Original Research Article

On Hyperbolic Generalized Woodall Numbers

Abstract. In this work, we introduce the generalized hyperbolic Woodall numbers. As special cases, we study with hyperbolic Woodall, hyperbolic modified Woodall, hyperbolic Cullen numbers and hyperbolic modified Cullen numbers. We present Binet's formulas, generating functions and the summation formulas for these numbers. Besides, we give Catalan's and Cassini's identities and present matrices related with these sequences.

Keywords. Woodall numbers, Cullen numbers, hyperbolic numbers, hyperbolic Woodall numbers, hyperbolic Cullen numbers.

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1. Introduction

To start with, we give some information which we need about the definition and properties of Woodall numbers.

1.1. Woodall Numbers. The generalized Woodall sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, 5, -8, 4)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 5W_{n-1} - 8W_{n-2} + 4W_{n-3} \tag{1.1}$$

with the initial values W_0, W_1, W_2 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - \frac{5}{4}W_{-(n-2)} + \frac{1}{4}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

In the following theorem, we give Binet formula of generalized Woodall numbers.

THEOREM 1. *[30, Theorem 1.1] Binet formula of generalized Woodall numbers can be given as*

$$W_n = (A_1 + A_2 n) \times 2^n + A_3$$

where

$$\begin{aligned} A_1 &= -W_2 + 4W_1 - 3W_0, \\ A_2 &= \frac{W_2 - 3W_1 + 2W_0}{2}, \\ A_3 &= W_2 - 4W_1 + 4W_0, \end{aligned}$$

that is,

$$W_n = ((-W_2 + 4W_1 - 3W_0) + \frac{W_2 - 3W_1 + 2W_0}{2} n) \times 2^n + (W_2 - 4W_1 + 4W_0). \tag{1.2}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 5x^2 + 8x - 4 = (x-2)^2 (x-1) = 0,$$

where $\alpha = \beta = 2$, $\gamma = 1$.

Then, the first few generalized Woodall numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Woodall numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{4}(8W_0 - 5W_1 + W_2)$
2	W_2	$\frac{1}{4}(11W_0 - 9W_1 + 2W_2)$
3	$4W_0 - 8W_1 + 5W_2$	$\frac{1}{16}(52W_0 - 47W_1 + 11W_2)$
4	$20W_0 - 36W_1 + 17W_2$	$\frac{1}{16}(57W_0 - 54W_1 + 13W_2)$
5	$68W_0 - 116W_1 + 49W_2$	$\frac{1}{64}(240W_0 - 233W_1 + 57W_2)$

Now, we define four specific cases of the sequence $\{W_n\}$.

The Woodall numbers $\{R_n\}$, sometimes called Riesel numbers, and also called Cullen numbers of the second kind, are numbers of the form

$$R_n = n \times 2^n - 1.$$

The first few Woodall numbers are:

$$1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, 106495, 229375, 491519, 1048575, \ldots$$

(sequence A003261 in the OEIS [26]). Woodall numbers were first studied by Allan J. C. Cunningham and H. J. Woodall in [7] in 1917, inspired by James Cullen's earlier study of the similarly-defined Cullen numbers.

The Cullen numbers $\{C_n\}$ are numbers of the form

$$C_n = n \times 2^n + 1.$$

The first few Cullen numbers are:

$$1, 3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, 229377, 491521, ...$$

(sequence A002064 in the OEIS). Woodall and Cullen sequences have been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [3,4,7,13,12,16,19,20,21,22,23] and references therein. Note that $\{R_n\}$ and $\{C_n\}$ hold the following relations:

$$\begin{aligned} R_n &= 4R_{n-1} - 4R_{n-2} - 1, \\ C_n &= 4C_{n-1} - 4C_{n-2} + 1. \end{aligned}$$

Note also that the sequences $\{R_n\}$ and $\{C_n\}$ satisfy the following third order linear recurrences:

$$\begin{aligned} R_n &= 5R_{n-1} - 8R_{n-2} + 4R_{n-3}, \qquad R_0 = -1, R_1 = 1, R_2 = 7, \\ C_n &= 5C_{n-1} - 8C_{n-2} + 4C_{n-3}, \qquad C_0 = 1, C_1 = 3, C_2 = 9. \end{aligned}$$

If we set $G_0 = 0$, $G_1 = 1$, $G_2 = 5$ then $\{G_n\}$ is the well-known modified Woodall sequence, if we set $H_0 = 3$, $H_1 = 5$, $H_2 = 9$ then $\{H_n\}$ is the well-known modified Cullen sequence. In other words, modified Woodall sequence $\{G_n\}_{n\geq 0}$ and modified Cullen sequence $\{H_n\}_{n\geq 0}$ are defined by the third-order recurrence relations

$$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}, \qquad G_0 = 0, G_1 = 1, G_2 = 5, \tag{1.3}$$

$$H_n = 5H_{n-1} - 8H_{n-2} + 4H_{n-3}, \qquad H_0 = 3, H_1 = 5, H_2 = 9, \tag{1.4}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= 2G_{-(n-1)} - \frac{5}{4}G_{-(n-2)} + \frac{1}{4}G_{-(n-3)}, \\ H_{-n} &= 2H_{-(n-1)} - \frac{5}{4}H_{-(n-2)} + \frac{1}{4}H_{-(n-3)}, \\ R_{-n} &= 2R_{-(n-1)} - \frac{5}{4}R_{-(n-2)} + \frac{1}{4}R_{-(n-3)}, \\ C_{-n} &= 2C_{-(n-1)} - \frac{5}{4}C_{-(n-2)} + \frac{1}{4}C_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.3) and (1.4) hold for all integer n.

Using the initial conditions in (1.2), Binet's formula of modified Woodall, modified Cullen, Woodall and Cullen sequences are

$$\begin{aligned} G_n &= (n-1)\,2^n + 1, \\ H_n &= 2^{n+1} + 1, \\ R_n &= n \times 2^n - 1, \\ C_n &= n \times 2^n + 1. \end{aligned}$$

Now, we give the generating function and the Cassini identity for generalized Woodall numbers.

