Solution of Inhomogeneous Differential Equations with Polynomial Coefficients in Nonstandard Analysis, in Terms of the Green's Function

Abstract

Discussions are presented by Morita and Sato on the problem of obtaining the particular solution of an inhomogeneous differential equation with polynomial coefficients in terms of the Green's function. In a preceding paper, solution is given without using the Green's function, on the basis of nonstandard analysis, for a restricted class of inhomogeneous terms. In the present paper, the corresponding solutions are given in terms of the Green's function. It is applied to Kummer's and the hypergeometric differential equation.

Keywords: Green's function; differential equations with polynomial coefficients; nonstandard analysis; Kummer's differential equation; hypergeometric differential equation

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1 Introduction

In the present paper, we treat the problem of obtaining the particular solutions of a differential equation with polynomial coefficients in terms of the Green's function.

In a preceding paper [4], this problem is studied in the framework of distribution theory, where the method is applied to Kummer's and the hypergeometric differential equation. In another paper [5], this problem is studied in the framework of nonstandard analysis, where a recipe of solution of the present problem is presented, and it is applied to a simple fractional and a first-order ordinary differential equation.

In a recent paper [6], a compact recipe based on nonstandard analysis, which is obtained by revising the one given in [5], is presented, and is applied to Kummer's differential equation.

In the preceding paper [7], we consider the problem of solving the equation which has a special class of inhomogeneous part, and we adopt a recipe without the Green's functuion, which is applied

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to the hypergeometric differential equation, the differential equations treated in [5] and the Hermite differential equation.

In the present paper, we solve the problem considered in [7], but now the results are expressed in terms of the Green's function. It is applied to Kummer's and the hypergeometric differential equation.

The presentation in this paper follows those in [4, 5, 6], in Introduction and in many descriptions in the following sections.

We consider a fractional differential equation, which takes the form:

$$p_n(t, {}_RD_t)u(t) = \sum_{l=0}^{n} a_l(t) {}_RD_t^{\rho_l} u(t) = f(t), \tag{1.1}$$

where (i) $n \in \mathbb{Z}_{>-1}$, $t \in \mathbb{R}$, (ii) $a_l(t)$ for $l \in \mathbb{Z}_{[0,n]}$ are polynomials of t, (iii) $\rho_l \in \mathbb{C}$ for $l \in \mathbb{Z}_{[0,n]}$ satisfy $\mathrm{Re}\ \rho_0 > \mathrm{Re}\ \rho_1 \geq \cdots \geq \mathrm{Re}\ \rho_n$ and $\mathrm{Re}\ \rho_0 > 0$.

Here $\mathbb{Z}$ is the set of all integers, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all real numbers and all complex numbers, respectively, and $\mathbb{Z}_{>a} = \{n \in \mathbb{Z} \mid n > a\}$, $\mathbb{Z}_{<b} = \{n \in \mathbb{Z} \mid n < b\}$ and $\mathbb{Z}_{[a,b]} = \{n \in \mathbb{Z} \mid a \leq n \leq b\}$ for $a, b \in \mathbb{Z}$ satisfying $a < b$. We also use $\mathbb{R}_{>a} = \{x \in \mathbb{R} \mid x > a\}$ for $a \in \mathbb{R}$, and $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Re}\ z > 0\}$.

We use Heaviside's step function $H(t)$, which is equal to 1 if $t > 0$ and, to 0 if $t \leq 0$. Here ${}_RD_t^{\rho_l}$ are the Riemann-Liouville fractional integrals and derivatives defined by the following definition; see [3, 8].

Definition 1.1. *Let* $t \in \mathbb{R}$, $\tau \in \mathbb{R}$, $u_0(t)$ *be locally integrable on* $\mathbb{R}_{>\tau}$, $u(t) = u_0(t)H(t-\tau)$, $\lambda \in \mathbb{C}_+$, $n \in \mathbb{Z}_{>-1}$ *and* $\rho = n - \lambda$. *Then* ${}_RD_t^{-\lambda}u(t)$ *is the Riemann-Liouville fractional integral defined by*

$$\begin{aligned} {}_RD_t^{-\lambda}u(t) &= \frac{1}{\Gamma(\lambda)} \int_{-\infty}^{t} (t-x)^{\lambda-1} u_0(x) H(x-\tau) dx \\ &= \frac{1}{\Gamma(\lambda)} \int_{\tau}^{t} (t-x)^{\lambda-1} u_0(x) dx \cdot H(t-\tau), \end{aligned} \tag{1.2}$$

and ${}_RD_t^{-\lambda}u(t) = 0$ *for* $t \leq \tau$, *where* $\Gamma(\lambda)$ *is the gamma function,* ${}_RD_t^{\rho}u(t) = {}_RD_t^{n-\lambda}u(t)$ *is the Riemann-Liouville fractional derivative defined by*

$$ {}_RD_t^{\rho}u(t) = {}_RD_t^{n-\lambda}u(t) = \frac{d^n}{dt^n}[{}_RD_t^{-\lambda}u_0(t)] \cdot H(t-\tau), \tag{1.3}$$

when $n \geq \mathrm{Re}\ \lambda$, *and* ${}_RD_t^{n}u(t) = \frac{d^n}{dt^n}u_0(t) \cdot H(t-\tau)$ *when* $\rho = n \in \mathbb{Z}_{>-1}$.

