

Solution of Inhomogeneous Differential Equations with Polynomial Coefficients in Nonstandard Analysis, in Terms of the Green's Function

Abstract

Discussions are presented by Morita and Sato on the problem of obtaining the particular solution of an inhomogeneous differential equation with polynomial coefficients in terms of the Green's function. In a preceding paper, solution is given without using the Green's function, on the basis of nonstandard analysis, for a restricted class of inhomogeneous terms. In the present paper, the corresponding solutions are given in terms of the Green's function. It is applied to Kummer's and the hypergeometric differential equation.

Keywords: Green's function; differential equations with polynomial coefficients; nonstandard analysis; Kummer's differential equation; hypergeometric differential equation

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1 Introduction

In the present paper, we treat the problem of obtaining the particular solutions of a differential equation with polynomial coefficients in terms of the Green's function.

In a preceding paper [4], this problem is studied in the framework of distribution theory, where the method is applied to Kummer's and the hypergeometric differential equation. In another paper [5], this problem is studied in the framework of nonstandard analysis, and it is applied to a simple fractional and a first-order ordinary differential equation.

In a recent paper [6], a compact recipe based on nonstandard analysis, is presented, and is applied to Kummer's differential equation.

In the preceding paper [7], solutions of an equation which has a special class of inhomogeneous part, are given, without using the Green's function. It is applied to the hypergeometric differential equation, the differential equations treated in [5] and the Hermite differential equation.

In the present paper, we solve the problem considered in [7], but now the results are expressed in terms of the Green's function. It is applied to Kummer's and the hypergeometric differential equation. The presentation in this paper follows those in [4, 5, 6, 7], in Introduction and in many descriptions in the following sections.

We consider a fractional differential equation, which takes the form:

$$p_n(t; {}_R D_t)u(t) =$$

$$X_n$$

$$I=0$$

$$a_l(t) {}_R D_t^{-l}$$

$${}_t u(t) = f(t); \quad (1.1)$$

where (i) $n \in \mathbb{Z}_{>0}$, $t \in \mathbb{R}$, (ii) $a_l(t)$ for $l \in \mathbb{Z}_{[0;n]}$ are polynomials of t , (iii) $\alpha_l \in \mathbb{C}$ for $l \in \mathbb{Z}_{[0;n]}$ satisfy $\operatorname{Re} \alpha_0 > \operatorname{Re} \alpha_1 > \dots > \operatorname{Re} \alpha_n$ and $\operatorname{Re} \alpha_0 > 0$.

Here \mathbb{Z} is the set of all integers, \mathbb{R} and \mathbb{C} are the sets of all real numbers and all complex numbers, respectively, and $\mathbb{Z}_{>a} = \{n \in \mathbb{Z} \mid n > a\}$, $\mathbb{Z}_{<b} = \{n \in \mathbb{Z} \mid n < b\}$ and $\mathbb{Z}_{[a;b]} = \{n \in \mathbb{Z} \mid a \leq n \leq b\}$ for $a, b \in \mathbb{Z}$ satisfying $a < b$. We also use $\mathbb{R}_{>a} = \{x \in \mathbb{R} \mid x > a\}$ for $a \in \mathbb{R}$, and $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$.

We use Heaviside's step function $H(t)$, which is equal to 1 if $t > 0$ and, to 0 if $t \leq 0$. Here ${}_R D_t^{-l}$

are the Riemann-Liouville fractional integrals and derivatives defined by the following definition; see [3, 8].

Definition 1.1. Let $t \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $u_0(t)$ be locally integrable on $\mathbb{R}_{>-\alpha}$, $u(t) = u_0(t)H(t - \alpha)$, $\alpha \in \mathbb{C}_+$, $n \in \mathbb{Z}_{>0}$ and $\alpha = n - \beta$. Then ${}_R D_t^{-\alpha}$

