0)^{\wp}.
\end{aligned} \tag{14}$$

Denote $f(t) = \chi^{\wp-\rho}(t_0) - \frac{\mu\Gamma(1+\wp-\rho)}{\Gamma(1-\rho)\Gamma(1+\wp)}(1 - \varpi^{\wp})(t - t_0)^{\wp}$, function $\Gamma(x) > 0$ $(x > 0)$, $1 - \varpi^{\wp} > 0$, $f(t)$ is a continuous function with respect to t which is strictly decreasing.

For $f(t) = 0$, if and only if

$$\tilde{t} = t_0 + \big(\frac{\chi^{\wp-\rho}(t_0)\Gamma(1-\rho)\Gamma(1+\wp)}{\mu\Gamma(1+\wp-\rho)\Gamma(1-\varpi^{\wp})}\big)^{\frac{1}{\wp}}, t \geq 0.$$

The above Lemma 9 proof is completed.

Remark 1. In [18], the authors gave a fractional-order differential inequality in Lemma 10 for ensuring the FTS of neural networks, and gave the settling time $t = t_0 + (\frac{\alpha}{\lambda}\chi^{\alpha-\gamma}(t_0)B(\alpha, 1-\gamma))^{\frac{1}{\alpha}}$. In [17], the researchers studied the inequality $D^{\alpha}\chi(t) \leq -\rho V^{\beta}(t)$ with the initial time $t_0 = 0$ and obtained the settling time $t = (\frac{\chi^{\alpha-\beta}(0)\Gamma(1-\alpha)\Gamma(1-\beta)}{\rho\Gamma(1+\alpha-\beta)})^{\frac{1}{\alpha}}$. In fact, the faster the convergence time becomes, the more helpful it becomes for our actual life. In Lemma 9 above, we establish two sets of fractional differential inequalities in each intermittent times and get the settling time $\tilde{t} = t_0 + \big(\frac{\chi^{\wp-\rho}(t_0)\Gamma(1-\rho)\Gamma(1+\wp)}{\mu\Gamma(1+\wp-\rho)\Gamma(1-\varpi^{\wp})}\big)^{\frac{1}{\wp}}$. Obviously, the introduced parameter ϖ can make the settling time $\tilde{t}$ decreases. In addition, it should be noted that Lemma 9 plays a significant role in the theorem below. Through the proposed aperiodically intermittent control, the condition of FTS of the neural network is guaranteed, and an accurate settling time is obtained.

Lemma 10 For $x \in \mathbb{Q}$, $0 < m < \wp < 1$, we have

(i) : $[\bar{x}][x] = sign^2(x^R) + sign^2(x^I) + sign^2(x^J) + sign^2(x^K) = |[x]|_1$,

(ii) : $|x|_1^m[\bar{x}][x] = |x|_1^m|[x]|_1 \geq |x|_1^m$.

Proof.

$$\begin{aligned}[x][\bar{x}] &= (sign(x^R) + isign(x^I) + jsign(x^J) + ksign(x^K)) \\ &\quad \times (sign(x^R) - isign(x^I) - jsign(x^J) - ksign(x^K)) \\ &= |sign(x^R)| + |sign(x^I)| + |sign(x^J)| + |sign(x^K)| \\ &= |[x]|_1.\end{aligned} \tag{15}$$

If $x = 0$, it can be obtained $|x|_1^m|[x]|_1 = 0 = |x|_1^m$, else $x \neq 0$, it can be got $|x|_1^m|[x]|_1 \geq |x|_1^m$. Therefore, for any $x \in \mathbb{Q}$, $|x|_1^m|[x]|_1 \geq |x|_1^m$.

Remark 2. Definition 3 provides a definition of a quaternion function $[x] = sign(x^R) + isign(x^I) + jsign(x^J) + ksign(x^K)$, and based on this definition, we obtain some related properties in above Lemma 10, which are an important factor in ensureing the FTS of FOQVNNs.

3. System description

In this section, the model of FOQVNNs is considered and we establish a response system to investigate the synchronization.

The drive-response systems studied in this paper are given as

$$\begin{aligned} D^{\wp}\imath_r(t) &= -m_r\imath_r(t) + \sum_{q=1}^{N} a_{rq}f_q(\imath_q(t)) + \sum_{q=1}^{N} b_{rq}g_q(\imath_q(t-\tau(t))) + \ell_r. \\ D^{\wp}\jmath_r(t) &= -m_r\jmath_r(t) + \sum_{q=1}^{N} a_{rq}f_q(\jmath_q(t)) + \sum_{q=1}^{N} b_{rq}g_q(\jmath_q(t-\tau(t))) + \ell_r + \Im_r(t). \end{aligned} \tag{16}$$

in which $\wp \in (0,1)$, state vectors $\imath_r(t), \jmath_r(t), \imath_q(t), \jmath_q(t), \imath_q(t-\tau(t)), \jmath_q(t-\tau(t)) \in \mathbb{Q}$, $m_r \in \mathbb{R}$ denote the self-feedback coeffcient, $a_{rq}, b_{rq} \in \mathbb{Q}$ is connection weights, $\tau(t)$ is the time-varying delay, ℓ_r represents the external input, $\Im_r(t)$ is the controller, $f_q(\cdot)$ and $g_q(\cdot) : \mathbb{Q} \longrightarrow \mathbb{Q}$ is the activation function.