In accordance with Definition 1.1, when $u_0(t) = \frac{1}{\Gamma(\nu)}(t-\tau)^{\nu-1}$, we adopt

$$ {}_RD_t^{\rho} \frac{(t-\tau)^{\nu-1}}{\Gamma(\nu)} H(t-\tau) = \begin{cases} \frac{(t-\tau)^{\nu-\rho-1}}{\Gamma(\nu-\rho)} H(t-\tau), & \nu - \rho \in \mathbb{C} \backslash \mathbb{Z}_{<1}, \\ 0, & \nu - \rho \in \mathbb{Z}_{<1}, \end{cases} \tag{1.4}$$

for $\nu \in \mathbb{C} \backslash \mathbb{Z}_{<1}$ and $\tau \in \mathbb{R}$. Here ${}_RD_t$ is used in place of usually used notation ${}_{\tau}D_R$, in order to show that the variable is t.

Remark 1.1. Let $g_\nu(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1} H(t)$ for $\nu \in \mathbb{C}$. Then $g_\nu(t) = 0$ if $\nu \in \mathbb{Z}_{<1}$, and Equation (1.4) shows that if $\nu \notin \mathbb{Z}_{<1}$, ${}_RD_t^{\rho} g_\nu(t) = g_{\nu-\rho}(t)$. As a consequence, we have ${}_RD_t^{\nu+n} g_\nu(t) = g_{-n}(t) = 0$ for $n \in \mathbb{Z}_{>-1}$.

In distribution theory [4, 9, 2, 10], we use distribution $\tilde{H}(t)$, which corresponds to function $H(t)$, differential operator D, and distribution $\delta(t) = D\tilde{H}(t)$, which is called Dirac's delta function.

1.1 Preliminaries on Nonstandard Analysis

In nonstandard analysis [1], where infinitesimal numbers appear. We denote the set of all infinitesimal real numbers by $\mathbb{R}^0$. We also use $\mathbb{R}^0_{>0} = \{\epsilon \in \mathbb{R}^0 \mid \epsilon > 0\}$, which is such that if $\epsilon \in \mathbb{R}^0_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ satisfying $\epsilon < \frac{1}{N}$. We use $\mathbb{R}^{ns}$, which has subsets $\mathbb{R}$ and $\mathbb{R}^0$. If $x \in \mathbb{R}^{ns}$ and $x \notin \mathbb{R}$, x is expressed as $x_1 + \epsilon$ by $x_1 \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^0$, where x_1 may be $0 \in \mathbb{R}$. Equation $x \simeq y$ for $x \in \mathbb{R}^{ns}$ and $y \in \mathbb{R}^{ns}$, is used, when $x - y \in \mathbb{R}^0$. We denote the set of all infinitesimal complex numbers by $\mathbb{C}^0$, which is the set of complex numbers z which satisfy $|\mathrm{Re}\, z| + |\mathrm{Im}\, z| \in \mathbb{R}^0$. We use $\mathbb{C}^{ns}$, which has subsets $\mathbb{C}$ and $\mathbb{C}^0$. If $z \in \mathbb{C}^{ns}$ and $z \notin \mathbb{C}$, z is expressed as $z_1 + \epsilon$ by $z_1 \in \mathbb{C}$ and $\epsilon \in \mathbb{C}^0$, where z_1 may be $0 \in \mathbb{C}$.

In place of (1.4), we now use

$$ {}_RD_t^\rho g_{\nu+\epsilon}(t) = {}_RD_t^\rho \frac{1}{\Gamma(\nu+\epsilon)} t^{\nu-1+\epsilon} H(t) = g_{\nu-\rho+\epsilon}(t) = \frac{1}{\Gamma(\nu-\rho+\epsilon)} t^{\nu-\rho-1+\epsilon} H(t), \tag{1.5} $$

for all $\rho \in \mathbb{C}$ and $\nu \in \mathbb{C}$, where $\epsilon \in \mathbb{R}^0_{>0}$.

Lemma 1.1. *Let* $\rho_1 \in \mathbb{C}$, $\rho_2 \in \mathbb{C}$, $\nu \in \mathbb{C}$, $\epsilon \in \mathbb{R}^0_{>0}$ *and* $g_{\nu+\epsilon}(t) = \frac{1}{\Gamma(\nu+\epsilon)} t^{\nu+\epsilon-1} H(t)$. *Then the index law:*

$$ {}_RD_t^{\rho_1}\, {}_RD_t^{\rho_2} g_{\nu+\epsilon}(t) = {}_RD_t^{\rho_1+\rho_2} g_{\nu+\epsilon}(t) = g_{\nu-\rho_1-\rho_2+\epsilon}(t), \tag{1.6} $$

always holds.

In the present study in nonstandard analysis, in place of $\tilde{H}(t)$ and $\delta(t)$ in distribution theory, $H_\epsilon(t)$ and $\delta_\epsilon(t)$ are used, which are given by

$$ H_\epsilon(t) = {}_RD_t^{-\epsilon} H(t) = g_{1+\epsilon}(t) = \frac{1}{\Gamma(\epsilon+1)} t^\epsilon H(t), \quad \delta_\epsilon(t) = \frac{d}{dt} H_\epsilon(t), \tag{1.7} $$

for $\epsilon \in \mathbb{R}^0_{>0}$. We note that they tend to $H(t)$ and 0, respectively, in the limit of $\epsilon \to 0$.