${}_t u(t)$ is the Riemann-Liouville fractional integral defined by

$${}_R D_t^{-\alpha}$$

$${}_t u(t) =$$

$$1$$

$$\int_{-\alpha}^t$$

$$\int_{-\alpha}^t$$

$$\int_{-\alpha}^t$$

$$(t - x)^{\alpha-1} u_0(x) H(x - \alpha) dx$$

$$=$$

$$1$$

$$\int_{-\alpha}^t$$

$$\int_{-\alpha}^t$$

$$-$$

$$({}^t \square x)_{\square 1} u_0(x) dx \square H(t \square \square); \quad (1.2)$$

and RD_{\square}

${}^t U(t) = 0$ for $t \square \square$, where $\square(\square)$ is the gamma function, RD_{\square}

${}^t U(t) = RD_{n \square}$

${}^t U(t)$ is the

Riemann-Liouville fractional derivative defined by

RD_{\square}

${}^t U(t) = RD_{n \square}$

${}^t U(t) =$

d_n

$dt_n [RD_{\square}$

${}^t U_0(t)] \square H(t \square \square); \quad (1.3)$

when $n \square Re \square$, and RD_n

${}^t U(t) = d_n$

$dt_n U_0(t) \square H(t \square \square)$ when $\square = n \square Z > \square 1$.

In accordance with Definition 1.1, when $u_0(t) = 1$

$\square(\square) (t \square \square)_{\square 1}$, we adopt

RD_{\square}

t

$(t \square \square)_{\square 1}$

$\square(\square)$

$H(t \square \square) =$

$($

$(t \square \square)_{\square 1}$

$\square(\square) H(t \square \square); \square \square \square 2 CnZ < 1;$

$0; \square \square \square 2 Z < 1;$

(1.4)

for $\square \square 2 CnZ < 1$ and $\square \square R$. Here RD_t is used in place of usually used notation DR , in order to show

that the variable is t .

Remark 1.1. Let $g_{\square}(t) = 1$

$\square(\square) {}^t \square 1 H(t)$ for $\square \square C$. Then $g_{\square}(t) = 0$ if $\square \square 2 Z < 1$, and Equation (1.4) shows

that if $\square \square 2 Z < 1$, RD_{\square}

${}^t g_{\square}(t) = g_{\square \square}(t)$. As a consequence, we have $RD_{\square+n}$

${}^t g_{\square}(t) = g_{\square n}(t) = 0$ for

$n \square 2 Z > \square 1$.

In distribution theory [4, 9, 2, 10], we use distribution $\sim H(t)$, which corresponds to function $H(t)$, differential operator D , and distribution $\delta(t) = D \sim H(t)$, which is called Dirac's delta function.

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1.1 Preliminaries on Nonstandard Analysis

In nonstandard analysis [1], where infinitesimal numbers appear. We denote the set of all infinitesimal real numbers by R_0 . We also use R_0

$> 0 = f \square 2 R_0 \text{ j } \square > 0g$, which is such that if $\square \square 2 R_0$

> 0 , there exists

$N \square 2 Z_{> 0}$ satisfying $\square < 1$

N . We use R_{ns} , which has subsets R and R_0 . If $x \square 2 R_{ns}$ and $x = 2 R$, x is

expressed as $x_1 + \square$ by $x_1 \square 2 R$ and $\square \square 2 R_0$, where x_1 may be $0 \square 2 R$. Equation $x' y$ for $x \square 2 R_{ns}$ and

$y \square 2 R_{ns}$, is used, when $x \square y \square 2 R_0$. We denote the set of all infinitesimal complex numbers by C_0 ,

which is the set of complex numbers z which satisfy $jRe z \text{ j} + |Im z| \square 2 R_0$. We use C_{ns} , which has

subsets C and C_0 . If $z \square 2 C_{ns}$ and $z = 2 C$, z is expressed as $z_1 + \square$ by $z_1 \square 2 C$ and $\square \square 2 C_0$, where z_1

may be $0 \square 2 C$.

In place of (1.4), we now use

RD_{\square}

${}^t g_{\square+}(t) = RD_{\square}$

t

1

$\square(\square + \square)$

${}^t \square 1+ H(t) = g_{\square \square+}(t) =$

1

$\square(\square \square \square + \square)$

${}^t \square \square 1+ H(t); \quad (1.5)$

for all $\square \square 2 C$ and $\square \square 2 C$, where $\square \square 2 R_0$

$\alpha > 0$.