Definition 4. If $\lim_{t\to T}\|\jmath_r(t) - \imath_r(t)\| = 0$, that is $\|\jmath_r(t) - \imath_r(t)\| = 0$ for any $t \geq T$ are satisfied, then the drive-response systems (16) can achieve FTS.

Assumption 1. For any $r = \{1, 2, \cdots, n\}$, $k = 1, 2.$ and $\imath, \jmath \in \mathbb{Q}$, there exist positive constants L_{kr}, P_{kr}, such that

$$\begin{aligned} |f_r(\imath) - f_r(\jmath)|_k &\leq L_{kr}|\imath - \jmath|_k, \\ |g_r(\imath) - g_r(\jmath)|_k &\leq P_{kr}|\imath - \jmath|_k. \end{aligned} \tag{17}$$

In order to synchronize the drive-response system (16), satisfying definition 4 and assumption 1, we can get an error system:

$$D^{\wp}(\Xi_r(t)) = -m_r\Xi_r(t) + \sum_{q=1}^{N} a_{rq}\tilde{f}_q(\Xi_q(t)) + \sum_{q=1}^{N} b_{rq}\tilde{g}_q(\Xi_q(t-\tau(t))) + \Im_r(t), \tag{18}$$

in which $\Xi_r(t) = \jmath_r(t) - \imath_r(t)$, $\tilde{f}_q(\Xi_q(t)) = f_q(\jmath_q(t)) - f_q(\imath_q(t))$, $\tilde{g}_q(\Xi_q(t-\tau(t))) = g_q(\jmath_q(t-\tau(t))) - g_q(\imath_q(t-\tau(t)))$.

4.Main results

In this part, a quaternion numerical aperiodically intermittent controller composed of symbolic functions and norms are studied. Among them, 1-norm are composed of the real part of the quaternion and the three imaginary parts, respectively. In the theorems 1, we establish the Lyapunov function to simplify the controller. Through the condition of lemma 9, the FTS of FOQVNNs is realized.

Firstly, consider the 1-norm controller

$$\begin{cases} \Im_r(t) = -n_r\Xi_r(t) - \lambda[\Xi_r(t)]|\Xi_r(t)|_1^\beta, & t_m \le t < s_m, \\ \Im_r(t) = 0, & s_m \le t < t_{m+1}, \end{cases} \tag{19}$$

in which $n_r \in \mathbb{Q}$, $n_r^R > 0$, $\lambda > 0$, $0 < \beta < \wp$.

Based on derive-response systems (16) and the controller (19), the error system is

$$D^\wp(\Xi_r(t)) = -m_r\Xi_r(t) + \sum_{q=1}^{N} a_{rq}\tilde{f}_q(\Xi_q(t)) + \sum_{q=1}^{N} b_{rq}\tilde{g}_q(\Xi_q(t-\tau(t))) - n_r\Xi_r(t) - \lambda[\Xi_r(t)]|\Xi_r(t)|_1^\beta.$$

In order to simplify of the calculation process, the following denotations are introduced

$$d_r = m_r + n_r. \tag{20}$$

Theorem 1. Under the controller (19), and Assumption 1 holds, if there exist positive constants d_r^R, L_{1r}, P_{1r}, η, ϑ, $|a_{qr}|_1$, $|b_{qr}|_1$, $|d_r^I|$, $|d_r^J|$, $|d_r^R|$ satisfying following conditions:
(i)$-d_r^R + |d_r^I| + |d_r^J| + |d_r^K| + \sum_{q=1}^{N} L_{1r}|a_{qr}|_1 + \sum_{q=1}^{N} P_{1r}|b_{qr}|_1\eta < 0$,
(ii)$-m_r^R + |m_r^I| + |m_r^J| + |m_r^K| + \sum_{q=1}^{N} L_{1r}|a_{qr}|_1 + \sum_{q=1}^{N} P_{1r}|b_{qr}|_1\vartheta < 0$,
then the drive-response systems (16) with aperiodically intermittent controllers (19) will be synchronized in finite-time, moreover the settling time of synchronization T_1 satifies

$$T_1 = t_0 + \Big(\frac{\chi_1^{\wp-\beta}(t_0)\Gamma(1-\beta)\Gamma(1+\wp)}{\lambda\Gamma(1+\wp-\beta)\Gamma(1-\varpi^\beta)}\Big)^{\frac{1}{\wp}},$$

where $\chi_1(t_0) = \sum_{r=1}^{N} |\Xi_r(t_0)|_1$, $\varpi \in (0,1)$, ϖ is defined in Definition 2.