Lemma 1.2. *In the notation in Remark 1.1,* $H_\epsilon(t) = g_{1+\epsilon}(t)$, $\delta_\epsilon(t) = g_\epsilon(t)$, *and*

$$ {}_RD_t^\epsilon H_\epsilon(t) = {}_RD_t^\epsilon g_{1+\epsilon}(t) = g_1(t) = H(t), \quad {}_RD_t^\epsilon \delta_\epsilon(t) = {}_RD_t^\epsilon g_\epsilon(t) = g_0(t) = 0. \tag{1.8} $$

1.2 Summary of the Following Sections

In Section 2, a recipe of solution of Equation (1.1), in nonstandard analysis, is presented. We there consider the solution of the following equation for $\tilde{u}(t)$:

$$ \tilde{p}_{n,\epsilon}(t, {}_RD_t)\tilde{u}(t) = \tilde{f}(t), \tag{1.9} $$

where $\epsilon \in \mathbb{R}^0_{>0}$ and

$$ \tilde{p}_{n,\epsilon}(t, {}_RD_t) := {}_RD_t^{-\epsilon} p_n(t, {}_RD_t)\, {}_RD_t^\epsilon. \tag{1.10} $$

In [6], Condition 2 on p. 383 is adopted, where the inhomogeneous terms $f(t)$ and $\tilde{f}(t)$ are assumed to satisfy one of the following four conditions.

Condition 1.1. *Let* $\epsilon \in \mathbb{R}^0_{>0}$ *and* $\beta \in \mathbb{C}$.

(i) $f(t) = f_0(t)H(t)$ *and* $\tilde{f}(t) = {}_R D_t^{-\epsilon} f(t) + c_\epsilon \delta_\epsilon(t)$*, where* $f_0(t)$ *is locally integrable on* $\mathbb{R}_{>0}$ *and* c_ϵ *is a constant.*

(ii) $f(t) = {}_R D_t^{\beta} f_\beta(t)$ *and* $\tilde{f}(t) = {}_R D_t^{\beta} \tilde{f}_\beta(t)$*, where*

$$\tilde{f}_\beta(t) = {}_R D_t^{-\epsilon} f_\beta(t) + c_{\beta,\epsilon}\delta_\epsilon(t), \quad f_\beta(t) = f_{\beta,0}(t)H(t), \tag{1.11}$$

$f_{\beta,0}(t)$ *is locally integrable on* $\mathbb{R}_{>0}$*, and* $c_{\beta,\epsilon}$ *is a constant.*

(iii) $\tilde{f}(t) = {}_R D_t^{\beta} \tilde{f}_\beta(t)$*, where* $\tilde{f}_\beta(t) = {}_R D_t H_\epsilon(t) = \delta_\epsilon(t)$*. When* $\beta \in \mathbb{Z}_{>-1}$*,* $f(t) = 0$*, and when* $\beta \notin \mathbb{Z}_{>-1}$*,* $f(t) = {}_R D_t^{\beta+1} H(t)$*.*

(iv) $\tilde{f}(t)$ *and* $f(t)$ *are expressed as follows:*

$$\tilde{f}(t) = \sum_{l=1}^{\infty} c_l \cdot {}_R D_t^{\beta_l} \delta_\epsilon(t) = \sum_{l=1}^{\infty} c_l \cdot \frac{t^{\epsilon-1-\beta_l}}{\Gamma(\epsilon-\beta_l)} H(t), \quad f(t) = \sum_{l=1}^{\infty} d_l \cdot {}_R D_t^{\beta_l+1} H(t), \tag{1.12}$$

respectively, where $c_l \in \mathbb{C}$ *are constants,* $\beta_l \in \mathbb{C}$ *satisfy* $-\mathrm{Re}\,\beta_l \geq -\mathrm{Re}\,\beta_1 \in \mathbb{R}$*, for all* $l \in \mathbb{Z}_{>0}$*, and* $d_l = c_l$ *if* $\beta_l \notin \mathbb{Z}_{>-1}$*, and* $d_l = 0$ *if* $\beta_l \in \mathbb{Z}_{>-1}$*.*

In [7], Conditions 1.2 and 1.1 on p. 52, are adopted. It is

Condition 1.2. *Let* $\epsilon \in \mathbb{R}^0_{>0}$ *and* $\beta \in \mathbb{C}$.

(i) $\tilde{f}(t) = \delta_\epsilon(t)$ *and* $f(t) = 0$.

(ii) $\tilde{f}_\beta(t) = {}_R D_t H_\epsilon(t) = \delta_\epsilon(t)$*, and* $\tilde{f}(t) = {}_R D_t^{\beta} \tilde{f}_\beta(t) = g_{\epsilon-\beta}(t)$*. When* $\beta \notin \mathbb{Z}_{>-1}$*,* $f(t) = {}_R D_t^{\beta+1} H(t)$*, and when* $\beta \in \mathbb{Z}_{>-1}$*,* $f(t) = 0$*.*

(iii) $\tilde{f}(t)$ *and* $f(t)$ *are expressed as follows:*

$$\tilde{f}(t) = \sum_{l=1}^{\infty} c_l \cdot {}_R D_t^{\beta_l} \delta_\epsilon(t) = \sum_{l=1}^{\infty} c_l \cdot g_{\Gamma(\epsilon-\beta_l)}(t), \quad f(t) = \sum_{l=1}^{\infty} d_l \cdot {}_R D_t^{\beta_l} \delta_\epsilon(t). \tag{1.13}$$

respectively, where $c_l \in \mathbb{C}$ *are constants,* $\beta_l \in \mathbb{C}$ *satisfy* $-\mathrm{Re}\,\beta_l \geq -\mathrm{Re}\,\beta_1 \in \mathbb{R}$*, for all* $l \in \mathbb{Z}_{>0}$*, and* $d_l = c_l$ *if* $\beta_l \notin \mathbb{Z}_{>-1}$*, and* $d_l = 0$ *if* $\beta_l \in \mathbb{Z}_{>-1}$*.*

Remark 1.2. Lemma 1.2 shows that when Condition 1.2(i) is satisfied, ${}_R D_t^{\epsilon} \tilde{f}(t) = f(t)$, and $\tilde{f}(t) = {}_R D_t^{-\epsilon} f(t)$ does not always hold, and when Condition 1.2(ii) is satisfied, ${}_R D_t^{\epsilon} \tilde{f}_\beta(t) = 0$.