Lemma 1.1. Let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $\gamma \in \mathbb{C}$, $\delta \in \mathbb{R}_0$

$\alpha > 0$ and $g_{\alpha+\beta}(t) = 1$

$\square_{(\alpha+\beta)} t_{\alpha+\beta} H(t)$. Then the index

law:

$\mathbb{R}D_{\alpha}$

$t \mathbb{R}D_{\beta}$

$t g_{\alpha+\beta}(t) = \mathbb{R}D_{\alpha+\beta}$

$t g_{\alpha+\beta}(t) = g_{\alpha+\beta}(t)$; (1.6)

always holds.

In the present study in nonstandard analysis, in place of $\sim H(t)$ and $\square(t)$ in distribution theory, $H_{\alpha}(t)$ and $\square_{\alpha}(t)$ are used, which are given by

$H_{\alpha}(t) = \mathbb{R}D_{\alpha}$

$t H(t) = g_{\alpha+\beta}(t) =$

1

$\square_{\alpha}(t)$

$t H(t)$; (1.7)

$\square_{\alpha}(t) = g_{\alpha}(t) =$

d

dt

$H_{\alpha}(t) =$

1

$\square_{\alpha}(t)$

$t_{\alpha} H(t) =$

—

$\square_{\alpha}(t)$

$t_{\alpha} H(t)$; (1.8)

for $\alpha \in \mathbb{R}_0$

$\alpha > 0$. We note that they tend to $H(t)$ and 0, respectively, in the limit of $\alpha \rightarrow 0$.

Lemma 1.2. In the notation in Remark 1.1, $H_{\alpha}(t) = g_{\alpha+\beta}(t)$, $\square_{\alpha}(t) = g_{\alpha}(t)$, and

$\mathbb{R}D_{\alpha}$

$t H_{\alpha}(t) = \mathbb{R}D_{\alpha}$

$t g_{\alpha+\beta}(t) = g_{\alpha}(t) = H(t)$; $\mathbb{R}D_{\alpha}$

$t \square_{\alpha}(t) = \mathbb{R}D_{\alpha}$

$t g_{\alpha}(t) = g_{\alpha}(t) = 0$; (1.9)

1.2 Summary of the Following Sections

In solving Equation (1.1) in nonstandard analysis, we consider the solution of the following equation

for $\sim u(t) = \mathbb{R}D_{\alpha}$

$t u(t)$:

$\sim p_{\alpha}(t; \mathbb{R}D_{\alpha}) \sim u(t) = \sim f(t)$; (1.10)

where $\alpha \in \mathbb{R}_0$

$\alpha > 0$ and

$\sim p_{\alpha}(t; \mathbb{R}D_{\alpha}) := \mathbb{R}D_{\alpha}$

$t p_{\alpha}(t; \mathbb{R}D_{\alpha}) \mathbb{R}D_{\alpha}$

t ; (1.11)

In [7], Conditions 1.2 and 1.1 on p. 52, are adopted. It is

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Condition 1.1. Let $\alpha \in \mathbb{R}_0$

$\alpha > 0$ and $\alpha \in \mathbb{C}$.

(i) $\sim f(t) = \square_{\alpha}(t)$ and $f(t) = 0$.

(ii) $\sim f(t) = \mathbb{R}D_{\alpha}$

$t \square_{\alpha}(t) = g_{\alpha}(t)$. When $\alpha \in \mathbb{Z}_{>0}$, $f(t) = \mathbb{R}D_{\alpha+1}$

$t H(t)$, and when $\alpha \in \mathbb{Z}_{>0}$,

$f(t) = 0$.

(iii) $\sim f(t)$ and $f(t)$ are expressed as follows:

$\sim f(t) =$

\int_0^X

$l=1$

$\square_{\alpha} \mathbb{R}D_{\alpha}$

$t \square_{\alpha}(t) =$

$$f(t) = \sum_{l=1}^{\infty} c_l g_{\alpha, \beta}(t); \quad (1.12)$$

respectively, where $c_l \in \mathbb{C}$ are constants, $\alpha, \beta \in \mathbb{C}$ satisfy $\operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0$, for all $l \in \mathbb{Z}_{>0}$, and $d_l = c_l$ if $l \in \mathbb{Z}_{>0}$, and $d_l = 0$ if $l \in \mathbb{Z}_{<0}$.