Proof. Consider the Lyapunov function below:

$$\chi_1(t) = \sum_{r=1}^{N} |\Xi_r(t)|_1,$$

when $t_m \le t < s_m$, it has

$$\begin{aligned} D^\wp\chi_1(t) &\le \frac{1}{2}\sum_{r=1}^{N}([\bar{\Xi}_r(t)]D^\wp\Xi_r(t) + D^\wp\bar{\Xi}_r(t)[\Xi_r(t)]) \\ &= \frac{1}{2}\sum_{r=1}^{N}\Big[\big\{[\bar{\Xi}_r(t)](-m_r\Xi_r(t) + \sum_{q=1}^{N} a_{rq}\tilde{f}_q(\Xi_q(t)) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{q=1}^{N} b_{rq}\tilde{g}_q(\Xi_q(t-\tau(t))) - n_r\Xi_r(t) - \lambda[\Xi_r(t)]|\Xi_r(t)|_1^{\beta})\big\} \\
&+\big\{(-\bar{m}_r\bar{\Xi}_r(t) + \sum_{q=1}^{N}\bar{a}_{rq}\bar{\tilde{f}}_q(\Xi_q(t)) \\
&+\sum_{q=1}^{N}\bar{b}_{rq}\bar{\tilde{g}}_q(\Xi_q(t-\tau(t))) - \bar{n}_r\bar{\Xi}_r(t) - \lambda[\bar{\Xi}_r(t)]|\bar{\Xi}_r(t)|_1^{\beta})[\Xi_r(t)]\big\}\Big] \\
=&-\frac{1}{2}\sum_{r=1}^{N}\big\{(m_r+n_r)[\bar{\Xi}_r(t)]\Xi_r(t) + (\bar{m}_r+\bar{n}_r)\bar{\Xi}_r(t)[\Xi_r(t)]\big\} \\
&+\frac{1}{2}\sum_{r=1}^{N}\sum_{q=1}^{N}\big\{a_{rq}\tilde{f}_q(\Xi_q(t))[\bar{\Xi}_r(t)] + \bar{a}_{rq}\bar{\tilde{f}}_q(\Xi_q(t))[\Xi_r(t)]\big\} \\
&+\frac{1}{2}\sum_{r=1}^{N}\sum_{q=1}^{N}\big\{b_{rq}[\bar{\Xi}_r(t)]\tilde{g}_q(\Xi_q(t-\tau(t))) + \bar{b}_{rq}[\Xi_r(t)]\bar{\tilde{g}}_q(\Xi_q(t-\tau(t)))\big\} \\
&-\frac{\lambda}{2}\sum_{r=1}^{N}\big\{([\bar{\Xi}_r(t)][\Xi_r(t)] + [\bar{\Xi}_r(t)][\Xi_r(t)])|\Xi_r(t)|_1^{\beta}\big\}.
\end{aligned} \tag{21}
$$

Using Lemma 5 and Lemma 6, we have

$$
\begin{aligned}
&-\frac{1}{2}\sum_{r=1}^{N}\big\{(m_r+n_r)[\bar{\Xi}_r(t)]\Xi_r(t) + (\bar{m}_r+\bar{n}_r)\bar{\Xi}_r(t)[\Xi_r(t)]\big\} \\
=&-\frac{1}{2}\sum_{r=1}^{N}\big\{d_r[\bar{\Xi}_r(t)]\Xi_r(t) + \bar{d}_r\bar{\Xi}_r(t)[\Xi_r(t)]\big\} \\
=&-\sum_{r=1}^{N} Re\big\{d_r[\bar{\Xi}_r(t)]\Xi_r(t)\big\} \\
=&-\sum_{r=1}^{N} Re\big\{(d_r^R + id_r^I + jd_r^J + kd_r^K)(sign(\Xi_r^R(t)) \\
&-isign(\Xi_r^I(t)) - jsign(\Xi_r^J(t)) - ksign(\Xi_r^K(t))) \\
&\times(\Xi_r^R(t)) + i\Xi_r^I(t) + j\Xi_r^J(t) + k\Xi_r^K(t))\big\} \\
\leq&\sum_{r=1}^{N}\big\{(-d_r^R + |d_r^I| + |d_r^J| + |d_r^K|)|\Xi_r(t)|_1\big\}.
\end{aligned} \tag{22}
$$