In Sections 3 and 4, full expressions of the Green's functions and the solutions, are derived along the recipe given in Section 2, for Kummer's differential equation, and in Sections 5, the corresponding results are given for the hypergeometric differential equation.

Section 6 is for Conclusion.

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2 Recipe of Solution of Differential Equation, in Nonstandard Analysis

In obtaining a particular solution of Equation (1.1) for $\tilde{f}(t)$ satisfying Condition 1.2(i), we use it defined in the following definition.

Definition 2.1. *Let* $\tilde{p}_{n,\epsilon}(t, {}_RD_t)$ *be given by Equation* (1.10)*. Then for* $\epsilon \in \mathbb{R}^0_{>0}$ *and* $\tau \in \mathbb{R}$*, the Green's function* $G_\epsilon(t,\tau)$ *for Equation* (1.1) *satisfies*

$$\tilde{p}_{n,\epsilon}(t, {}_RD_t)G_\epsilon(t,\tau) = \delta_\epsilon(t-\tau). \tag{2.1}$$

Lemma 2.1. *Let* $G_\epsilon(t,\tau)$ *be defined as in Definition 2.1, and* $G_0(t,\tau) := {}_RD_t^\epsilon G_\epsilon(t,\tau)$*. Then* $G_0(t,\tau)$ *is a complementary solution of Equation* (1.1) *on* $\mathbb{R}_{>\tau}$*, and* ${}_RD_t^{-1}p_n(t, {}_RD_t)G_0(t,\tau) = 1$ *at any value of* t *satisfying* $t > \tau$*.*

Proof. These are confirmed by applying ${}_RD_t^\epsilon$ and ${}_RD_t^{-1+\epsilon}$ to Equation (2.1), by noting Lemma 1.2. ☐

Theorem 2.1. *Let Condition 1.2(i) be satisfied,* $G_\epsilon(t,\tau)$ *and* $G_0(t,\tau)$ *be given as in Lemma 2.1. Then* $\tilde{u}_\epsilon(t)$ *given by*

$$\tilde{u}_\epsilon(t) = G_\epsilon(t,0), \tag{2.2}$$

is the particular solution of Equation (1.9) *for the term* $\tilde{f}(t) = \delta_\epsilon(t)$*, and* $u_0(t)$ *given by*

$$u_0(t) == G_0(t,0), \tag{2.3}$$

is a complementary solution of Equation (1.1).

Proof. By using Equations (2.2) and (2.1), we obtain

$$\tilde{p}_{n,\epsilon}(t, {}_RD_t)\tilde{u}_\epsilon(t) = \tilde{p}_{n,\epsilon}(t, {}_RD_t)G_\epsilon(t,0) = \delta_\epsilon(t), \tag{2.4}$$

which is a proof for $\tilde{u}_\epsilon(t)$. ☐

3 Solution of Kummer's Differential Equation, I

Kummer's differential equation is described by

$$p_K(t, {}_RD_t)u(t) := [t\frac{d^2}{dt^2} + (c-bt)\frac{d}{dt} - ab]u(t) = f(t), \tag{3.1}$$

where a, b and c are constants satifying $a \neq 0$ and $b \neq 0$.

Lemma 3.1. *When Condition 1.2(i) is satisfied, we construct the following transformation of Equation* (3.1) *for* $\tilde{u}(t) = {}_RD_t^{-\epsilon}u(t)$ *and* $\tilde{f}(t) = {}_RD_t^{-\epsilon}f(t)$*:*

$$\begin{aligned}\tilde{p}_{K,\epsilon}(t, {}_RD_t)\tilde{u}(t) &:= {}_RD_t^{-\epsilon}p_K(t, {}_RD_t){}_RD_t^{\epsilon}\tilde{u}(t) = {}_RD_t^{-\epsilon}[t\frac{d^2}{dt^2} + (c - bt)\frac{d}{dt} - ab]{}_RD_t^{\epsilon}\tilde{u}(t) \\ &= [t\frac{d^2}{dt^2} + (c - \epsilon - bt)\frac{d}{dt} - (a - \epsilon)b]\tilde{u}(t) = \tilde{f}(t). \qquad (3.2)\end{aligned}$$

In obtaining this equation from Equation (3.1), we use the following lemma given in [7].

Lemma 3.2. *Let* $\lambda \in \mathbb{C}_+$, $m \in \mathbb{Z}_{>-1}$ *and* $\rho = m - \lambda$. *Then*

$$\begin{aligned}{}_RD_t^{\rho}[tu(t)] &= t \cdot {}_RD_t^{\rho}u(t) + \rho \cdot {}_RD_t^{\rho-1}u(t), &(3.3)\\ {}_RD_t^{\rho}[t^2u(t)] &= t^2 \cdot {}_RD_t^{\rho}u(t) + 2\rho t \cdot {}_RD_t^{\rho-1}u(t) + \rho(\rho - 1) \cdot {}_RD_t^{\rho-2}u(t). &(3.4)\end{aligned}$$

In accordance with Definition 2.1, we define the Green's function $G_{K,\epsilon}(t, \tau)$, which satisfies

$$\tilde{p}_{K,\epsilon}(t, {}_RD_t)G_{K,\epsilon}(t, \tau) = \delta_\epsilon(t - \tau), \qquad (3.5)$$

for $\tau \in \mathbb{R}$. The solutions of Equations (3.2) and (3.1) are then given with the aid of Theorem 2.1 and the following lemma.