The generating function for generalized Woodall numbers is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 5W_0)x + (W_2 - 5W_1 + 8W_0)x^2}{1 - 5x + 8x^2 - 4x^3}. \tag{1.5}$$

The Cassini identity for generalized Woodall numbers is

$$\begin{aligned}
W_{n+1}W_{n-1} - W_n^2 &= \frac{1}{4}2^n(A + B2^n + Cn). \\
A &= 4W_1^2 + W_2^2 - 4W_0W_1 + 4W_0W_2 - 5W_1W_2. \\
B &= -4W_0^2 - 9W_1^2 - W_2^2 + 12W_0W_1 - 4W_0W_2 + 6W_1W_2. \\
C &= 8W_0^2 + 12W_1^2 + W_2^2 - 20W_0W_1 + 6W_0W_2 - 7W_1W_2.
\end{aligned}$$

For further information about generalized Woodall numbers, see [30].

Now, in 1989, I. Kantor is worked the hypercomplex numbers systems, [18]. This numbers systems are extensions of real numbers. Some commutative some of hypercomplex number systems are defined below.

Complex numbers are

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\},$$

hyperbolic (double, split-complex) numbers [27] are

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}$$

and dual numbers [11] are

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

One of the non-commutative examples of hypercomplex number systems are quaternions, [15],

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ i^2 = j^2 = k^2 = ijk = -1\},$$

octonions [2] and sedenions [28]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n-ions (see for example [5], [17], [24]).

Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) [15] as an extension to the complex numbers. Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848, [6].

Now, we will give some information related to hyperbolic numbers. We given hyperbolic numbers above as follows:

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R},\ j^2 = 1,\ j \neq \pm 1\}.$$

The base elements $\{1, j\}$ of hyperbolic numbers satisfy the following properties (commutative multiplications):

$$1.j = j,\ j^2 = j.j = 1$$

where j symbolizes the hyperbolic unit ($j^2 = 1$).

The multiplication of two hyperbolic numbers $m = a_0 + ja_1$ and $n = b_0 + jb_1$ is

$$mn = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0).$$

Sum of hyperbolic numbers is defined as componentwise. To give an example for q and p:

$$m + n = a_0 + b_0 + j(a_1 + b_1).$$

Now, we give details about hyperbolic and some information related to hyperbolic sequences from the literature.

- Richter, [25] worked On Hyperbolic Complex Numbers.
- Gürses, Şentürk and Yüce, [14] studied A Study on Dual-Generalized Complex and Hyperbolic-Generalized Complex numbers.
- Cockle [6] worked the Hyperbolic numbers with complex coefficients.
- Aydın, [1] worked hyperbolic Fibonacci numbers given by

  $$\widetilde{F}_n = F_n + hF_{n+1},\ (h^2 = 1)$$

  where Fibonacci numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$ with the initial condition $F_1 = F_2 = 1$, $(n \geq 3)$.
- Dikmen, [8] worked hyperbolic Jacobsthal numbers given by

  $$\widehat{J}_n = J_n + hJ_{n+1}\ ,\ (h^2 = 1)$$

  where Jacobsthal numbers, respectively, given by $J_n = J_{n-1} + 2J_{n-2},\ J_0 = 0,\ J_1 = 1$.
- Taş, [33] worked on hyperbolic Jacobsthal-Lucas sequence given by

  $$HJ_n = J_n + hJ_{n+1}, (h^2 = 1)$$

  where Jacobsthal-Lucas numbers, respectively, given by $J_{n+2} = J_{n+1} + 2J_n$, with the initial condition $J_0 = 2,\ J_1 = 1$.
- Soykan and Taşdemir, [32] worked a study on hyperbolic numbers with generalized Jacobsthal given by

  $$\widetilde{V}_n = V_n + hV_{n+1}\ ,\ (h^2 = 1)$$

  where generalized Jacobsthal numbers are given by $V_n = V_{n-1} + 2V_{n-2},\ V_0 = a,\ V_1 = b\ (n \geq 2)$ with the initial values $V_0,\ V_1$ not all being zero.

- Dişkaya, Menken, Catarino, [10] worked on the hyperbolic Leonardo and hyperbolic Francois quaternions given by

$$\begin{aligned} H\mathcal{L}_n &= \mathcal{L}_n e_0 + \mathcal{L}_{n+1} e_1 + \mathcal{L}_{n+2} e_2 + \mathcal{L}_{n+3} e_3, \\ H\mathcal{F}_n &= \mathcal{F}_n e_0 + \mathcal{F}_{n+1} e_1 + \mathcal{F}_{n+2} e_2 + \mathcal{F}_{n+3} e_3 \end{aligned}$$

  where Francois and Leonardo numbers, respectively, given by $\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2} + 1$, with the initial condition $\mathcal{F}_0 = 2$, $\mathcal{F}_1 = 1$ and $\mathcal{L}_{n+2} = \mathcal{L}_{n+1} + \mathcal{L}_n$, with the initial condition $\mathcal{L}_0 = 1$, $\mathcal{L}_1 = 1$.
- Dikmen and Altınsoy, [9] worked on third order hyperbolic Jacobsthal numbers given by

$$\begin{aligned} \widehat{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)}, \\ \widehat{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)} \end{aligned}$$

  where Jacobsthal numbers, respectively, given by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}$, $J_0^{(3)} = 0$, $J_1^{(3)} = 1$, $J_2^{(3)} = 1$, $j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)}$, $j_0^{(3)} = 2$, $j_1^{(3)} = 1$, $j_2^{(3)} = 5$.

Next section, we present the hyperbolic generalized Woodall numbers and their generating functions and Binet's formulas.

2. Hyperbolic Generalized Woodall Numbers

In this chapter, we define hyperbolic generalized Woodall numbers and present generating functions and Binet's formulas for them.

We now define hyperbolic generalized Woodall numbers over $\mathbb{H}$. The nth hyperbolic generalized Woodall number is

$$\mathcal{H}W_n = W_n + jW_{n+1}. \tag{2.1}$$

with the initial values $\mathcal{H}W_0$, $\mathcal{H}W_1$, $\mathcal{H}W_2$. (2.1) can be written to negative subscripts by defining,

$$\mathcal{H}W_{-n} = W_{-n} + jW_{-n+1}.$$

so identity (2.1) holds for all integers n.