Using Lemma 6 and Assumption 1, it follows that

$$
\frac{1}{2}\sum_{r=1}^{N}\sum_{q=1}^{N}\big\{a_{rq}\tilde{f}_q(\Xi_q(t))[\bar{\Xi}_r(t)] + \bar{a}_{rq}\bar{\tilde{f}}_q(\Xi_q(t))[\Xi_r(t)]\big\}
$$

$$
\begin{aligned}
&= \sum_{r=1}^{N}\sum_{q=1}^{N} Re\big\{a_{rq}\tilde{f}_q(\Xi_q(t))[\bar{\Xi}_r(t)]\big\} \\
&= \sum_{r=1}^{N}\sum_{q=1}^{N} Re\big\{(a_{rq}^R + ia_{rq}^I + ja_{rq}^J + ka_{rq}^K)(\tilde{f}_q^R(\Xi_q(t)) + i\tilde{f}_q^I(\Xi_q(t)) + j\tilde{f}_q^J(\Xi_q(t)) \\
&\quad + k\tilde{f}_q^K(\Xi_q(t))) \times (sign(\Xi_r^R(t)) - isign(\Xi_r^I(t)) - jsign(\Xi_r^J(t)) - ksign(\Xi_r^K(t)))\big\} \\
&\le \sum_{r=1}^{N}\sum_{q=1}^{N} \big\{|a_{rq}^R||\tilde{f}_q(\Xi_q(t))|_1 + |a_{rq}^I||\tilde{f}_q(\Xi_q(t))|_1 + |a_{rq}^J||\tilde{f}_q(\Xi_q(t))|_1 + |a_{rq}^K||\tilde{f}_q(\Xi_q(t))|_1\big\} \\
&\le \sum_{r=1}^{N}\sum_{q=1}^{N} |a_{rq}|_1|\tilde{f}_q(\Xi_q(t))|_1 \\
&\le \sum_{r=1}^{N}\sum_{q=1}^{N} L_{1r}|a_{qr}|_1|\Xi_r(t)|_1.
\end{aligned} \tag{23}
$$

Similar to inequality (23), using Lemma 6 and Assumption 1, we can obtain

$$
\begin{aligned}
&\frac{1}{2}\sum_{r=1}^{N}\sum_{q=1}^{N} \big\{b_{rq}[\bar{\Xi}_r(t)]\tilde{g}_q(\Xi_q(t-\tau(t))) + \bar{b}_{rq}[\Xi_r(t)]\bar{\tilde{g}}_q(\Xi_q(t-\tau(t)))\big\} \\
&= \sum_{r=1}^{N}\sum_{q=1}^{N} Re\big\{b_{rq}[\bar{\Xi}_r(t)]\tilde{g}_q(\Xi_q(t-\tau(t)))\big\} \\
&\le \sum_{r=1}^{N}\sum_{q=1}^{N} |b_{rq}|_1|\tilde{g}_q(\Xi_q(t-\tau(t)))|_1 \\
&\le \sum_{r=1}^{N}\sum_{q=1}^{N} P_{1r}|b_{qr}|_1|\Xi_r(t-\tau(t))|_1.
\end{aligned} \tag{24}
$$

Using Lemma 6, Lemma 8 and Lemma 10. it has

$$
\begin{aligned}
&-\frac{\lambda}{2}\sum_{r=1}^{N} \big\{([\bar{\Xi}_r(t)][\Xi_r(t)] + [\bar{\Xi}_r(t)][\Xi_r(t)])|\Xi_r(t)|_1^{\beta}\big\} \\
&= -\lambda\sum_{r=1}^{N} \big\{([\bar{\Xi}_r(t)][\Xi_r(t)])|\Xi_r(t)|_1^{\beta}\big\} \\
&\le -\lambda\sum_{r=1}^{N} |\Xi_r(t)|_1^{\beta} \\
&\le -\lambda(\sum_{r=1}^{N} |\Xi_r(t)|_1)^{\beta}.
\end{aligned} \tag{25}
$$

According to the above

$$\begin{aligned} D^{\wp}\chi_1(t) \le & \sum_{r=1}^{N}(-d_r^R + |d_r^I| + |d_r^J| + |d_r^K|)|\Xi_r(t)|_1 + \sum_{r=1}^{N}\sum_{q=1}^{N} L_{1r}|a_{qr}|_1|\Xi_r(t)|_1 \\ & + \sum_{r=1}^{N}\sum_{q=1}^{N} P_{1r}|b_{qr}|_1|\Xi_r(t-\tau(t))|_1 - \lambda(\sum_{r=1}^{N}|\Xi_r(t)|_1)^{\beta} \\ & = (-d_r^R + |d_r^I| + |d_r^J| + |d_r^K|)\chi_1(t) + \sum_{q=1}^{N} L_{1r}|a_{qr}|_1\chi_1(t) \\ & + \sum_{q=1}^{N} P_{1r}|b_{qr}|_1\chi_1(t-\tau(t)) - \lambda\chi_1^{\beta}(t). \end{aligned} \tag{26}$$