Lemma 3.3. *Two complementary solutions of Equation* (3.1)*, given by*

$$K_1(t) = {}_1F_1(a; c; bt) := \sum_{k=0}^{\infty}\frac{(a)_k b^k}{k!(c)_k}t^k, \quad t > 0, \qquad (3.6)$$

$$K_2(t) = \frac{1}{\Gamma(2 - c)}t^{1-c} \cdot {}_1F_1(a - c + 1; 2 - c; bt) = \sum_{k=0}^{\infty}\frac{(a - c + 1)_k b^k}{k!\Gamma(2 - c + k)}t^{1-c+k}, \quad t > 0, \qquad (3.7)$$

exist, when $c \notin \mathbb{Z}_{<1}$ *and* $c \notin \mathbb{Z}_{>1}$*, respectively. Here* $(a)_k$ *for* $k \in \mathbb{Z}_{>0}$ *and* $k = 0$*, denote* $(a)_k = \prod_{l=0}^{k-1}(a + l) = \frac{\Gamma(a+k)}{\Gamma(a)}$ *and* $(a)_0 = 1$*, respectively.*

In the present paper, these equations are proved in Remark 3.1 and Lemma 3.4 given below.

Theorem 3.1. *Let* $\tilde{f}(t)$ *satisfy Condition 1.2(i), and* $K_1(t)$ *be given by* (3.6)*. Then* $\tilde{u}_\epsilon(t)$ *and* $G_{K,\epsilon}(t, 0)$*, given by*

$$\tilde{u}_\epsilon(t) = G_{K,\epsilon}(t, 0) = \frac{1}{-1 + c}\sum_{k=0}^{\infty}\frac{(a)_k b^k}{(c)_k\Gamma(k + \epsilon + 1)}t^{k+\epsilon}H(t), \qquad (3.8)$$

are particular solutions of Equations (3.2) *and* (3.5) *for* $\tau = 0$*, and* $u_0(t)$ *and* $G_{K,0}(t, 0)$*, given by*

$$u_0(t) = G_{K,0}(t, 0) = {}_RD_t^{\epsilon}G_{K,\epsilon}(t, 0) = \frac{1}{-1 + c}K_1(t)H(t), \qquad (3.9)$$

are complementary solutions of Equation (3.1)*.*

This result is derived with the aid of the complementary solutions given by Equations (3.6) *and* (3.7)*, and hence by assuming* $c \notin \mathbb{Z}_{<1}$*.*

Remark 3.1. The derivation of Equation (3.8) with the aid of Frobenius method, is given in Section 3.1 of [6], and then the statement for $G_{K,0}(t,0)$ is due to Lemma 2.1.

Lemma 3.4. *Let* $K_2(t)$ *be given by* (3.7)*. Then* $u_c(t)$*, given by*

$$u_c(t) = K_2(t)H(t), \tag{3.10}$$

is a complementary solution of Equation (3.1).

Remark 3.2. We now give a derivation of Equation (3.10), by modifying the above mentioned proof of Equation (3.8) given in [6]. We assume that the solution of Equation (3.1) is expressed by Equation (52-c), which is obtained from Equation (52) in [6] by replacing $\tilde{u}$ by u. Then Equation (3.1) is expressed by Equation (53-c), which is obtained from Equation (53) in [6] by replacing $\tilde{u}$ by u, ϵ by 0, and $\tilde{f}$ by f. We then note that when $f(t) = 0$, Equation (53-c) is satisfied by Equation (55-c), which is obtained from Equation (55) in [6] by replacing ϵ by 0. By using these in Equation (52-c) and putting $u(t) = p_0 u_c(t)$, we obtain Equation (3.10).

4 Solution of Kummer's Differential Equation, II

Lemma 4.1. *When Condition 1.2(ii) is satisfied, we construct the following transformed differential equations of Equations* (3.1) *and* (3.2)*, for* $w(t) = {}_RD_t^{-\beta}u(t)$ *and* $\tilde{w}(t) = {}_RD_t^{-\beta}\tilde{u}(t)$*, respectively, with the aid of Lemma 3.2:*

$$\begin{aligned}\tilde{p}_{K,\beta}(t,{}_RD_t)w(t) &:= {}_RD_t^{-\beta}p_K(t,{}_RD_t){}_RD_t^{\beta}w(t)\\ &= [t\frac{d^2}{dt^2} + (c-\beta-bt)\frac{d}{dt} - (a-\beta)b]w(t) = 0,\end{aligned} \tag{4.1}$$

$$\begin{aligned}\tilde{p}_{K,\beta+\epsilon}(t,{}_RD_t)\tilde{w}(t) &:= {}_RD_t^{-\beta}p_{K,\epsilon}(t,{}_RD_t){}_RD_t^{\beta}\tilde{w}(t)\\ &= [t\frac{d^2}{dt^2} + (c-\beta-\epsilon-bt)\frac{d}{dt} - (a-\beta-\epsilon)b]\tilde{w}(t) = \delta_\epsilon(t).\end{aligned} \tag{4.2}$$

Remark 4.1. In this section, we consider Equations (4.1) and (4.2) in place of Equations (3.1) and (3.2), respectively, and hence the equations in this section are obtained from the corresponding equations in Section 3, by replacing c by $c-\beta$, a by $a-\beta$, f by f_β, $\tilde{f}$ by $\tilde{f}_\beta$, u by w, and $\tilde{u}$ by $\tilde{w}$. They will be given without derivation.