For four special cases of the nth dual hyperbolic generalized Woodall numbers are given as

$$\begin{aligned} \mathcal{H}G_n &= G_n + jG_{n+1}, \\ \mathcal{H}H_n &= H_n + jH_{n+1}, \\ \mathcal{H}R_n &= R_n + jR_{n+1}, \\ \mathcal{H}C_n &= C_n + jC_{n+1}. \end{aligned}$$

It is clear that

$$\mathcal{H}W_n = 5\mathcal{H}W_{n-1} - 8\mathcal{H}W_{n-2} + 4\mathcal{H}W_{n-3}. \tag{2.2}$$

The sequence $\{\mathcal{H}W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{H}W_{-n} = -2\mathcal{H}W_{-(n-1)} - \frac{5}{4}\mathcal{H}W_{-(n-2)} + \frac{1}{4}\mathcal{H}W_{-(n-3)}.$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n.

The initial several hyperbolic generalized Woodall numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few hyperbolic generalized Woodall numbers.

n	$\mathcal{H}W_n$	$\mathcal{H}W_{-n}$
0	$\mathcal{H}W_0$	$\mathcal{H}W_0$
1	$\mathcal{H}W_1$	$\frac{1}{4}(8\mathcal{H}W_0 - 5\mathcal{H}W_1 + \mathcal{H}W_2)$
2	$\mathcal{H}W_2$	$\frac{1}{4}(11\mathcal{H}W_0 - 9\mathcal{H}W_1 + 2\mathcal{H}W_2)$
3	$4\mathcal{H}W_0 - 8\mathcal{H}W_1 + 5\mathcal{H}W_2$	$\frac{1}{16}(52\mathcal{H}W_0 - 47\mathcal{H}W_1 + 11\mathcal{H}W_2)$
4	$20\mathcal{H}W_0 - 36\mathcal{H}W_1 + 17\mathcal{H}W_2$	$\frac{1}{16}(57\mathcal{H}W_0 - 54\mathcal{H}W_1 + 13\mathcal{H}W_2)$
5	$68\mathcal{H}W_0 - 116\mathcal{H}W_1 + 49\mathcal{H}W_2$	$\frac{1}{64}(240\mathcal{H}W_0 - 233\mathcal{H}W_1 + 57\mathcal{H}W_2)$

Note that

$$\begin{aligned}
\mathcal{H}W_0 &= W_0 + jW_1, \\
\mathcal{H}W_1 &= W_1 + jW_2, \\
\mathcal{H}W_2 &= W_2 + jW_3 = W_2 + j(4W_0 - 8W_1 + 5W_2).
\end{aligned}$$

For hyperbolic modified Woodall numbers (taking $W_n = G_n$, $G_0 = 0$, $G_1 = 1$, $G_2 = 5$) we get

$$\begin{aligned}
\mathcal{H}G_0 &= G_0 + jG_1 = j, \\
\mathcal{H}G_1 &= G_1 + jG_2 = 1 + 5j, \\
\mathcal{H}G_2 &= G_2 + jG_3 = 5 + 17j
\end{aligned}$$

and for hyperbolic modified Cullen numbers (taking $W_n = H_n$, $H_0 = 3$, $H_1 = 5$, $H_2 = 9$) we get

$$\begin{aligned}
\mathcal{H}H_0 &= H_0 + jH_1 = 3 + 5j, \\
\mathcal{H}H_1 &= H_1 + jH_2 = 5 + 9j, \\
\mathcal{H}H_2 &= H_2 + jH_3 = 9 + 17j
\end{aligned}$$

and for hyperbolic Woodall numbers (taking $W_n = R_n$, $R_0 = -1$, $R_1 = 1$, $R_2 = 7$) we get

$$\begin{aligned}
\mathcal{H}R_0 &= R_0 + jR_1 = -1 + j, \\
\mathcal{H}R_1 &= R_1 + jR_2 = 1 + 7j, \\
\mathcal{H}R_2 &= R_2 + jR_3 = 7 + 23j
\end{aligned}$$

and for hyperbolic Cullen numbers (taking $W_n = C_n$, $C_0 = 1$, $C_1 = 3$, $C_2 = 9$) we get

$$\begin{aligned} \mathcal{H}C_0 &= C_0 + jC_1 = 1 + 3j, \\ \mathcal{H}C_1 &= C_1 + jC_2 = 3 + 9j, \\ \mathcal{H}C_2 &= C_2 + jC_3 = 9 + 25j. \end{aligned}$$

A few hyperbolic modified Woodall numbers, hyperbolic modified Cullen numbers, hyperbolic Woodall numbers and hyperbolic Cullen numbers with positive subscript and negative subscript are given in the following Table 3, Table 4, Table 5 and Table 6.

Table 3. Hyperbolic modified Woodall numbers

n	$\mathcal{H}G_n$	$\mathcal{H}G_{-n}$
0	j	j
1	$1+5j$	0
2	$5+17j$	$\frac{1}{4}$
3	$17+49j$	$\frac{1}{2}+\frac{1}{4}j$
4	$49+129j$	$\frac{11}{16}+\frac{1}{2}j$
5	$129+321j$	$\frac{13}{16}+\frac{11}{16}j$

Table 4. Hyperbolic modified Cullen numbers

n	$\mathcal{H}H_n$	$\mathcal{H}H_{-n}$
0	$3+5j$	$3+5j$
1	$5+9j$	$2+3j$
2	$9+17j$	$\frac{3}{2}+3j$
3	$17+33j$	$\frac{5}{4}+\frac{3}{2}j$
4	$33+65j$	$\frac{9}{8}+\frac{3}{2}j$
5	$65+129j$	$\frac{17}{16}+\frac{9}{8}j$

Table 5. Hyperbolic Woodall numbers

n	$\mathcal{H}R_n$	$\mathcal{H}R_{-n}$
0	$-1+j$	$-1+j$
1	$1+7j$	$-\frac{3}{2}-j$
2	$7+23j$	$-\frac{3}{2}-\frac{3}{2}j$
3	$23+63j$	$-\frac{11}{8}-\frac{3}{2}j$
4	$63+159j$	$-\frac{5}{4}-\frac{11}{8}j$
5	$159+383j$	$-\frac{37}{32}-\frac{5}{4}j$

Table 6. Hyperbolic Cullen numbers

n	$\mathcal{H}C_n$	$\mathcal{H}C_{-n}$
0	$1+3j$	$1+3j$
1	$3+9j$	$\frac{1}{2}+j$
2	$9+25j$	$\frac{1}{2}+j$
3	$25+65j$	$\frac{5}{8}+\frac{1}{2}j$
4	$65+161j$	$\frac{3}{4}+\frac{1}{2}j$
5	$161+385j$	$\frac{27}{32}+\frac{3}{4}j$