By the fractional-order Razumikhin theorem ([29]) and combining the condition (i), there is a $\eta > 0$, such that $\chi_1(t-\tau(t)) < \eta\chi_1(t)$, it yields

$$D^{\wp}\chi_1(t) \le (-d_r^R + |d_r^I| + |d_r^J| + |d_r^K| + \sum_{q=1}^{N} L_{1r}|a_{qr}|_1 + \sum_{q=1}^{N} P_{1r}|b_{qr}|_1\eta)\chi_1(t) - \lambda\chi_1^{\beta}(t)$$

thus,

$$D^{\wp}\chi_1(t) \le -\lambda\chi_1^{\beta}(t). \tag{27}$$

when $s_m \le t < t_{m+1}$, it has

$$\begin{aligned} D^{\wp}\chi_1(t) \le & \frac{1}{2}\sum_{r=1}^{N}([\bar{\Xi}_r(t)]D^{\wp}\Xi_r(t) + D^{\wp}\bar{\Xi}_r(t)[\Xi_r(t)]) \\ & = \frac{1}{2}\sum_{r=1}^{N}\Big[\big\{[\bar{\Xi}_r(t)](-m_r\Xi_r(t) + \sum_{q=1}^{N} a_{rq}\tilde{f}_q(\Xi_q(t)) \\ & + \sum_{q=1}^{N} b_{rq}\tilde{g}_q(\Xi_q(t-\tau(t)))\big\} + \big\{(-\bar{m}_r\bar{\Xi}_r(t) + \sum_{q=1}^{N}\bar{a}_{rq}\bar{\tilde{f}}_q(\Xi_q(t)) \\ & + \sum_{q=1}^{N}\bar{b}_{rq}\bar{\tilde{g}}_q(\Xi_q(t-\tau(t)))\big\}[\Xi_r(t)]\Big] \\ & = -\frac{1}{2}\sum_{r=1}^{N}\big\{m_r[\bar{\Xi}_r(t)]\Xi_r(t) + \bar{m}_r\bar{\Xi}_r(t)[\Xi_r(t)]\big\} \\ & + \frac{1}{2}\sum_{r=1}^{N}\sum_{q=1}^{N}\big\{a_{rq}\tilde{f}_q(\Xi_q(t))[\bar{\Xi}_r(t)] + \bar{a}_{rq}\bar{\tilde{f}}_q(\Xi_q(t))[\Xi_r(t)]\big\} \\ & + \frac{1}{2}\sum_{r=1}^{N}\sum_{q=1}^{N}\big\{b_{rq}[\bar{\Xi}_r(t)]\tilde{g}_q(\Xi_q(t-\tau(t))) + \bar{b}_{rq}[\Xi_r(t)]\bar{\tilde{g}}_q(\Xi_q(t-\tau(t)))\big\}. \end{aligned} \tag{28}$$

Similar to the inequality (22), it has

$$
\begin{aligned}
&-\frac{1}{2}\sum_{r=1}^{N}\big\{m_r[\bar{\Xi}_r(t)]\Xi_r(t)+\bar{m}_r\bar{\Xi}_r(t)[\Xi_r(t)]\big\}\\
=&-\sum_{r=1}^{N}Re\big\{m_r[\bar{\Xi}_r(t)]\Xi_r(t)\big\}\\
=&-\sum_{r=1}^{N}Re\big\{(m_r^R+im_r^I+jm_r^J+km_r^K)(sign(\Xi_r^R(t))\\
&-isign(\Xi_r^I(t))-jsign(\Xi_r^J(t))-ksign(\Xi_r^K(t)))\\
&\times(\Xi_r^R(t)+i\Xi_r^I(t)+\Xi_r^J(t)+k\Xi_r^K(t))\big\}\\
\le&\sum_{r=1}^{N}\big\{(-m_r^R+|m_r^I|+|m_r^J|+|m_r^K|)|\Xi_r(t)|_1\big\}.
\end{aligned}
\tag{29}
$$

Similar to the inequality (26), it can be seen that

$$
\begin{aligned}
D^{\wp}\chi_1(t)\le&\sum_{r=1}^{N}(-m_r^R+|m_r^I|+|m_r^J|+|m_r^K|)|\Xi_r(t)|_1+\sum_{r=1}^{N}\sum_{q=1}^{N}L_{1r}|a_{qr}|_1|\Xi_r(t)|_1\\
&+\sum_{r=1}^{N}\sum_{q=1}^{N}P_{1r}|b_{qr}|_1|\Xi_r(t-\tau(t))|_1\\
=&(-m_r^R+|m_r^I|+|m_r^J|+|m_r^K|)\chi_1(t)+\sum_{q=1}^{N}L_{1r}|a_{qr}|_1\chi_1(t)\\
&+\sum_{q=1}^{N}P_{1r}|b_{qr}|_1\chi_1(t-\tau(t)).
\end{aligned}
\tag{30}
$$