Remark 1.2. Condition 1.1(i) is satisfied, when Condition 1.1(ii) is satisfied and $\alpha = 0$. In Sections 2 and 3, full expressions of the Green's functions and the solutions, are derived for Kummer's and the hypergeometric differential equation, respectively. Section 4 is for Conclusion.

2 Solution of Kummer's Differential Equation

Kummer's differential equation is described by

$$t^2 \frac{d^2 u}{dt^2} + (c - bt) \frac{du}{dt} - ab u = f(t); \quad (2.1)$$

where a, b and c are constants satisfying $a \neq 0$ and $b \neq 0$.

Lemma 2.1. When Condition 1.1(ii) is satisfied, we construct the following transformed differential equations of Equation (2.1), for $\tilde{u}(t) = t^{-\alpha} u(t)$

$$\tilde{t} \tilde{u}(\tilde{t}), \quad \tilde{t} \tilde{w}(\tilde{t}) = t^{-\alpha} u(t), \quad \tilde{t} \tilde{w}(\tilde{t}) = t^{-\alpha} u(t);$$

$$\tilde{t} \tilde{u}(\tilde{t}), \quad \tilde{t} \tilde{w}(\tilde{t}) := t^{-\alpha} u(t);$$

$$\tilde{t} \tilde{u}(\tilde{t}) = g_{\alpha, \beta}(\tilde{t}); \quad (2.2)$$

$$\tilde{t} \tilde{w}(\tilde{t}) := t^{-\alpha} u(t);$$

$$\tilde{t} \tilde{w}(\tilde{t}) = \tilde{u}(\tilde{t}); \quad (2.3)$$

$$\tilde{t} \tilde{w}(\tilde{t}) := t^{-\alpha} u(t);$$

$$\tilde{t} \tilde{w}(\tilde{t}) = 0; \quad (2.4)$$

$$\tilde{t} \tilde{w}(\tilde{t}) = 0; \quad (2.4)$$

In obtaining these equations from Equation (2.1), we use Equation (2.5) in the following lemma, which is given in [7].

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Lemma 2.2. Let $\alpha \in \mathbb{C}^+$, $m \in \mathbb{Z}_{>0}$ and $\beta = m - \alpha$. Then

$${}_{t_0}^R D_{\alpha} [tu(t)] = t_{\alpha} {}_{t_0}^R D_{\alpha} u(t) + \alpha {}_{t_0}^R D_{\alpha-1} u(t); \quad (2.5)$$

$${}_{t_0}^R D_{\alpha} [t^2 u(t)] = t^2_{\alpha} {}_{t_0}^R D_{\alpha} u(t) + 2 t_{\alpha-1} {}_{t_0}^R D_{\alpha-1} u(t) + \alpha(\alpha-1) {}_{t_0}^R D_{\alpha-2} u(t); \quad (2.6)$$

Lemma 2.3. Two complementary solutions of Equation (2.4), which are given by

$$K_{\alpha,1}(t) = {}_{t_0}^R I_{\alpha} (a - c; c - \alpha; bt) =$$

$$\sum_{k=0}^{\infty} \frac{(a - c)_k b^k}{k! (c - \alpha)_k} t^k; \quad t > 0; \quad (2.7)$$

$$K_{\alpha,2}(t) = \frac{1}{\alpha(2 - c - \alpha)} {}_{t_0}^R I_{\alpha} (a - c + 1; 2 - c - \alpha; bt) =$$

$$\sum_{k=0}^{\infty} \frac{(a - c + 1)_k b^k}{k! \alpha(2 - c - \alpha + k)} t^{k+\alpha}; \quad t > 0; \quad (2.8)$$

exist, when $c - \alpha = 2 - Z < 1$ and when $c - \alpha = 2 - Z > 1$, respectively.

Following Equation (60) in [6], we define the Green's function $G_{K_{\alpha,1}}(t; \tau)$, which satisfies

$$-{}_{\tau}^R D_{\alpha} G_{K_{\alpha,1}}(t; \tau) = \delta(t - \tau); \quad (2.9)$$

for $\tau \in \mathbb{R}$.

In the present section, formulas are derived with the aid of two complementary solutions given by (2.7) and (2.8), and hence they hold when $c - \alpha = 2 - Z < 1$.