Lemma 4.2. *Lemma 3.3 and Remark 4.1 show that two complementary solutions of Equation* (4.1)*, which are given by*

$$K_{\beta,1}(t) = {}_1F_1(a-\beta;c-\beta;bt) = \sum_{k=0}^{\infty}\frac{(a-\beta)_k b^k}{k!(c-\beta)_k}t^k, \quad t>0, \tag{4.3}$$

$$\begin{aligned}K_{\beta,2}(t) &= \frac{1}{\Gamma(2-c+\beta)}t^{1-c+\beta}\cdot {}_1F_1(a-c+1;2-c+\beta;bt)\\ &= \sum_{k=0}^{\infty}\frac{(a-c+1)_k b^k}{k!\Gamma(2-c+\beta+k)}t^{1-c+\beta+k} = {}_RD_t^{-\beta}K_2(t)H(t), \quad t>0,\end{aligned} \tag{4.4}$$

exist, when $c-\beta \notin \mathbb{Z}_{<1}$ *and when* $c-\beta \notin \mathbb{Z}_{>1}$*, respectively.*

Corresponding to Equation (3.5), we define the Green's function $G_{K,\beta,\epsilon}(t,\tau)$, which satisfies

$$\tilde{p}_{K,\beta+\epsilon}(t, {}_RD_t)G_{K,\beta,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau), \tag{4.5}$$

for $\tau \in \mathbb{R}$. The solutions of Equations (4.2), (4.1), (3.2) and (3.1) are then given with the aid of Theorem 3.1 and Lemma 3.4.

Remark 4.2. Equation (4.5) is obtained from Equation (3.5), by replacing c by $c-\beta$, a by $a-\beta$, and $G_{K,\epsilon}$ by $G_{K,\beta,\epsilon}$.

In the present section, formulas are derived with the aid of two complementary solutions given by (4.3) and (4.4), and hence they hold when $c-\beta \notin \mathbb{Z}_{<1}$.

Lemma 4.3. *Let* $K_{\beta,1}(t)$ *be given by Equation* (4.3)*. Then Lemma 3.1, Remark 4.1 and Lemmas 4.2 and 2.1 show that* $G_{K,\beta,\epsilon}(t,0)$ *and* $G_{K,\beta,0}(t,0)$*, given by*

$$\tilde{w}_{\beta,\epsilon}(t) = G_{K,\beta,\epsilon}(t,0) = {}_RD_t^{-\epsilon}G_{K,\beta,0}(t,0), \quad w_{\beta,0}(t) = G_{K,\beta,0}(t,0) = \frac{1}{-1+c-\beta}K_{\beta,1}(t)H(t), \tag{4.6}$$

are a particular solution of Equation (4.5) *for* $\tau = 0$*, and a complementary solution of Equation* (4.1)*, respectively.*

With the aid of Remark 4.2, we have the following lemma for $G_{K,\beta,\epsilon}(t,\tau)$ for $\tau > 0$.

Lemma 4.4. *The lemma, which is obtained from Lemma 3.1 by replacing* K_1 *by* $K_{\beta,1}$*, Lemma 3.3 by Lemma 4.2,* K_2 *by* $K_{\beta,2}$*,* $G_{K,\epsilon}$ *by* $G_{K,\beta,\epsilon}$*, and* $G_{K,0}$ *by* $G_{K,\beta,0}$*, holds.*

Theorem 4.1. *Let Condition 1.2(ii) be satisfied, and* $G_{K,\beta,\epsilon}(t,0)$ *and* $G_{K.0}(t,0)$ *be given in Equations* (4.6) *and* (3.9)*, respectively. Then Lemmas 4.2 and 3.1 show that*

(i) $\tilde{u}_{\epsilon-\beta}(t) = {}_RD_t^\beta \tilde{w}_{\beta,\epsilon}(t)$*, given by*

$$\tilde{u}_{\epsilon-\beta}(t) = {}_RD_t^\beta G_{K,\beta,\epsilon}(t,0) = \frac{1}{-1+c-\beta}\sum_{k=0}^{\infty}\frac{(a-\beta)_k b^k}{(c-\beta)_k\Gamma(k-\beta+1+\epsilon)}t^{k-\beta+\epsilon}H(t), \tag{4.7}$$

is a particular solution of Equation (3.2)*,*

(ii) *if* $\beta = n \in \mathbb{Z}_{>-1}$*,* $u_{-n}(t) = {}_RD_t^\epsilon \tilde{u}_{\epsilon-n}(t)$*, expressed by*

$$\begin{aligned} u_{-n}(t) &= \sum_{k=n}^{\infty}\frac{(a-n)_k b^k}{(-1+c-n)_{k+1}}\frac{1}{(k-n)!}t^{k-n}H(t) \\ &= \sum_{l=0}^{\infty}\frac{(a-n)_{n+l}b^{n+l}}{(-1+c-n)_{n+l+1}}\frac{1}{l!}t^l H(t) = C_n G_{K,0}(t,0), \end{aligned} \tag{4.8}$$

is a complementary solution of Equation (3.1)*, where* $C_n = \frac{(a-n)_n}{(-1+c-n)_{n+1}}b^n = \frac{\Gamma(a)\Gamma(c-n-1)}{\Gamma(a-n)\Gamma(c)}b^n$*, and*

(iii) *if* $\beta \notin \mathbb{Z}_{>-1}$*,* $u_{-\beta}(t)$*, given by* $u_{-\beta}(t) = {}_RD_t^\epsilon \tilde{u}_{\epsilon-\beta}(t) = {}_RD_t^\beta G_{K,\beta,0}(t,0)$ *with the aid of Equation* (4.7)*, is a particular solution of Equation* (3.1)*.*

Theorem 4.1 shows that if $\tilde{f}(t) = {}_RD_t^\beta \delta_\epsilon(t)$, the particular solution of Equation (3.2) is given by Equation (4.7). As a consequence, we have the following theorem.