Now, we will state Binet's formula for the hyperbolic generalized Woodall numbers and in the rest of the paper, we fix the following notations:

$$\begin{aligned} \widehat{\alpha} &= 1 + 2j, \\ \widehat{\beta} &= 2j, \\ \widehat{\gamma} &= 1 + j. \end{aligned}$$

Note that we have the following identities:

$$\begin{aligned} \widehat{\alpha}^2 &= 5+4j, \\ \widehat{\beta}^2 &= 4, \\ \widehat{\gamma}^2 &= 2+2j, \\ \widehat{\alpha}\widehat{\beta} &= 4+2j, \\ \widehat{\alpha}\widehat{\gamma} &= 3+3j, \\ \widehat{\beta}\widehat{\gamma} &= 2+2j, \\ \widehat{\alpha}\widehat{\beta}\widehat{\gamma} &= 6+6j. \end{aligned}$$

2.1. Binet's Formula. Now, we present Binet's formula in the following theorem.

THEOREM 2. *(Binet's Formula) For any integer n, the nth hyperbolic generalized Woodall number is*

$$\mathcal{H}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \tag{2.3}$$

Proof. Using Binet's formula

$$W_n = (A_1 + A_2 n)2^n + A_3$$

of the generalized Woodall numbers, we obtain

$$\begin{aligned} \mathcal{H}W_n &= W_n + jW_{n+1} \\ &= (A_1 + A_2 n)2^n + A_3 + j((A_1 + A_2(n+1))2^{n+1} + A_3) \\ &= A_1 2^n + A_2 n 2^n + A_3 \\ &\quad + jA_1 2^{n+1} + jA_2 n 2^{n+1} + jA_2 2^{n+1} + jA_3 \\ &= A_1 2^n(1+2j) + A_2 n 2^n(1+2j) + A_2 2^n(2j) + A_3(1+j) \\ &= A_1 2^n\widehat{\alpha} + A_2 n 2^n\widehat{\alpha} + A_2 2^n\widehat{\beta} + A_3\widehat{\gamma} \\ &= (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \end{aligned}$$

This proves (2.3). □

As special cases, for any integer n, the Binet's Formula of nth hyperbolic modified Woodall number, hyperbolic modified Cullen number, hyperbolic Woodall number and hyperbolic Cullen number are

- $\mathcal{H}G_n = (-\widehat{\alpha} + \widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma}$,

  $\mathcal{H}G_n = 1 + (n-1)2^n + j(1 + n2^{n+1})$.
- $\mathcal{H}H_n = (2\widehat{\alpha})2^n + \widehat{\gamma}$,

  $\mathcal{H}H_n = 1 + 2^{n+1} + j(1 + 2^{n+2})$.
- $\mathcal{H}R_n = (\widehat{\beta} + n\widehat{\alpha})2^n - \widehat{\gamma}$,

  $\mathcal{H}R_n = -1 + n2^n + j(-1 + 2^{n+1} + n2^{n+1})$.

- $\mathcal{H}C_n = (\widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma}$,

$$\mathcal{H}C_n = 1 + n2^n + j(1 + 2^{n+1} + n2^{n+1}).$$

Next, we present generating function.

2.2. Generating Function.

THEOREM 3. *The generating function for the hyperbolic generalized Woodall numbers is*

$$\sum_{n=0}^{\infty} \mathcal{H}W_n x^n = \frac{\mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2}{1 - 5x + 8x^2 - 4x^3}. \tag{2.4}$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} \mathcal{H}W_n x^n$$

be generating function of the hyperbolic generalized Woodall numbers. Then, using the definition of the hyperbolic generalized Woodall numbers, and substracting $xg(x)$, $x^2g(x)$ and $x^3g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1 - 5x + 8x^2 - 4x^3)g(x) &= \sum_{n=0}^{\infty} \mathcal{H}W_n x^n - 5x\sum_{n=0}^{\infty} \mathcal{H}W_n x^n + 8x^2\sum_{n=0}^{\infty} \mathcal{H}W_n x^n - 4x^3\sum_{n=0}^{\infty} \mathcal{H}W_n x^n \\
&= \sum_{n=0}^{\infty} \mathcal{H}W_n x^n - 5\sum_{n=0}^{\infty} \mathcal{H}W_n x^{n+1} + 8\sum_{n=0}^{\infty} \mathcal{H}W_n x^{n+2} - 4\sum_{n=0}^{\infty} \mathcal{H}W_n x^{n+3} \\
&= \sum_{n=0}^{\infty} \mathcal{H}W_n x^n - 5\sum_{n=1}^{\infty} \mathcal{H}W_{n-1} x^n + 8\sum_{n=2}^{\infty} \mathcal{H}W_{n-2} x^n - 4\sum_{n=3}^{\infty} \mathcal{H}W_{n-3} x^n \\
&= (\mathcal{H}W_0 + \mathcal{H}W_1 x + \mathcal{H}W_2 x^2) - 5(\mathcal{H}W_0 x + \mathcal{H}W_1 x^2) + 8\mathcal{H}W_0 x^2 \\
&\quad + \sum_{n=3}^{\infty} (\mathcal{H}W_n - 5\mathcal{H}W_{n-1} + 8\mathcal{H}W_{n-2} - 4\mathcal{H}W_{n-3}) x^n \\
&= \mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2.
\end{aligned}$$

Note that we used the recurrence relation $\mathcal{H}W_n = 5\mathcal{H}W_{n-1} - 8\mathcal{H}W_{n-2} + 4\mathcal{H}W_{n-3}$. Rearranging above equation, we get

$$g(x) = \frac{\mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2}{1 - 5x + 8x^2 - 4x^3}.$$

The proof is finished. □

As special cases, the generating functions for the hyperbolic modified Woodall, hyperbolic modified Cullen, hyperbolic Woodall and hyperbolic Cullen numbers are

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{H}G_n x^n &= \frac{j + x}{1 - 5x + 8x^2 - 4x^3}, \\
\sum_{n=0}^{\infty} \mathcal{H}H_n x^n &= \frac{5j + 3 + (-16j - 10)\,x + (12j + 8)\,x^2}{1 - 5x + 8x^2 - 4x^3}, \\
\sum_{n=0}^{\infty} \mathcal{H}R_n x^n &= \frac{-1 + j + (2j + 6)x + (-4j - 6)x^2}{1 - 5x + 8x^2 - 4x^3}
\end{aligned}$$

and

$$\sum_{n=0}^{\infty} \mathcal{H}C_n x^n = \frac{3j+1+(-6j-2)\,x+(4j+2)\,x^2}{1-5x+8x^2-4x^3}$$

respectively.