Lemma 2.4. Let $K_{\alpha,1}(t)$ be given by Equation (2.7). Then Lemmas 2.1 and 2.3 show that $G_{K_{\alpha,1}}(t; 0)$ and $G_{K_{\alpha,2}}(t; 0)$, given by

$$\sim w_{\alpha,1}(t) = G_{K_{\alpha,1}}(t; 0) = {}_{t_0}^R D_{\alpha} K_{\alpha,1}(t; 0); \quad (2.10)$$

$$w_{\alpha,2}(t) = G_{K_{\alpha,2}}(t; 0) =$$

$$\frac{1}{\alpha(1 + c - \alpha)} K_{\alpha,1}(t) H(t); \quad (2.11)$$

are a particular solution of Equations (2.9) for $\tau = 0$, and also of Equation (2.3), and a complementary solution of Equation (2.4), respectively.

Remark 2.1. The derivation of Equations (2.10) with (2.11) for $\tau = 0$, with the aid of Frobenius method, is given in Section 3.1 of [6].

Theorem 2.1. Let Condition 1.1(ii) be satisfied, and $G_{K_{\alpha,1}}(t; 0)$ and $G_{K_{\alpha,2}}(t; 0)$ be given by Equation (2.10) with (2.11). Then Lemmas 2.1, 2.4 and 2.3 show that

(i) when $\alpha \in \mathbb{C}$ and $\sim f(t) = g_{\alpha}(t)$, by using Equation (2.12), we obtain $\sim u_{\alpha}(t) = {}_{t_0}^R D_{\alpha}$

$$t \sim w_{\alpha}(t),$$

given by

$$\sim u_{\alpha}(t) = {}_{t_0}^R D_{\alpha} G_{K_{\alpha,1}}(t; 0) =$$

$$\frac{1}{\alpha(1 + c - \alpha)}$$

$$\sum_{k=0}^{\infty} \frac{(a - c)_k b^k}{k! (c - \alpha)_k} t^k$$

$$+ \frac{(c - \alpha)_k}{k! (k - \alpha + 1 + \alpha)}$$

$$t^{k+\alpha} H(t); \quad (2.12)$$

which is a particular solution of Equation (2.2),

(ii) if $\alpha = n + 2$, $Z > 1$, by using Equation (2.12), we obtain $u_{\alpha}(t) = {}_R D_{\alpha}^{-1} u_{\alpha}(t)$, expressed by

$$\begin{aligned} u_{\alpha}(t) &= \sum_{k=n}^{\infty} \frac{(a - n)_k b_k}{(\alpha + c - n)_{k+1}} \\ &= \sum_{l=0}^{\infty} \frac{(a - n)_{n+l} b_{n+l}}{(\alpha + c - n)_{n+l+1}} \\ &= {}_t H(t) = C_n G_{K,0,0}(t; 0); \end{aligned} \quad (2.13)$$

which is a complementary solution of Equation (2.1), where $C_n = \frac{(a - n)_n}{(\alpha + c - n)_{n+1}}$

$$b_n = \frac{(a)_n (c - n)_1}{(a - n)_n (c)_n}$$

and

(iii) if $\alpha = 2$, $Z > 1$, by using Equation (2.12), we obtain $u_{\alpha}(t)$, given by $u_{\alpha}(t) = {}_R D_{\alpha}^{-1} G_{K,0,0}(t; 0)$, which is a particular solution of Equation (2.1).

Theorem 2.1 shows that if $\tilde{f}(t) = {}_R D_{\alpha}^{-1} f(t)$, the particular solution of Equation (2.2) is given by Equation (2.12). As a consequence, we have the following theorem.

