Simple Tests for *n*-th Roots of Natural Numbers being Natural Numbers and Elementary Methods to Determine Their Values

Abstract

In this paper we tackle the challenging problem to determine, in a simple but reliable way, whether – for a given, arbitrary number x, $x \geq 2$ – the n-th root of x produces a rational or an irrational result, i.e. we determine whether $\sqrt[n]{x} \in \mathbf{Q}$ or $\sqrt[n]{x} \notin \mathbf{Q}$. To solve this problem in a straight-forward manner we make use of the prime factorization of x. As a main contribution we design several tests which can be applied to determine, for which values of n, $\sqrt[n]{x} \in \mathbf{Q}$ if the natural number x satisfies a given set of properties. Quite often our tests proposed will allow us to answer the question "$\sqrt[n]{x} \in \mathbf{Q}$?" within just a few seconds. Finally, we demonstrate that, for an astonishingly high percentage of all natural numbers x, $x \geq 2$, it is impossible to find even a single $n \in \mathbf{N}$, $n \geq 2$ such that $\sqrt[n]{x} \in \mathbf{Q}$.

Keywords: Number theory, prime factorization, efficient tests for $\sqrt[n]{x}$ ($n,x \in \mathbf{N}$, $n.x \geq 2$) being a rational number, replacement of Newton's method to determine a rational value of $\sqrt[n]{x}$.

1. Introduction

In [5] it has been demonstrated that it is astonishingly simple to answer the question "for which $n \in \mathbf{N}$, $n \geq 2$ the n-th root of x yields to a rational result, i.e. $\sqrt[n]{x} \in \mathbf{Q}$?" This question can be answered in a straight-forward manner if one makes use of the (unique) prime factorization of x possessing the following structure:

$$x = p_1^{k\_1} \bullet p_2^{k\_2} \bullet\bullet\bullet\, p_i^{k\_i} \bullet\bullet\bullet\, p_m^{k\_m} \qquad (1),$$

where p_i are prime numbers $\forall$ i$\in${1, 2, …, m} and $p_i \neq p_j$ $\forall$ i$\neq$j, m $\geq$ 1, $k_i \in \mathbf{N}$ $\forall$ i$\in${1, 2, …, m}.

Note: k_i to be read as k_i.

The result derived in [5] was:

For any $n,x \in \mathbf{N}$, $n.x \geq 2$:

$$\sqrt[n]{x} \in \mathbf{Q} \iff n \in CD(\{k_1, k_2, \ldots, k_m\}), \qquad (2),$$

where CD(M), for M being a subset of **N**, denotes the set of common divisors c, $c \geq 2$, of all elements of the set M.

Therefore, *if* m $\geq$ 2:

$$CD(\{k_1, k_2, \ldots, k_m\}) := \{c \in \mathbf{N}, c \geq 2 \mid \forall k_i \,\exists\, \nu_i = \nu_i(k_i) \in \mathbf{N}: c \bullet \nu_i = k_i\},$$

and, *if* m=1:

$$CD(\{k_1\}) := \{c \in \mathbf{N}, c \geq 2 \mid \exists\, \nu \in \mathbf{N}: c \bullet \nu = k_1\}.$$

It should be noted that for answering our question "$\sqrt[n]{x} \in \mathbf{Q}$?" by applying eq. (2) the values of the p_i appearing in eq. (1) are not of interest but only the values of the exponents k_i are relevant.

EXAMPLES:

- **Ex1.1**: $x = p^{10}$, p denoting an arbitrary prime number
  $\Rightarrow$ CD({10}) = {2, 5, 10} and, therefore, $\sqrt{p^{10}} \in$ **Q**, $\sqrt[5]{p^{10}} \in$ **Q**, $\sqrt[10]{p^{10}} \in$ **Q**.

  In [5] it has also been proven that, if $\sqrt[n]{x} \in$ **Q** and $x \in$ **N**, then this also directly implies that $\sqrt[n]{x} \in$ **N**. This will be taken into account by us in the following.