Now let's show that we obtained the Binet formula using the generating function.

2.3. Obtaining Binet's Formula From Generating Function. We obtain Binet's formula of hyperbolic generalized Woodall number $\{\mathcal{H}W_n\}$ by the use of generating function for $\mathcal{H}W_n$.

THEOREM 4. *(Binet's formula of hyperbolic generalized Woodall numbers)*

$$\mathcal{H}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \tag{2.5}$$

Proof. Let

$$\sum_{n=0}^{\infty} \mathcal{H}W_n x^n = \frac{\mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2}{1-5x+8x^2-4x^3}.$$

Then we write

$$\frac{\mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2}{(1-x)\,(1-2x)^2} = \frac{d_1}{(1-x)} + \frac{d_2}{(1-2x)} + \frac{d_3}{(1-2x)^2}. \tag{2.6}$$

So

$$\begin{aligned}
\mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2 &= (d_1+d_2+d_3) + (-4d_1 - 3d_2 - d_3)x \\
&\quad + (4d_1 + 2d_2)x^2.
\end{aligned}$$

We get

$$\begin{aligned}
\mathcal{H}W_0 &= d_1 + d_2 + d_3, \\
\mathcal{H}W_1 - 5\mathcal{H}W_0 &= -4d_1 - 3d_2 - d_3, \\
\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0 &= 4d_1 + 2d_2.
\end{aligned}$$

If we solve these simultanious equation,

$$\begin{aligned}
d_1 &= 4\mathcal{H}W_0 - 4\mathcal{H}W_1 + \mathcal{H}W_2, \\
d_2 &= -4\mathcal{H}W_0 + \frac{11}{2}\mathcal{H}W_1 - \frac{3}{2}\mathcal{H}W_2, \\
d_3 &= \mathcal{H}W_0 - \frac{3}{2}\mathcal{H}W_1 + \frac{1}{2}\mathcal{H}W_2.
\end{aligned}$$

Thus (2.6) can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{H}W_n x^n &= d_1\frac{1}{(1-x)}+d_2\frac{1}{(1-2x)}+d_3\frac{1}{(2x-1)^2}, \\
&= d_1\sum_{n=0}^{\infty}x^n+d_2\sum_{n=0}^{\infty}2^n x^n+d_3\sum_{n=0}^{\infty}2^n(n+1)x^n, \\
&= \sum_{n=0}^{\infty}(d_1+d_2 2^n+d_3 2^n(n+1))x^n, \\
&= \sum_{n=0}^{\infty}(4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2+(-4\mathcal{H}W_0+\frac{11}{2}\mathcal{H}W_1-\frac{3}{2}\mathcal{H}W_2)2^n \\
&\quad +(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)2^n(n+1))x^n, \\
&= \sum_{n=0}^{\infty}(4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2+(-4\mathcal{H}W_0+\frac{11}{2}\mathcal{H}W_1-\frac{3}{2}\mathcal{H}W_2)2^n \\
&\quad +(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)2^n+(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)2^n n)x^n, \\
&= \sum_{n=0}^{\infty}(4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2+(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)n2^n \\
&\quad +(-3\mathcal{H}W_0+4\mathcal{H}W_1-\mathcal{H}W_2)2^n)x^n, \\
&= \sum_{n=0}^{\infty}((-3\mathcal{H}W_0+4\mathcal{H}W_1-\mathcal{H}W_2)+(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)n)2^n \\
&\quad +4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2)x^n.
\end{aligned}$$

This gives

$$\mathcal{H}W_n=(\mathcal{H}A_1+\mathcal{H}A_2 n)2^n+\mathcal{H}A_3$$

where

$$\begin{aligned}
\mathcal{H}A_1 &= -3\mathcal{H}W_0+4\mathcal{H}W_1-\mathcal{H}W_2, \\
\mathcal{H}A_2 &= \mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2, \\
\mathcal{H}A_3 &= 4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2.
\end{aligned}$$

Note that the following equalities are true:

$$\begin{aligned}
A_1\widehat{\alpha}+A_2\widehat{\beta} &= (-W_2+4W_1-3W_0)(1+2j)+(\frac{W_2-3W_1+2W_0}{2})(2j) \\
&= -3W_0+4W_1-W_2+j(-4W_0+5W_1-W_2).
\end{aligned}$$

$$\begin{aligned}
A_2\widehat{\alpha} &= \frac{W_2-3W_1+2W_0}{2}(1+2j) \\
&= W_0-\frac{3}{2}W_1+\frac{1}{2}W_2+j(2W_0-3W_1+W_2).
\end{aligned}$$

$$A_3\widehat{\gamma}=W_2-4W_1+4W_0+j(W_2-4W_1+4W_0).$$

Therefore, we can write the following equalition:

$$\mathcal{H}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}.$$

The proof is finished. □

Next, using Theorem 4, we present the Binet's formulas of hyperbolic modified Woodall, hyperbolic modified Cullen, hyperbolic Woodall and hyperbolic Cullen numbers.

3. Some Identities For Hyperbolic Generalized Woodall Numbers

We now present a few special identities for the hyperbolic generalized Woodall sequence $\{\mathcal{H}W_n\}$. The following theorem presents the Simpson's identity for the hyperbolic generalized Woodall numbers.