By the fractional-order Razumikhin theorem ([29]) and combining the condition (ii), there is a $\vartheta>0$, such that $\chi_1(t-\tau(t))<\vartheta\chi_1(t)$, we can get

$$
D^{\wp}\chi_1(t)\le(-m_r^R+|m_r^I|+|m_r^J|+|m_r^K|+\sum_{q=1}^{N}L_{1r}|a_{qr}|_1+\sum_{q=1}^{N}P_{1r}|b_{qr}|_1\vartheta)\chi_1(t).
$$

therefore

$$
D^{\wp}\chi_1(t)\le 0. \tag{31}
$$

From Lemma 9, the drive-response systems (16) can achieve FTS and the settling time T_1 is given by

$$T_1 = t_0 + \Big(\frac{\chi_1^{\wp-\beta}(t_0)\Gamma(1-\beta)\Gamma(1+\wp)}{\lambda\Gamma(1+\wp-\beta)\Gamma(1-\varpi^{\beta})}\Big)^{\frac{1}{\wp}}.$$

This proof is completed.

Remark 3. According to the designed controllers (19), we can see that they contain 1-norm $|\Xi_r(t)|_1$ terms. Therefore, based on the characteristics of the controllers containing 1-norm, the selected Lyapunov functions in Theorems 1 are also presented in the form of 1-norm. Meanwhile according to the sign function properties of the defined complex function, some synchronization criteria are obtained to ensure FTS of FOQVNNs.

5.Numerical simulations

In this section, some simulation examples are used to validate the effectiveness of our derived theoretical results.

Example 1. Consider the drive-response FOQVNNs with two neurons in the following forms:

$$D^{\wp}\imath_r(t) = -m_r\imath_r(t) + \sum_{q=1}^{2} a_{rq}f_q(\imath_q(t)) + \sum_{q=1}^{2} b_{rq}g_q(\imath_q(t-\tau(t))) + \ell_r, \tag{32}$$

$$D^{\wp}\jmath_r(t) = -m_r\jmath_r(t) + \sum_{q=1}^{2} a_{rq}f_q(\jmath_q(t)) + \sum_{q=1}^{2} b_{rq}g_q(\jmath_q(t-\tau(t))) + \ell_r + \Im_r(t), \tag{33}$$

where $\wp = 0.98$, $r = 1,2$, $\imath_r(t) = \imath_r^R(t) + i\imath_r^I(t) + j\imath_r^J(t) + k\imath_r^K(t)$, $\imath_q^R, \imath_q^I, \imath_q^J, \imath_q^K \in \mathbb{R}$, $f_q(\imath_q(t)) = sin(\imath_q^R(t)) + isin(\imath_q^I(t)) + jsin(\imath_q^J(t)) + ksin(\imath_q^K(t))$ and $g_q(\imath_q(t)) = tanh(\imath_q^R(t)) + itanh(\imath_q^I(t)) + jtanh(\imath_q^J(t)) + ktanh(\imath_q^K(t))$, $\tau(t) = 0.6|sin(t)|$, $\Im_r(t) = -n_r\Xi_r(t) - \lambda[\Xi_r(t)]|\Xi_r(t)|_1^{\beta}$ and

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix},$$

$$N = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} = \begin{pmatrix} 5-i-j-k & 0 \\ 0 & 5-i-j-k \end{pmatrix},$$

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} 10-i-j-k & 0 \\ 0 & 10-i-j-k \end{pmatrix},$$

$$A = \begin{pmatrix} 0.3 + 0.3i + 0.3j + 0.1k & 0.2 + 0.3i + 0.1j + 0.3k \\ 0.3 + 0.3j + 0.2k & 0.1 + 0.1i + 0.2j \end{pmatrix},$$

$$B = \begin{pmatrix} 0.3 + 0.3i + 0.3j + 0.3k & 0.2 + 0.1i - 0.1j \\ 0.3 + 0.1i - 0.1k & 0.1 + 0.3i + 0.3j + 0.3k \end{pmatrix}.$$