- **Ex1.2**: $x = 2^{10}$,
  $\Rightarrow$ CD({10}) = {2, 5, 10} cf. above and, therefore,
  $\sqrt{2^{10}} = 2^{10:2} = 2^5 = 32 \in$ **N**, $\sqrt[5]{2^{10}} = 2^{10:5} = 2^2 = 4 \in$ **N**, $\sqrt[10]{2^{10}} = 2^{10:10} = 2^1 = 2 \in$ **N**.

- **Ex1.3**: $x = 331776 = 2^{12} \bullet 3^4$. Therefore, we have to determine CD({12, 4}) = {2, 4}. This implies that $\sqrt{331776} \in$**N** and $\sqrt[4]{331776} \in$**N**. But, $\sqrt[n]{331776} \notin$**Q** $\forall n \notin \{2,4\}$.

One of the main goals of this contribution is to develop simple tests to determine – for arbitrary natural numbers n, $x \in$ **N**, $n \geq 2$ and $x \geq 2$ – whether $\sqrt[n]{x} \in$ **Q** and therefore even $\sqrt[n]{x} \in$ **N**. For our tests we need the prime factorization of x, according to eq. (1).

The prime factorization of x can be derived (in particular with the help of a computer) without much expenditure, if the value of x is not very large. However, it is well-known that if x is the product of two extremely large prime numbers, prime factorization of x may become very difficult or even practically infeasible. This fact is used, e.g., in the field of cryptography, in particular to construct very secure encryption algorithms, such as the RSA algorithm (see [3]).

To simplify our notation and argumentation throughout this paper let us introduce some basic notations and abbreviations.

We denote by:

$\mathbf{N}_{\geq 2} := \{x \in \mathbf{N} \mid x \geq 2\}$.

Moreover, for a natural number $x \in \mathbf{N}_{\geq 2}$ we want to introduce notations to denote the set of natural numbers $n \in \mathbf{N}_{\geq 2}$ for which $\sqrt[n]{x}$ results in a rational root, i.e. $\sqrt[n]{x} \in$**Q** . So let denote

$\mathbf{N}_{rat\_R}$ (x) the set of natural numbers $n \in \mathbf{N}_{\geq 2}$ such that for given $x \in \mathbf{N}_{\geq 2}$: $\sqrt[n]{x} \in \mathbf{Q}$ $\forall$ $n \in \mathbf{N}_{rat\_R}$ (x)

and – analogously – let denote

$\mathbf{N}_{irrat\_R}$ (x) the set of natural numbers $n \in \mathbf{N}_{\geq 2}$ such that for given $x \in \mathbf{N}_{\geq 2}$: $\sqrt[n]{x} \notin \mathbf{Q}$ $\forall$ $n \in \mathbf{N}_{irrat\_R}$ (x).

Our notations $\mathbf{N}_{rat\_R}$ (x) and $\mathbf{N}_{irrat\_R}$ (x), respectively, are used to denote – for a specific $x \in \mathbf{N}_{\geq 2}$ – the set of natural numbers $x \in \mathbf{N}_{\geq 2}$ leading to rational or irrational results, respectively, of the n-th root $\sqrt[n]{x}$. We generalize this notation by replacing x by a set S $\subseteq \mathbf{N}_{\geq 2}$, e.g., by a set the elements of which have common properties. Then, $\mathbf{N}_{rat\_R}$ (S) denotes the subset of $\mathbf{N}_{\geq 2}$ such that $\forall$ $x \in$S and $\forall$ $n \in \mathbf{N}_{rat\_R}$ (S): $\sqrt[n]{x} \in$ **Q**. And the meaning of $\mathbf{N}_{irrat\_R}$ (S) will be analogous.

The rest of this paper is structured as follows: In Section 2, we introduce simple tests which allow us, in a fast and efficient manner, to exactly determine – for any given $x \in \mathbf{N}_{\geq 2}$ – the sets $\mathbf{N}_{rat\_R}$ (x) and $\mathbf{N}_{irrat\_R}$ (x). We will demonstrate the ease of use of our tests by various examples. In Section 3, investigations will be presented regarding the set of natural numbers the n-th roots of which are all irrational. The paper will conclude with a short summary and outlook.