THEOREM 5. *(Simpson's formula for hyperbolic generalized Woodall sequence)For all integers n we have*

$$\begin{vmatrix} \mathcal{H}W_{n+2} & \mathcal{H}W_{n+1} & \mathcal{H}W_n \\ \mathcal{H}W_{n+1} & \mathcal{H}W_n & \mathcal{H}W_{n-1} \\ \mathcal{H}W_n & \mathcal{H}W_{n-1} & \mathcal{H}W_{n-2} \end{vmatrix} = 4^n \begin{vmatrix} \mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0 \\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1} \\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2} \end{vmatrix}.$$

Proof. For the proof we use mathematical induction. For $n = 0$ identity is true. Now we obtain is true for $n = k$. Hence we write the following identity

$$\begin{vmatrix} \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ \mathcal{H}W_{k+1} & \mathcal{H}W_k & \mathcal{H}W_{k-1} \\ \mathcal{H}W_k & \mathcal{H}W_{k-1} & \mathcal{H}W_{k-2} \end{vmatrix} = 4^k \begin{vmatrix} \mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0 \\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1} \\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2} \end{vmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned}
\begin{vmatrix} \mathcal{H}W_{k+3} & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ \mathcal{H}W_{k+1} & \mathcal{H}W_k & \mathcal{H}W_{k-1} \end{vmatrix} &= \begin{vmatrix} 5\mathcal{H}W_{k+2} - 8\mathcal{H}W_{k+1} + 4\mathcal{H}W_k & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ 5\mathcal{H}W_{k+1} - 8\mathcal{H}W_k + 4\mathcal{H}W_{k-1} & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ 5\mathcal{H}W_k - 8\mathcal{H}W_{k-1} + 4\mathcal{H}W_{k-2} & \mathcal{H}W_k & \mathcal{H}W_{k-1} \end{vmatrix} \\
&= 5\begin{vmatrix} \mathcal{H}W_{k+2} & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ \mathcal{H}W_{k+1} & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ \mathcal{H}W_k & \mathcal{H}W_k & \mathcal{H}W_{k-1} \end{vmatrix} - 8\begin{vmatrix} \mathcal{H}W_{k+1} & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ \mathcal{H}W_k & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ \mathcal{H}W_{k-1} & \mathcal{H}W_k & \mathcal{H}W_{k-1} \end{vmatrix} \\
&\quad +4\begin{vmatrix} \mathcal{H}W_k & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ \mathcal{H}W_{k-1} & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ \mathcal{H}W_{k-2} & \mathcal{H}W_k & \mathcal{H}W_{k-1} \end{vmatrix} \\
&= 4\begin{vmatrix} \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ \mathcal{H}W_{k+1} & \mathcal{H}W_k & \mathcal{H}W_{k-1} \\ \mathcal{H}W_k & \mathcal{H}W_{k-1} & \mathcal{H}W_{k-2} \end{vmatrix} = 4^{k+1}\begin{vmatrix} \mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0 \\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1} \\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2} \end{vmatrix}.
\end{aligned}$$

Thus, the proof is finished. □

From prewious theorem, we get following corollary.

COROLLARY 6. *(Simpson's formula for hyperbolic generalized Woodall sequence's special cases)*

(a): $\begin{vmatrix} \mathcal{H}G_{k+2} & \mathcal{H}G_{k+1} & \mathcal{H}G_{k} \\ \mathcal{H}G_{k+1} & \mathcal{H}G_{k} & \mathcal{H}G_{k-1} \\ \mathcal{H}G_{k} & \mathcal{H}G_{k-1} & \mathcal{H}G_{k-2} \end{vmatrix} = -4^{n-1}(9+9j).$

(b): $\begin{vmatrix} \mathcal{H}H_{k+2} & \mathcal{H}H_{k+1} & \mathcal{H}H_{k} \\ \mathcal{H}H_{k+1} & \mathcal{H}H_{k} & \mathcal{H}H_{k-1} \\ \mathcal{H}H_{k} & \mathcal{H}H_{k-1} & \mathcal{H}H_{k-2} \end{vmatrix} = 0.$

(c): $\begin{vmatrix} \mathcal{H}R_{k+2} & \mathcal{H}R_{k+1} & \mathcal{H}R_{k} \\ \mathcal{H}R_{k+1} & \mathcal{H}R_{k} & \mathcal{H}R_{k-1} \\ \mathcal{H}R_{k} & \mathcal{H}R_{k-1} & \mathcal{H}R_{k-2} \end{vmatrix} = 4^{n-1}(9+9j).$

(d): $\begin{vmatrix} \mathcal{H}C_{k+2} & \mathcal{H}C_{k+1} & \mathcal{H}C_{k} \\ \mathcal{H}C_{k+1} & \mathcal{H}C_{k} & \mathcal{H}C_{k-1} \\ \mathcal{H}C_{k} & \mathcal{H}C_{k-1} & \mathcal{H}C_{k-2} \end{vmatrix} = -4^{n-1}(9+9j).$

THEOREM 7. *(Catalan's identity) For all integers n and m, the following identity holds*

$\mathcal{H}W_{n+m}\mathcal{H}W_{n-m} - \mathcal{H}W_n^2 = 2^{n-m}(-2^{m+n}m^2\widehat{\alpha}^2A_2^2 + A_2A_3(-2^{m+1}\widehat{\beta}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} + 2^{2m}\widehat{\beta}\widehat{\gamma} - m\widehat{\alpha}\widehat{\gamma} + n\widehat{\alpha}\widehat{\gamma} - 2^{m+1}n\widehat{\alpha}\widehat{\gamma} + 2^{2m}m\widehat{\alpha}\widehat{\gamma} + 2^{2m}n\widehat{\alpha}\widehat{\gamma}) + A_1A_3(\widehat{\alpha}\widehat{\gamma} - 2^{m+1}\widehat{\alpha}\widehat{\gamma} + 2^{2m}\widehat{\alpha}\widehat{\gamma})).$

Proof. Using the Binet's formula $\mathcal{H}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2n\widehat{\alpha})2^n + A_3\widehat{\gamma}$, we get the required identity. □

As special cases of the above theorem, we give Catalan's identity of hyperbolic modified Woodall, hyperbolic modified Cullen, hyperbolic Woodall and hyperbolic Cullen numbers. Firstly, we present Catalan's identity of hyperbolic Woodall numbers.