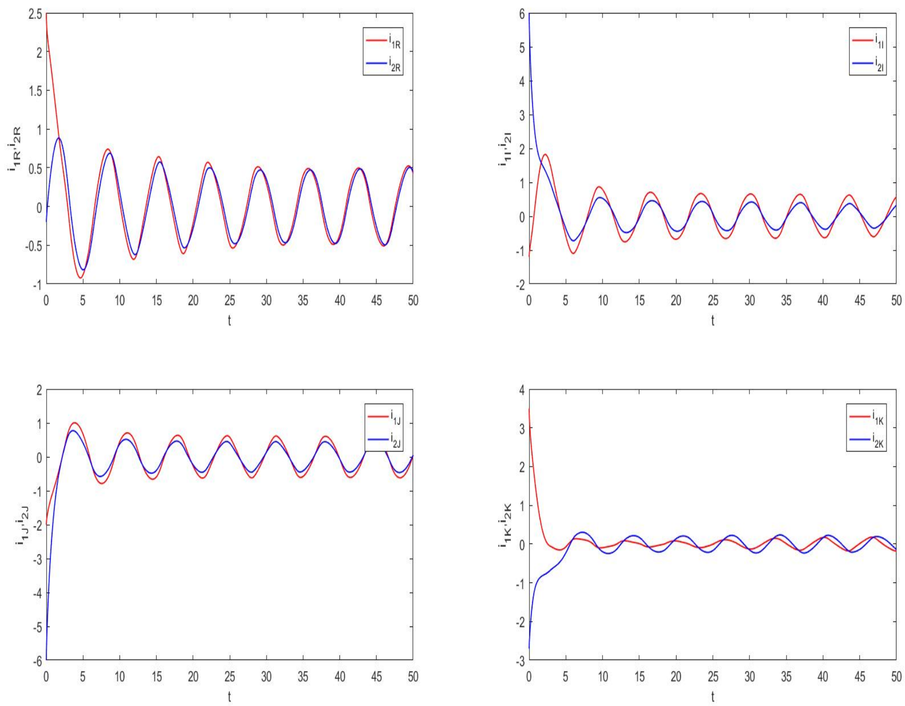

Figure 1: Transient states of R, I, J, K part of derive systems (32) with 2-neurons.

Let the initial values: $\imath_1(0) = 2.5 - 1.2i - 2j + 3.5k$, $\imath_2(0) = -0.2 + 6i - 6j - 2.7k$, the external input $\omega_1 = \omega_2 = 0$, then the four part trajectories of the drive system (32) are depicted in Figure 1. Obviously, the four part trajectories do not converge. In order to verify the validity of Theorem 1 with controller $\Im_r(t) = -n_r \Xi_r(t) - \lambda[\Xi_r(t)]|\Xi_r(t)|_1^{\beta}$, we perform the following steps. Firstly, let $L_{11} = L_{12} = P_{11} = P_{12} = 1$, $\eta = \vartheta = 1$. They can satisfy the condition of Theorem 1 that $-d_r^R + |d_r^I| + |d_r^J| + |d_r^K| + \sum_{q=1}^{2} L_{1r}|a_{qr}|_1 + \sum_{q=1}^{2} P_{1r}|b_{qr}|_1 \eta < 0$, $-m_r^R + |m_r^I| + |m_r^J| + |m_r^K| + \sum_{q=1}^{2} L_{1r}|a_{qr}|_1 + \sum_{q=1}^{2} P_{1r}|b_{qr}|_1 \vartheta < 0$. Choose $\lambda = 0.9$, $\beta = 0.5$. Secondly, set the initial values: $\jmath_1(0) = 0.2 - 0.7i - 8j + 8.5k$, $\jmath_2(0) = 4.1 + 0.9i + j - 3.3k$, $\omega_1 = \omega_2 = 0$, the time response of the four parts of (33) are depicted in Figure 2.

Thereupon, we can get that the initial values of the error system are $\Xi_1(0) = -2.3 + 0.5i - 6j + 5k$, $\Xi_2(0) = 4.3 - 5.1i + 7j - 0.5k$, the period width $t_{m+1} - t_m$ is randomly generated between 0.4s and 0.6s, and the ratio of the control width $s_m - t_m (m > 0)$ is randomly generated between 0.3 and 0.7, $\varpi = 0.7$. Thirdly, according to Theorem 1, the drive system (32) and response system (33) can reach FTS at $\tilde{t} = 2.0352$, and their error system trajectories are shown in Figure 3.