2. Tests for $\sqrt[n]{x}$ Being an Irrational Number (for $n, x \in \mathbf{N}$, $n, x \geq 2$)

In our Introduction we have already demonstrated that for an arbitrary $x \in \mathbf{N}_{\geq 2}$ possessing a prime factorization according to eq. (1) only the exponents $k_1, k_2, \ldots, k_m$ appearing in the factorization are relevant and the set $\mathbf{N}_{rat\_R}(x)$ can be determined by $\mathbf{N}_{rat\_R}(x) = CD(\{k_1, k_2, \ldots, k_m\})$.
And, similarly, $\mathbf{N}_{irrat\_R}(x) = \mathbf{N}_{\geq 2} \setminus \mathbf{N}_{rat\_R}(x)$.

Here, we make use of these elementary insights, to construct simple and easy to apply tests which will allow us to recognize immediately – for an adequately chosen subset $S \subset \mathbf{N}_{\geq 2}$ – whether $\sqrt[n]{x} \in \mathbf{N}$ or $\sqrt[n]{x} \notin \mathbf{Q}$ for $x \in S$ and a natural number $n \in \mathbf{N}_{\geq 2}$, arbitrarily chosen.

- Test **T1**:

Let us start by considering the set

$S_1 = \{ x \in \mathbf{N}_{\geq 2} \mid x = p_1 \bullet p_2^{k\_2} \bullet\bullet\bullet p_i^{k\_i} \bullet\bullet\bullet p_m^{k\_m}$; where p_i are prime numbers $\forall\, i \in \{1, 2, \ldots, m\}$ and $p_i \neq p_j\ \forall\, i \neq j$, $m \geq 1$, $k_j \in \mathbf{N}\ \forall\, j \in \{2, 3, \ldots, m\} \}$.

It should be noted that, without loss of generality, it is sufficient that for at least one of the k_i in eq. (1) the condition $k_i = 1$ holds. Then, the form required by S_1 can be achieved by just renumbering the k_i.

TEST T1: *If* $x \in S_1$ *then* $CD(\{k_1, k_2, \ldots, k_m\}) = \emptyset$.
Therefore, $\mathbf{N}_{rat\_R}(S_1) = \emptyset$ and $\mathbf{N}_{irrat\_R}(S_1) = \mathbf{N}_{\geq 2}$, i.e. $\sqrt[n]{x} \notin \mathbf{Q}\ \forall\, x \in S_1,\ \forall\, n \in \mathbf{N}_{\geq 2}$.

Remark: A special case for S_1 is $S_1^* = \{ p \in \mathbf{N}_{\geq 2} \mid p$ is a prime number$\}$ and, therefore, it is proven that $\sqrt[n]{p} \notin \mathbf{Q}\ \forall\, n \in \mathbf{N}_{\geq 2}$ and $\forall p$, p being a prime number.

- Test **T2**:

Let us now consider the set

$S_2 = \{ x \in \mathbf{N}_{\geq 2} \mid x = p_1^{k\_1} \bullet p_2^{k\_2} \bullet\bullet\bullet p_m^{k\_m}$; where p_i are prime numbers $\forall\, i \in \{1, 2, \ldots, m\}$ and $p_i \neq p_j\ \forall\, i \neq j$, $m \geq 2$, $k_i \in \mathbf{N}\ \forall\, i$, and $\exists\, \alpha, \beta \in \{1, 2, \ldots, m\}$: k_α and k_β having no common divisors $d>1 \}$.

TEST T2: *If* $x \in S_2$ *then* $CD(\{k_1, k_2, \ldots, k_m\}) = \emptyset$.
Therefore, $\mathbf{N}_{rat\_R}(S_2) = \emptyset$ and $\mathbf{N}_{irrat\_R}(S_2) = \mathbf{N}_{\geq 2}$, i.e. $\sqrt[n]{x} \notin \mathbf{Q}\ \forall\, x \in S_2,\ \forall\, n \in \mathbf{N}_{\geq 2}$.