COROLLARY 8. *(Catalan's identity for the hyperbolic modified Woodall numbers) For all integers n and m, the following identity holds*

$$\begin{aligned} \mathcal{H}G_{n+m}\mathcal{H}G_{n-m} - \mathcal{H}G_n^2 &= -2^{n-m}(\widehat{\alpha}\widehat{\gamma} - \widehat{\beta}\widehat{\gamma} + 2^{2m}\widehat{\alpha}\widehat{\gamma} - 2^{2m}\widehat{\beta}\widehat{\gamma} - 2^{m+1}\widehat{\alpha}\widehat{\gamma} + 2^{m+1}\widehat{\beta}\widehat{\gamma} \\ &\quad + m\widehat{\alpha}\widehat{\gamma} - n\widehat{\alpha}\widehat{\gamma} + 2^{m+n}m^2\widehat{\alpha}^2 - 2^{2m}m\widehat{\alpha}\widehat{\gamma} - 2^{2m}n\widehat{\alpha}\widehat{\gamma} + 2^{m+1}n\widehat{\alpha}\widehat{\gamma}). \end{aligned}$$

Proof. Take $W_n = G_n$ in Theorem 7. □

Secondly, we give Catalan's identity of hyperbolic modified Cullen numbers.

COROLLARY 9. *(Catalan's identity for the hyperbolic modified Cullen numbers) For all integers n and m, the following identity holds*

$$\mathcal{H}H_{n+m}\mathcal{H}H_{n-m} - \mathcal{H}H_n^2 = 2^{n-m}(2\widehat{\alpha}\widehat{\gamma} + 2 \times 2^{2m}\widehat{\alpha}\widehat{\gamma} - 2 \times 2^{m+1}\widehat{\alpha}\widehat{\gamma}).$$

Proof. Take $W_n = H_n$ in Theorem 7. □

Thirdly, we give Catalan's identity of hyperbolic Woodall numbers.

COROLLARY 10. *(Catalan's identity for the hyperbolic Woodall numbers) For all integers n and m, the following identity holds*

$$\begin{aligned}\mathcal{H}R_{n+m}\mathcal{H}R_{n-m}-\mathcal{H}R_n^2 &= -2^{n-m}(\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\beta}\widehat{\gamma}-2^{m+1}\widehat{\beta}\widehat{\gamma}-m\widehat{\alpha}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma}+2^{m+n}m^2\widehat{\alpha}^2\\ &\quad +2^{2m}m\widehat{\alpha}\widehat{\gamma}+2^{2m}n\widehat{\alpha}\widehat{\gamma}-2^{m+1}n\widehat{\alpha}\widehat{\gamma}).\end{aligned}$$

Proof. Take $W_n=R_n$ in Theorem 7. □

Fourthly, we give Catalan's identity of hyperbolic Cullen numbers.

COROLLARY 11. *(Catalan's identity for the hyperbolic Cullen numbers) For all integers n and m, the following identity holds*

$$\begin{aligned}\mathcal{H}C_{n+m}\mathcal{H}C_{n-m}-\mathcal{H}C_n^2 &= 2^{n-m}(\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\beta}\widehat{\gamma}-2^{m+1}\widehat{\beta}\widehat{\gamma}-m\widehat{\alpha}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma}-2^{m+n}m^2\widehat{\alpha}^2\\ &\quad +2^{2m}m\widehat{\alpha}\widehat{\gamma}+2^{2m}n\widehat{\alpha}\widehat{\gamma}-2^{m+1}n\widehat{\alpha}\widehat{\gamma}).\end{aligned}$$

Proof. Take $W_n=C_n$ in Theorem 7. □

Note that for $m=1$ in Catalan's identity, we get the Cassini's identity for the hyperbolic generalized Woodall sequence.

COROLLARY 12. *(Cassini's identity) For all integers n, the following identity holds*

$$\mathcal{H}W_{n+1}\mathcal{H}W_{n-1}-\mathcal{H}W_n^2=2^{n-1}(A_2A_3(3\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma})-2^{n+1}A_2^2\widehat{\alpha}^2+A_1A_3\widehat{\alpha}\widehat{\gamma}).$$

As special cases of Cassini's identity, we give Cassini's identity of hyperbolic modified Woodall, hyperbolic modified Cullen, hyperbolic Woodall and hyperbolic Cullen numbers. Firstly, we present Cassini's identity of hyperbolic modified Woodall numbers.

COROLLARY 13. *(Cassini's identity of hyperbolic modified Woodall numbers) For all integers n, the following identity holds*

$$\mathcal{H}G_{n+1}\mathcal{H}G_{n-1}-\mathcal{H}G_n^2=2^{n-1}(2\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}-2^{n+1}\widehat{\alpha}^2+n\widehat{\alpha}\widehat{\gamma}).$$

Secondly, we give Cassini's identity of hyperbolic modified Cullen numbers.

COROLLARY 14. *(Cassini's identity of hyperbolic modified Cullen numbers) For all integers n, the following identity holds*

$$\mathcal{H}H_{n+1}\mathcal{H}H_{n-1}-\mathcal{H}H_n^2=2^n\widehat{\alpha}\widehat{\gamma}.$$

Fourth, we give Cassini's identity of hyperbolic Woodall numbers.

COROLLARY 15. *(Cassini's identity of hyperbolic Woodall numbers) For all integers n, the following identity holds*

$$\mathcal{H}R_{n+1}\mathcal{H}R_{n-1}-\mathcal{H}R_n^2=-2^{n-1}(3\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}+2^{n+1}\widehat{\alpha}^2+n\widehat{\alpha}\widehat{\gamma}).$$

Third, we give Cassini's identity of hyperbolic Cullen numbers.

Corollary 16. *(Cassini's identity of hyperbolic Cullen numbers) For all integers n, the following identity holds*

$$\mathcal{H}C_{n+1}\mathcal{H}C_{n-1}-\mathcal{H}C_n^2=2^{n-1}(3\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}-2^{n+1}\widehat{\alpha}^2+n\widehat{\alpha}\widehat{\gamma}).$$

Theorem 17. *For all integers m, n, G_n is woodall numbers, the following identity is true:*

$$\mathcal{H}W_{n+m}=\mathcal{H}W_nG_{m+1}+\mathcal{H}W_{n-1}(-8G_m+4G_{m-1})+4\mathcal{H}W_{n-2}G_m.$$