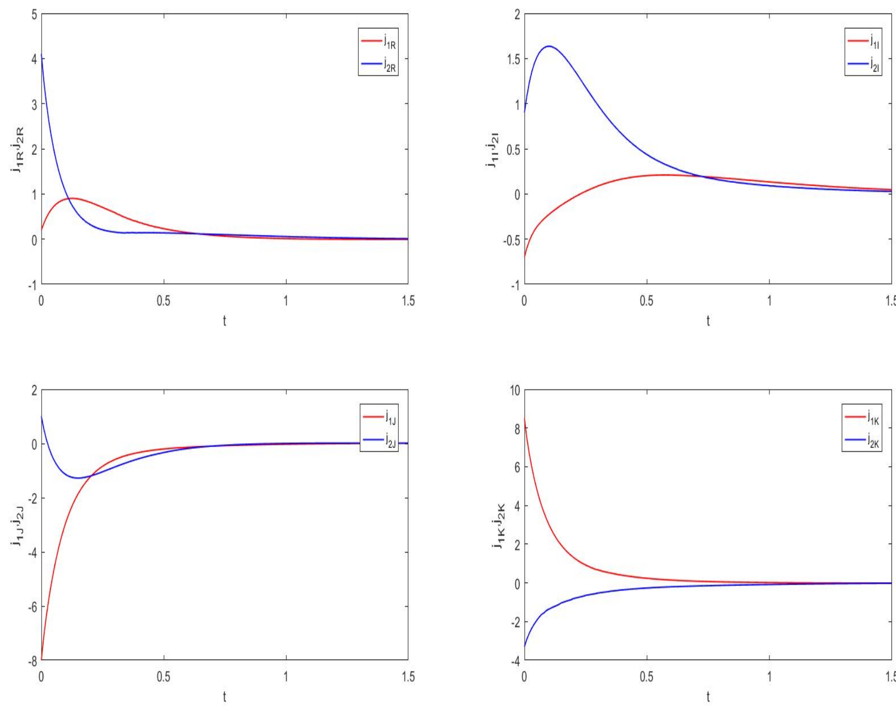

Figure 2: Transient states of R, I, J, K part of response systems (33) with 2-neurons.

Remark 4. In this paper, the FTS problem of FOQVNNs with time-varying delay under aperiodically intermittent control is discussed when $0 < \wp < 1$. By using the fractional derivative and inequality method, the Lyapunov function constructed by quadratic norm and linear norm is derived, and the FTS criterion of the error system is obtained. And numerical simulation shows that the derived synchronization criterion is correct and effective in Figures 1 and 3. In recent years, many QVNNs have attracted increasing attention from scholars. In this paper, we only study the FTS based on aperiodically intermittent control, fixed/predefined-time synchronization of FOQVNNs under aperiodically intermittent control will be discussed in the future, the biggest challenge we face is how to prove the fix/predefined-time stability lemma under intermittent time for FOQVNNs.

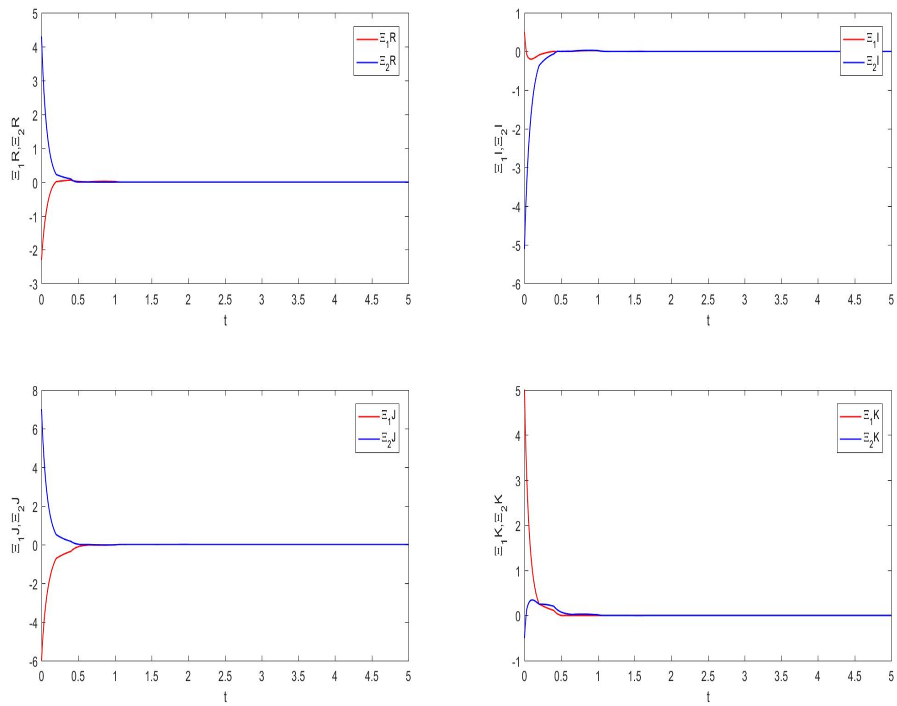

Figure 3: State trajectories of $\Xi_1 R, \Xi_1 I, \Xi_1 J, \Xi_1 K, \Xi_2 R, \Xi_2 I, \Xi_2 J, \Xi_2 K$.

6.Conclusion

In this article, we research the FTS between two FOQVNNs by using aperiodically intermittent control. In order to ensure the FTS of FOQVNNs, a new group of fractional differential inequalities during aperiodically intermittent times is given, and a control scheme composed of symbolic functions and norms are proposed by the non-decomposition method. Meanwhile, according to the actual needs, we can choose the control time by adjusting the parameter ϖ, and the synchronization time deduced is better than the synchronization time convergence effect obtained in the previous literature. The validity of the derived theoretical analysis is confirmed by numerical simulations.

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