Remark: A special case for S_2 is $S_2^* = \{ p_1^{k\_1} \bullet p_2^{k\_2} \mid p_1, p_2, k_1$ (resp. k_1) and k_2 (resp. k_2) are all prime numbers and $p_1 \neq p_2$ and $k_1 \neq k_2 \}$. Therefore, it is proven – for this special case of S_2 – that $\sqrt[n]{x} \notin \mathbf{Q}\ \forall\, n \in \mathbf{N}_{\geq 2}$ and $\forall\, x \in S_2^*$.

- Test **T3**:

We want to close this section with a test leading to the result that, for $x \in S$, $\mathbf{N}_{rat\_R}(x)$ is not an empty set as in tests T1 and T2 but now it contains just one element.

So, let us consider the set

$S_3 = \{ x \in \mathbf{N}_{\geq 2} \mid x = p_1^{k\_1} \bullet y$, $y=1$ *or* $y = p_2^{k\_2} \bullet p_3^{k\_3} \bullet\bullet\bullet p_m^{k\_m}$, $m \geq 2$; where p_i and k_1 (resp. k_1) are prime numbers $\forall\, i \in \{1, 2, \ldots, m\}$ and $p_i \neq p_j\ \forall\, i \neq j$, and, in the prime factorization of x, if $m \geq 2$, all exponents k_i (resp. k_i), $i \geq 1$, share the common divisor k_1 , but no other one$\}$.

TEST T3: *If* $x \in S_3$ *then* $\mathbf{N}_{rat\_R}(x) = \{k_1\}$ and $\mathbf{N}_{irrat\_R}(x) = \mathbf{N}_{\geq 2} \setminus \{k_1\}$

EXAMPLES:

- **Ex2.1**: $x = p_1^{101}$, e.g. p_1=211 (being a prime number), i.e. x represents a number possessing more than 200 digits. Applying test **T3** (where we set y=1) we find out in just a few seconds that $\mathbf{N}_{rat\_R}(x) = \{101\}$, i.e. $\sqrt[n]{x} \notin \mathbf{Q} \ \forall \ n \in \mathbf{N}_{\geq 2}$, $n \neq 101$ as well as that $\sqrt[101]{x} = p_1 \in \mathbf{N}$.

- **Ex2.2**: Let $S_{12} = (p^{12}$ | p representing an arbitrary prime number) $\Rightarrow \mathbf{N}_{rat\_R}(S_{12})$ = CD({12}) = {2,3,4,6,12}, i.e. $\sqrt[n]{x} \in \mathbf{N} \ \forall \ x \in S_{12}, \ \forall \ n \in \mathbf{N}_{rat\_R}(S_{12})$. And, therefore, $\mathbf{N}_{irrat\_R}(S_{12}) = \mathbf{N}_{\geq 2} \setminus \{2,3,4,6,12\}$.

Last but not least, let us shortly discuss the interesting question how to determine the value of $\sqrt[n]{x}$ if we know that $\sqrt[n]{x} \in \mathbf{N}$, i.e. $n \in \mathbf{N}_{rat\_R}(x)$. Assuming again that the prime factorization of *x* is given and is such as specified by eq. (1) then, of course, we immediately recognize that

$$\sqrt[n]{x} = p_1^{k\_1/n} \bullet p_2^{k\_2/n} \bullet\bullet\bullet \ p_m^{k\_m/n} .$$

Indeed, this is an astonishingly simple way to determine the value of any *n*-th root of any natural number $x \in \mathbf{N}_{\geq 2}$, if $\sqrt[n]{x}$ results in a rational (and in this case, in particular, even in a natural) number.