Theorem 2.2. Let $\tilde{f}(t)$ and $f(t)$ satisfy Condition 1.1(iii), so that they are given by Equation (1.12), and

$$\begin{aligned} {}_R D_{\alpha}^{-1} G_{K,l,l}(t; 0) &= \sum_{k=0}^{\infty} \frac{(a - l)_k b_k}{(c - l)_k (k - l + 1 + \alpha)} \\ {}_t G_{K,l,l}(t; 0) &= \sum_{l=1}^{\infty} \frac{d_l {}_R D_{\alpha}^{-1} G_{K,l,0}(t; 0)}{(c - l)_k (k - l + 1 + \alpha)} \end{aligned} \quad (2.14)$$

Then $\tilde{u}_f(t)$ and $u_f(t)$, given by

$$\begin{aligned} \tilde{u}_f(t) &= \sum_{l=1}^{\infty} c_l {}_R D_{\alpha}^{-1} G_{K,l,l}(t; 0); \\ u_f(t) &= \sum_{l=1}^{\infty} d_l {}_R D_{\alpha}^{-1} G_{K,l,0}(t; 0); \end{aligned} \quad (2.15)$$

are particular solutions of Equations (2.2) and (2.1), respectively. Condition $c - l = 2$, $Z < 1$ in Lemma 2.3 requires the condition $c - l = 2$, $Z < 1$ for all $l \geq 2$, in the present case.

Lemma 2.5. Lemmas 2.3 and 2.1 show that $w_c(t)$ and $u_c(t)$, given by

$$\begin{aligned} w_c(t) &:= K_{0,2}(t)H(t) = {}_R D_{\alpha}^{-1} K_{0,2}(t)H(t); \\ u_c(t) &= {}_R D_{\alpha}^{-1} w_c(t) = K_{0,2}(t)H(t); \end{aligned} \quad (2.16)$$

are a complementary solution of Equations (2.4) and (2.1), respectively.

Remark 2.2. We now give a derivation of Equation (2.16) for $\alpha = 0$, by modifying the above mentioned proof of Equation (45) given in [6]. We assume that the solution of Equation (2.1) is expressed by Equation (52-c), which is obtained from Equation (52) in [6] by replacing \tilde{u} by u . Then Equation (2.1) is expressed by Equation (53-c), which is obtained from Equation (53) in [6] by replacing \tilde{u} by u , α by 0, and \tilde{f} by f . We then note that when $f(t) = 0$, Equation (53-c) is satisfied

by Equation (55-c), which is obtained from Equation (55) in [6] by replacing α by 0. By using these in Equation (52-c) and putting $u(t) = p_0 u_c(t)$, we obtain Equation (46) in [6], which gives (2.16) for $\alpha = 0$.

3 Solution of the Hypergeometric Differential Equation

As stated in Introduction, solutions of the hypergeometric differential equation, are given in [7] without the Green's function. We now give them in terms of the Green's function.

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The hypergeometric differential equation is described by

$$p_H(t; \alpha, \beta, \gamma) u(t) = [t(1 - t) \frac{d^2}{dt^2} + (c - (a + b + 1)t) \frac{d}{dt} - ab] u(t) = f(t); \quad (3.1)$$

$$\text{where } a, b \text{ and } c \text{ are constants satisfying } a \neq 0 \text{ and } b \neq 0.$$

$\frac{d}{dt}$

$$[t(1 - t) \frac{d^2}{dt^2} + (c - (a + b + 1)t) \frac{d}{dt} - ab] u(t) = f(t); \quad (3.1)$$

where a, b and c are constants satisfying $a \neq 0$ and $b \neq 0$.

Lemma 3.1. When Condition 1.1(ii) is satisfied, we construct the following transformed differential equations of Equation (3.1), for $\tilde{u}(t) = \alpha D_{\alpha}^{-1} u(t)$

$$t \tilde{u}(t), \tilde{w}(t) = \alpha D_{\alpha}^{-1} u(t)$$

$$t \tilde{u}(t) \text{ and } w(t) = \alpha D_{\alpha}^{-1} u(t)$$

$$t \tilde{w}(t) =$$

$$\alpha D_{\alpha}^{-1} u(t)$$

$$t \tilde{u}(t):$$

$$\sim p_{H_{\alpha}}(t; \alpha, \beta, \gamma) \tilde{u}(t) = \alpha D_{\alpha}^{-1} f(t)$$

$$t \tilde{u}(t) = g_{\alpha}(t); \quad (3.2)$$

$$\sim p_{H_{\alpha}}(t; \alpha, \beta, \gamma) \tilde{w}(t) = \alpha D_{\alpha}^{-1} f(t); \quad (3.3)$$

$$\sim p_{H_{\alpha}}(t; \alpha, \beta, \gamma) w(t) = 0; \quad (3.4)$$

where the left hand side of Equation (3.2) is given by

$$\sim p_{H_{\alpha}}(t; \alpha, \beta, \gamma) = t(1 - t)$$

$\frac{d^2}{dt^2}$

$$[t(1 - t) \frac{d^2}{dt^2} + (c - (a + b + 1)t) \frac{d}{dt} - ab] \tilde{u}(t) = f(t); \quad (3.5)$$