Theorem 4.2. *Let $\tilde{f}(t)$ and $f(t)$ satisfy Condition 1.2(iii), so that they are given by Equation* (1.13)*, and*

$$ {}_RD_t^{\beta_l} G_{K,\beta_l,\epsilon}(t,0) = \frac{1}{-1+c-\beta_l}\sum_{k=0}^{\infty}\frac{(a-\beta_l)_k b^k}{(c-\beta_l)_k\Gamma(k-\beta_l+1+\epsilon)}t^{k-\beta_l+\epsilon}H(t). \tag{4.9}$$

Then $\tilde{u}_f(t)$ and $u_f(t)$, given by

$$\tilde{u}_f(t) = \sum_{l=1}^{\infty} c_l \cdot {}_RD_t^{\beta_l} G_{K,\beta_l,\epsilon}(t,0), \quad u_f(t) = \sum_{l=1}^{\infty} d_l \cdot {}_RD_t^{\beta_l} G_{K,\beta_l,0}(t,0), \tag{4.10}$$

are particular solutions of Equations (3.2) *and* (3.1)*, respectively. Condition $c-\beta \notin \mathbb{Z}_{<1}$ in Lemma 4.2 requires the condition $c-\beta_l \notin \mathbb{Z}_{<1}$ for all $l \in \mathbb{Z}_{>0}$, in the present case.*

Lemma 4.5. *Lemma 3.4 and Remark 4.1 show that $w_c(t)$, given by*

$$w_c(t) := K_{\beta,2}(t)H(t) = {}_RD_t^{-\beta} K_2(t)H(t), \tag{4.11}$$

is a complementary solution of Equation (4.1)*, and Lemma 4.1 shows that $u_c(t) = {}_RD_t^{\beta} w_c(t)$ is a complementary solution of Equation* (3.1)*, which is given in Lemma 3.4.*

5 Solution of the Hypergeometric Differential Equation

As stated in Introduction, solutions of the hypergeometric differential equation, are given in [7] without the Green's function. We now give them in terms of the Green's function.

The hypergeometric differential equation is described by

$$p_H(t, {}_RD_t)u(t) = [t(1-t)\frac{d^2}{dt^2} + (c-(a+b+1)t)\frac{d}{dt} - ab]u(t) = f(t), \tag{5.1}$$

where a, b and c are constants satifying $a \neq 0$ and $b \neq 0$.

Lemma 5.1. *Let $c \notin \mathbb{Z}_{<1}$. Then there exist two complementary solutions of Equation* (5.1)*, which are given by*

$$H_1(t) = {}_2F_1(a,b;c;t) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}t^k, \quad t>0, \tag{5.2}$$

$$H_2(t) = \frac{1}{\Gamma(2-c)}t^{1-c} \cdot {}_2F_1(1+a-c, 1+b-c; 2-c; t), \quad t>0. \tag{5.3}$$

In [7], with the aid of formulas (3.3) and (3.4) for $\rho = -\epsilon$, we obtain the following transformed differential equation of Equation (5.1), which is satisfied by $\tilde{u}(t) = {}_RD_t^{-\epsilon}u(t)$ and $\tilde{f}(t) = {}_RD_t^{-\epsilon}f(t)$:

$$\begin{aligned}\tilde{p}_{H,\epsilon}(t, {}_RD_t)\tilde{u}(t) &= {}_RD_t^{-\epsilon} p_H(t, {}_RD_t) {}_RD_t^{\epsilon}\tilde{u}(t) \\ &= {}_RD_t^{-\epsilon}[t(1-t)\frac{d^2}{dt^2} + (c-(a+b+1)t)\frac{d}{dt} - ab] {}_RD_t^{\epsilon}\tilde{u}(t) \\ &= [t(1-t)\frac{d^2}{dt^2} + (c-\epsilon-(a+b+1-2\epsilon)t)\frac{d}{dt} - (a-\epsilon)(b-\epsilon)\tilde{u}(t) = \tilde{f}(t). \end{aligned} \tag{5.4}$$

Theorem 5.1. *Let $H_1(t)$ and $H_2(t)$ be given by Equations* (5.2) *and* (5.3)*, respectively. Then*

(i) *when $\beta \in \mathbb{C}$ and $\tilde{f}(t) = g_{\epsilon-\beta}(t)$, $\tilde{u}_{\epsilon-\beta}(t)$, given by*

$$\tilde{u}_{\epsilon-\beta}(t) = {}_RD_t^{\beta} G_{H,\beta,\epsilon}(t,0) = \frac{1}{-1+c-\beta} \sum_{k=0}^{\infty} \frac{(a-\beta)_k (b-\beta)_k}{(c-\beta)_k \Gamma(k-\beta+\epsilon+1)} t^{k-\beta+\epsilon} H(t), \tag{5.5}$$

is a particular solution of Equation (5.4)*,*

(ii) *if $\beta = n \in \mathbb{Z}_{>-1}$, $u_{-n}(t) = {}_RD_t^{\epsilon} \tilde{u}_{\epsilon-n}(t)$, expressed by*

$$\begin{aligned} u_{-n}(t) &= \sum_{k=n}^{\infty} \frac{(a-n)_k (b-n)_k}{(-1+c-n)_{k+1}} \frac{1}{(k-n)!} t^{k-n} H(t) \\ &= \sum_{l=0}^{\infty} \frac{(a-n)_{n+l} (b-n)_{n+l}}{(-1+c-n)_{n+l+1}} \frac{1}{l!} t^{l} H(t) = C_n G_{H,0}(t,0), \end{aligned} \tag{5.6}$$

is a complementary solution of Equation (5.1)*, where $C_n = \frac{(a-n)_n (b-n)_n}{(-1+c-n)_{n+1}} = \frac{\Gamma(a)\Gamma(b)\Gamma(c-n-1)}{\Gamma(a-n)\Gamma(b-n)\Gamma(c)}$, and*