3. Investigating the Set of Natural Numbers Possessing Irrational *n*-th Roots

Let us now introduce an additional notation to name the total set of natural numbers, which possess irrational *n*-th roots only:

$$\mathbf{SN}\_irrat\_roots := \{x \in \mathbf{N}_{\geq 2} \mid \sqrt[n]{x} \notin \mathbf{Q} \ \forall \ n \in \mathbf{N}_{\geq 2}\}.$$

Taking into account test **T1** (cf. Section 2) it is evident, that the set $S_4 = \{x \in \mathbf{N}_{\geq 2} \mid x=4y-2, \ y \in \mathbf{N}\}$ characterizing the even numbers not being an integer multiple of 4 represent a strict subset of **SN**_irrat_roots. In particular, the elements *x* of S_4 possess a prime factorization such that

$$x = 2 \bullet p_1^{k\_1} \bullet p_2^{k\_2} \bullet\bullet\bullet \ p_m^{k\_m}, \ p_i \text{ denoting an odd prime number } \forall \ i \in \{1, 2, \ldots, m\}.$$

Therefore $S_4 \subset S_1$.

Testing whether a given (even) number is an integer multiple of 4 is very simple, indeed. We just have to test whether the last two digits are a multiple of 4. Therefore, we realize immediately that all natural numbers ending with 02, 06, 10, 14, …, 98 are part of set S_4 and for all the other even natural numbers $n : n \notin S_4$.

By finding the set S_4 we have proven the result (being non-evident a priori) that for $\forall \ x \in \mathbf{N}$ a number $z \in S_4$ can be found in the immediate neighborhood of *x* such that $| x-z | \leq 2$. Also, the fact $\mathbf{N}_{irrat\_R}(S_4) = \mathbf{N}_{\geq 2}$ directly implies that $z \in$ **SN**_irrat_roots. Therefore, we can say that the set **SN**_irrat_roots is a "dense coverage" of the set **N**.

Considering now, e.g., the set $\mathbf{N}_{\leq 1000} := \{n \in \mathbf{N} \mid n \leq 1000\}$ already the properties of set S_4 imply that at least 25% of all numbers of $\mathbf{N}_{\leq 1000}$ are part of **SN**_irrat_roots. A more detailed view shows that – defining regions of size 100 to partition the set $\mathbf{N}_{\leq 1000}$ – we find only rather few natural numbers *x*, being part of each partition, for which $x \notin$ **SN**_irrat_roots.

In detail, we observe that for

- Partition P1 = {2, 3, …, 99}:

$x \notin$ **SN**_irrat_roots $\Leftrightarrow$ $x \in M_1$ = {4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81}.

Thus, only for 11 numbers in P1: $\exists\, n \in \mathbf{N}_{\geq 2}$ with $\sqrt[n]{x} \in \mathbf{N}$, $x \in M_1$.

- Partition P2 = {100, 101, …, 199}:

  $x \notin$ **SN**_irrat_roots $\Leftrightarrow$ $x \in M_2$ = {100, 121, 125, 128, 144, 169, 196}.

  Thus, only for 7 numbers in P2: $\exists\, n \in \mathbf{N}_{\geq 2}$ with $\sqrt[n]{x} \in \mathbf{N}$, $x \in M_2$.

- Partition P3 = {200, 201, …, 299}:

  $x \notin$ **SN**_irrat_roots $\Leftrightarrow$ $x \in M_3$ = {216, 225, 243, 256, 289}.

  Thus, only for 5 numbers in P3: $\exists\, n \in \mathbf{N}_{\geq 2}$ with $\sqrt[n]{x} \in \mathbf{N}$, $x \in M_3$.

And in all the other partitions covering neighboring numbers (with a size of 100) between 300 and 1000 at least 97% of those 100 natural numbers are part of **SN**_irrat_roots. This implies that, considering an arbitrary natural number *x*, $300 \leq x \leq 1000$, in nearly all cases we will find a number *x** directly neighboring to *x* (distance=1) such that $x^* \in$ **SN**_irrat_roots. In almost all cases, however, even "$x \in$ **SN**_irrat_roots" would already be satisfied for *x* itself.