$\frac{d}{dt}$

$$[t(1 - t) \frac{d^2}{dt^2} + (c - (a + b + 1)t) \frac{d}{dt} - ab] \tilde{u}(t) = f(t); \quad (3.5)$$

and then those of Equations (3.3) and (3.4) are obtained from Equation (3.5) by replacing α by $\alpha + \beta$ and α , respectively.

In obtaining these equations from Equation (3.1), we use Lemma 2.2, which is given in [7].

Lemma 3.2. Let $c - \alpha - \beta = 2Z < 1$. Then there exist two complementary solutions of Equation (3.4), which are given by

$$H_{\alpha,1}(t) = {}_2F_1(a - \alpha; b - \alpha; c - \alpha; t) =$$

$$\sum_{k=0}^{\infty} X_k$$

$$(a - \alpha)_k (b - \alpha)_k$$

$$k! (c - \alpha)_k$$

$$t^k; \quad t > 0; \quad (3.6)$$

$$H_{\alpha,2}(t) =$$

$$1$$

$$\alpha(2 - c - \alpha)$$

$$t \alpha (c - \alpha) {}_2F_1(1 + a - \alpha; 1 + b - \alpha; 2 - c - \alpha; t)$$

$$= \alpha D_{\alpha}^{-1} f(t)$$

$$t H_{\alpha,2}(t) H(t); \quad t > 0; \quad (3.7)$$

Following Equation (2.20) in [7], we define the Green's function $G_{H_{\alpha}}(t; \alpha)$, which satisfies

$$\sim p_{H_{\alpha}}(t; \alpha, \beta, \gamma) G_{H_{\alpha}}(t; \alpha) = \alpha(t - \alpha); \quad (3.8)$$

for $\alpha \in \mathbb{R}$.

In the present section, formulas are derived with the aid of two complementary solutions given by (3.6) and (3.7), and hence they hold when $c - \alpha - \beta = 2Z < 1$.

Lemma 3.3. Let $H_{\alpha,1}(t)$ be given by Equation (3.6). Then Lemmas 3.1 and 3.2 show that $G_{H_{\alpha}}(t; 0)$

and $G_{H_{\alpha}}(t; 0)$, given by

$$\sim w_{\alpha}(t) = G_{H_{\alpha}}(t; 0) = \alpha D_{\alpha}^{-1} f(t)$$

$$t G_{H_{\alpha}}(t; 0); \quad (3.9)$$

$$w_{\alpha}(t) = G_{H_{\alpha}}(t; 0) =$$

1

$${}_{\alpha}^1 1 + c {}_{\alpha}^1 H(t); \quad (3.10)$$

are a particular solution of Equation (3.8) for $\alpha = 0$, and also of Equation (3.3), and a complementary solution of Equation (3.4), respectively.

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Remark 3.1. The derivation of Equations (3.9) with (3.10) for $\alpha = 0$, with the aid of Frobenius method, is given in p. 53 of [7].

Theorem 3.1. Let Condition 1.1(ii) be satisfied, and ${}_{\alpha}^1 G_{H;\alpha}(t; 0)$ and ${}_{\alpha}^1 G_{H;\alpha;0}(t; 0)$ be given by Equation (3.9) with (3.10). Then Lemmas 3.3 and 3.1 show that Then

(i) when $\alpha \in \mathbb{C}$ and $f(t) = g_{\alpha}(t)$, $\sim u_{\alpha}(t)$, given by

$$\sim u_{\alpha}(t) = {}_{\alpha}^1 D_{\alpha}$$

$${}_{\alpha}^1 G_{H;\alpha}(t; 0) =$$

1

$${}_{\alpha}^1 1 + c {}_{\alpha}^1$$

$${}_{\alpha}^1 X$$

$$k=0$$

$$(a - \alpha)_k (b - \alpha)_k$$

$$(c - \alpha)_k \Gamma(k - \alpha + 1)$$

$$t_{k-\alpha} H(t); \quad (3.11)$$

is a particular solution of Equation (3.2),

(ii) if $\alpha = n \in \mathbb{Z}_{>0}$, $u_{\alpha}(t) = {}_{\alpha}^1 D_{\alpha}$