(iii) *if $\beta \notin \mathbb{Z}_{>-1}$, $u_{-\beta}(t)$, given by $u_{-\beta}(t) = {}_RD_t^{\epsilon} \tilde{u}_{\epsilon-\beta}(t) = {}_RD_t^{\beta} G_{H,\beta,0}(t,0)$ with the aid of Equation* (5.5)*, is a particular solution of Equation* (5.1)*.*

Theorem 5.2. *Let $\tilde{f}(t)$ and $f(t)$ satisfy Condition 1.2(iii), so that they are given by Equation* (1.13)*, and*

$${}_RD_t^{\beta_l} G_{H,\beta_l,\epsilon}(t,0) = \frac{1}{-1+c-\beta_l} \sum_{k=0}^{\infty} \frac{(a-\beta_l)_k (b-\beta_l)_k}{(c-\beta_l)_k \Gamma(k-\beta_l+1+\epsilon)} t^{k-\beta_l+\epsilon} H(t). \tag{5.7}$$

Then $\tilde{u}_f(t)$ and $u_f(t)$, given by

$$\tilde{u}_f(t) = \sum_{l=1}^{\infty} c_l \cdot {}_RD_t^{\beta_l} G_{H,\beta_l,\epsilon}(t,0), \quad u_f(t) = \sum_{l=1}^{\infty} d_l \cdot {}_RD_t^{\beta_l} G_{H,\beta_l,0}(t,0), \tag{5.8}$$

are particular solutions of Equations (5.4) *and* (5.1)*, respectively. The condition, which corresponds to Condition $c-\beta \notin \mathbb{Z}_{<1}$ in Lemma 4.2, requires the condition $c-\beta_l \notin \mathbb{Z}_{<1}$ for all $l \in \mathbb{Z}_{>0}$, in the present case.*

Lemma 5.2. *Let $H_2(t)$ be given by* (5.3) *Then $u_c(t)$, given by*

$$u_c(t) = H_2(t) H(t), \tag{5.9}$$

is a complementary solution of Equation (5.1)*.*

6 Conclusion

In the preceding paper [7], we consider the problem of solving the equation which has a special class of inhomogeneous part, and we adopt a recipe without the Green's functuion, which is applied to the hypergeometric differential equation, the differential equations treated in [5] and the Hermite differential equation.

In the present paper, we solve the problem considered in [7], but now the results are expressed in terms of the Green's function. It is applied to Kummer's and the hypergeometric differential equation.

In Section 3, the results for Kummer's differential equation, in which Condition 1.2(i) is satisfied, are given in Theorem 3.1 and Lemma 3.4.

In Section 4, the results for Kummer's differential equation, in which Condition 1.2(ii) is satisfied, are given in Theorem 4.1 and Lemma 4.5, and the results, in which Condition 1.2(iii) is satisfied, are given in Theorem 4.2.

In Section 5, the corresponding results for the hypergeometric differential equation, are given in Theorems 5.1 and 5.2 and Lemma 5.2.

References

[1] Diener, F.; Diener, M. (1995). Tutorial, in Nonstandard Analysis in Practice. Springer, Berlin, pp. 1-21.

[2] Gelfand, I. M.; Silov, G. E. (1964). Generalized Functions. Vol. 1. Academic Press Inc., New York.

[3] Morita, T.; Sato, K. (2013). Liouville and Riemann-Liouville Fractional Derivatives via Contour Integrals. Frac. Calc. Appl. Anal. 16, 630–653.

[4] Morita, T.; Sato, K. (2017). Solution of Inhomogeneous Differential Equations with Polynomial Coefficients in Terms of the Green's Function. Mathematics. 5, 62, 1–24. https://doi.org/10.3390/math5040062

[5] Morita, T.; Sato, K. (2021). Solution of Inhomogeneous Fractional Differential Equations with Polynomial Coefficients in Terms of the Green's Function, in Nonstandard Analysis. Mathematics. 9, 1944, 1–24. https://doi.org/10.3390/math9161944

[6] Morita, T. (2022). Solution of Inhomogeneous Differential Equations with Polynomial Coefficients in Terms of the Green's Function, in Nonstandard Analysis. AppliedMath. 2, 379–392. https://doi.org/10.3390/appliedmath2030022

[7] Morita, T. (2023). Solution of Inhomogeneous Differential Equations with Polynomial Coefficients in Nonstandard Analysis. J. Adv. Math. Comput. Sci. 38 (3), 49-60. https://doi.org/10.9734/jamcs/2023/v38i31751

[8] Podlubny, I. (1999). Fractional Differential Equations. Academic Press, San Diego, Section 2.3.2.

[9] Schwartz, L. (1966). Théorie des Distributions. Hermann, Paris.

[10] Zemanian, A. H. (1965). Distribution Theory and Transform Analysis. Dover Publ. Inc., New York.