4. Conclusions and Outlook

This contribution showed that understanding the properties of *n*-th roots $\sqrt[n]{x}$ of natural and of positive rational numbers can be improved significantly if the argumentation directly relies on the prime factorization, resulting for *x*, as opposed to the usual argumentation based on polynomials. We demonstrated that, using the prime factorization of *x*, it is possible to design numerous tests which allow one to recognize in an astonishingly simple and efficient manner whether $\sqrt[n]{x}$, for arbitrary values $x \in \mathbf{N}_{\geq 2}$ and $n \in \mathbf{N}_{\geq 2}$, result in a rational or an irrational value. In case $\sqrt[n]{x} \in \mathbf{Q}$ (with the consequence $\sqrt[n]{x} \in \mathbf{N}$), by means of our innovative calculation method, it is typically possible to determine the exact result of $\sqrt[n]{x}$ within just a few seconds. Calculation methods to determine the result of $\sqrt[n]{x}$ which also allow one to approximate the value of $\sqrt[n]{x}$ (such as, e.g., Newton's method [1] or [2]) can now be limited to the approximation of $\sqrt[n]{x}$, if $\sqrt[n]{x} \notin \mathbf{Q}$. Similarly, usage of the Lemma of Gauß [4] can be restricted to some cases in which we want to prove that $\sqrt[n]{x} \notin \mathbf{Q}$ and, e.g., prime factorization of *x* could be practically infeasible.

The tests presented by us to cover the cases $\sqrt[n]{x}$, for $x \in \mathbf{N}_{\geq 2}$, can be extended in a straight-forward manner to cases $\sqrt[n]{y}$, in which $y \in \mathbf{Q}^+ \setminus \mathbf{N}$, for given $n \in \mathbf{N}_{\geq 2}$ (where $\mathbf{Q}^+$ denotes the set of positive rational numbers). Those modified tests could be used to prove that $\sqrt[n]{y} \in \mathbf{Q}$ (still for $y \in \mathbf{Q}^+ \setminus \mathbf{N}$) and, if so, to determine the value of $\sqrt[n]{y}$ (which, typically, will no longer be a natural number).

As it has been proven by the author already earlier [5]: $\sqrt[n]{x} \notin \mathbf{Q}$ $\forall\, x \in \mathbf{R}^+ \setminus \mathbf{Q}$, $\forall\, n \in \mathbf{N}_{\geq 2}$.

Therefore, it is evident, a priori, that for any given $n \in \mathbf{N}_{\geq 2}$ it is not (!) possible that: $\sqrt[n]{x} \in \mathbf{Q}$ for any $x \in \mathbf{R}^+ \setminus \mathbf{Q}$.

This contribution clearly shows that the decision whether the *n*-th root $\sqrt[n]{x}$ of a natural number *x* possesses a rational value becomes rather trivial, if the prime factorization of *x* is available. In the particular case that a given natural number *x* is relatively small, e.g. $x \leq 1000$, then obtaining the prime factorization of *x* is a very easy exercise, indeed.

References

[1] J.H. Hubbard, D. Schleicher, S. Sutherland: How to Find All Roots of Complex Polynomials by Newtons Method. Inventiones Mathematicae, Vol. 146 (2021), pp. 1-33.

[2] C.T. Kelley: Solving Nonlinear Equations with Newton's Method, no1 in Fundamentals of Algorithms, SIAM, 2003.

[3] R.L. Rivest, A. Shamir, L. Adleman: A Method for Obtaining Digital Signatures and Public-key Cryptosystems. C.ACM, Vol. 21, No. 2 (1978), pp. 120-126.

[4] E.W. Weisslein: Rational Zero Theorem. From MathWorld – A Wolfram Web Resource, https://mathworld.wolfram.com/RationalZeroTheorem.html

[5] B.E. Wolfinger: Simple Criteria for $\sqrt[n]{x}$ ($n \in \mathbf{N}$, $n \geq 2$, $x \in \mathbf{R}$) Being a Rational or an Irrational Number. J. Advances in Mathematics and Computer Science, Vol. 38, Issue 9 (2023), pp. 23-30.