$\sim u_{\alpha}(t)$, expressed by

$$u_{\alpha}(t) =$$

$${}_{\alpha}^1 X$$

$$k=n$$

$$(a - n)_k (b - n)_k$$

$$(\alpha + c - n)_{k+1}$$

1

$$(k - n)!$$

$$t_{k-n} H(t)$$

=

$${}_{\alpha}^1 X$$

$$l=0$$

$$(a - n)_{n+l} (b - n)_{n+l}$$

$$(\alpha + c - n)_{n+l+1}$$

1

$$l!$$

$$t_l H(t) = C_n G_{H;0;0}(t; 0); \quad (3.12)$$

is a complementary solution of Equation (3.1), where $C_n = (a - n)_n (b - n)_n$

$$(\alpha + c - n)_{n+1}$$

$$= (a - n)_n (b - n)_n (c - n)$$

$$(a - n)_n (b - n)_n (c),$$

and

(iii) if $\alpha \in \mathbb{Z}_{>0}$, $u_{\alpha}(t)$, given by $u_{\alpha}(t) = {}_{\alpha}^1 D_{\alpha}$

$$\sim u_{\alpha}(t) = {}_{\alpha}^1 D_{\alpha}$$

${}_{\alpha}^1 G_{H;\alpha;0}(t; 0)$ with the aid of

Equation (3.11), is a particular solution of Equation (3.1).

Theorem 3.2. Let $\sim f(t)$ and $f(t)$ satisfy Condition 1.1(iii), so that they are given by Equation (1.12),

and

$${}_{\alpha}^1 D_{\alpha}$$

$${}_{\alpha}^1 G_{H;\alpha}(t; 0) =$$

1

$${}_{\alpha}^1 1 + c {}_{\alpha}^1$$

$${}_{\alpha}^1 X$$

$$k=0$$

$$(a - \alpha)_k (b - \alpha)_k$$

$$(c - \alpha)_k \Gamma(k - \alpha + 1)$$

$$t_{k-\alpha} H(t); \quad (3.13)$$

Then $\sim u_{\alpha}(t)$ and $u_{\alpha}(t)$, given by

$$\sim u_{\alpha}(t) =$$

$${}_{\alpha}^1 X$$

$$\sum_{l=1}^{\infty} c_l {}_R D_{-}^l u_f(t) = \sum_{l=1}^{\infty} d_l {}_R D_{-}^l u_f(t); \quad (3.14)$$

are particular solutions of Equations (3.2) and (3.1), respectively. The condition, which corresponds to Condition $c_{-} = 2 Z < 1$ in Lemma 2.3, requires the condition $c_{-} = 2 Z < 1$ for all $l \geq 2 Z > 0$, in the present case.

Lemma 3.4. Lemmas 3.1 and 3.2 show that $u_c(t)$, given by

$$u_c(t) = H_{0,2}(t)H(t); \quad (3.15)$$

is a complementary solution of Equation (3.1).

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4 Conclusion

In the preceding paper [7], we consider the problem of solving the equation which has a special class of inhomogeneous part, and we adopt a recipe without the Green's function, which is applied to the hypergeometric differential equation, the differential equations treated in [5] and the Hermite differential equation.

In the present paper, we solve the problem considered in [7], but now the results are expressed in terms of the Green's function. It is applied to Kummer's and the hypergeometric differential equation. In Section 2, the results for Kummer's differential equation, in which Condition 1.1(ii) is satisfied, are given in Theorem 2.1 and Lemma 2.5, and the results, in which Condition 1.1(iii) is satisfied, are given in Theorem 2.2. Here Condition 1.1(i) is treated as a special case of Condition 1.1(ii).

In Section 3, the corresponding results for the hypergeometric differential equation, are given in Theorems 3.1 and 3.2 and Lemma 3.4.

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