Proof. The identity (17) can be proved by mathematical induction on m. First of all, we assume that $m\geq 0$ and $n\geq 0$. If $m=0$ we get

$$\mathcal{H}W_n=\mathcal{H}W_nG_1+\mathcal{H}W_{n-1}(-8G_0+4G_{-1})+4\mathcal{H}W_{n-2}G_0$$

which is true by seeing that $G_{-1}=0$, $G_{-2}=\frac{1}{4}$, $G_{-3}=\frac{1}{2}$. We assume that the identity given holds for $m=k$. For $m=k+1$, we get

$$\begin{aligned}
\mathcal{H}W_{(k+1)+n} &= 5\mathcal{H}W_{n+k}-8\mathcal{H}W_{n+k-1}+4\mathcal{H}W_{n+k-2}\\
&= 5(\mathcal{H}W_nG_{k+1}+\mathcal{H}W_{n-1}(-8G_k+4G_{k-1})+4\mathcal{H}W_{n-2}G_k)\\
&\quad -8(\mathcal{H}W_nG_k+\mathcal{H}W_{n-1}(-8G_{k-1}+4G_{k-2})+4\mathcal{H}W_{n-2}G_{k-1})\\
&\quad +4(\mathcal{H}W_nG_{k-1}+\mathcal{H}W_{n-1}(-8G_{k-2}+4G_{k-3})+4\mathcal{H}W_{n-2}G_{k-2})\\
&= \mathcal{H}W_n(5G_{k+1}-8G_k+4G_{k-1})+\mathcal{H}W_{n-1}(-8(5G_k-8G_{k-1}+4G_{k-2})\\
&\quad +4(5G_{k-1}-8G_{k-2}+4G_{k-3}))+4\mathcal{H}W_{n-2}(5G_k-8G_{k-1}+4G_{k-2})\\
&= \mathcal{H}W_nG_{k+2}+\mathcal{H}W_{n-1}(-8G_{k+1}+4G_k)+4\mathcal{H}W_{n-2}G_{k+1}\\
&= \mathcal{H}W_nG_{(k+1)+1}+\mathcal{H}W_{n-1}(-8G_{(k+1)}+4G_{(k+1)-1})+4\mathcal{H}W_{n-2}G_{(k+1)}.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves (17). Similarly, we can show for the other cases.
□

4. Linear Sums For Hyperbolic Generalized Woodall Numbers

In this section, we give the summation formulas of the hyperbolic generalized Woodall numbers with positive and negatif subscripts. Now, we present the summation formulas of the generalized Woodall numbers.

Proposition 18. *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n}W_k=\frac{1}{2}W_2(2n-2^{n+1}(n-1)+2^{n+2}(n-2)+6)-\frac{1}{2}W_1(8n-2^{n+1}(3n-5)+2^{n+2}(3n-8)+22)+W_0(4n-2^{n+1}(n-2)+2^{n+2}(n-3)+9)$.
- $\sum_{k=0}^{n}W_{k+1}=\frac{1}{2}W_2(2n+2^{n+3}(n-1)-2^{n+2}n+8)-\frac{1}{2}W_1(8n-2^{n+2}(3n-2)+2^{n+3}(3n-5)+30)+W_0(4n-2^{n+2}(n-1)+2^{n+3}(n-2)+12)$.

- $\sum_{k=0}^{n} W_{k+2} = \frac{1}{2}W_2(2n - 2^{n+3}(n+1) + 2^{n+4}n + 10) + W_0(4n + 2^{n+4}(n-1) - 2^{n+3}n + 16) - \frac{1}{2}W_1(8n - 2^{n+3}(3n+1) + 2^{n+4}(3n-2) + 40)$.
- $\sum_{k=0}^{n} W_{k+3} = W_0(4n - 2^{n+4}(n+1) + 2^{n+5}n + 20) - \frac{1}{2}W_1(8n + 2^{n+5}(3n+1) - 2^{n+4}(3n+4) + 48) + \frac{1}{2}W_2(2n - 2^{n+4}(n+2) + 2^{n+5}(n+1) + 10)$.

Proof. For the proof, see Soykan [29]. □

PROPOSITION 19. *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n} W_{2k} = \frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32)$.
- $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{18}W_2(18n - 2^{2n+3}(2n+1) + 2^{2n+5}(2n-1) + 40) - \frac{1}{18}W_1(72n - 2^{2n+3}(6n+1) + 2^{2n+5}(6n-5) + 150) + \frac{1}{9}W_0(36n + 2^{2n+5}(2n-2) - 2 \times 2^{2n+3}n + 64)$.
- $\sum_{k=0}^{n} W_{2k+2} = \frac{1}{9}W_0(36n - 2^{2n+4}(2n+1) + 2^{2n+6}(2n-1) + 80) - \frac{1}{18}W_1(72n - 2^{2n+4}(6n+4) + 2^{2n+6}(6n-2) + 192) + \frac{1}{18}W_2(18n - 2^{2n+4}(2n+2) + 2 \times 2^{2n+6}n + 50)$.
- $\sum_{k=0}^{n} W_{2k+3} = \frac{1}{18}W_2\left((18n - 2^{2n+5}(2n+3) + 2^{2n+7}(2n+1) + 58\right) - \frac{1}{18}W_1(72n + 2^{2n+7}(6n+1) - 2^{2n+5}(6n+7) + 240) + \frac{1}{9}W_0(36n - 2^{2n+5}(2n+2) + 2 \times 2^{2n+7}n + 100)$.
- $\sum_{k=0}^{n} W_{2k+4} = \frac{1}{18}W_2(18n - 2^{2n+6}(2n+4) + 2^{2n+8}(2n+2) + 50) + \frac{1}{9}W_0(36n - 2^{2n+6}(2n+3) + 2^{2n+8}(2n+1) + 116) - \frac{1}{18}W_1(72n + 2^{2n+8}(6n+4) - 2^{2n+6}(6n+10) + 264)$.

Proof. For the proof, see Soykan [29]. □

PROPOSITION 20. *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n} W_{-k} = 4W_0(n + \frac{1}{2^{n+1}}(n+4) - \frac{1}{2^{n+2}}(n+3) - 1) + 2W_1(\frac{1}{2^{n+2}}(3n+8) - 2n - \frac{1}{2^{n+1}}(3n+11) + \frac{7}{2}) + 2W_2(\frac{1}{2}n + \frac{1}{2^{n+1}}(n+3) - \frac{1}{2^{n+2}}(n+2) - 1)$.
- $\sum_{k=0}^{n} W_{-k+1} = 2W_2(\frac{1}{2}n + \frac{1}{2^n}(n+2) - \frac{1}{2^{n+1}}(n+1) - \frac{3}{2}) + 4W_0(n + \frac{1}{2^n}(n+3) - \frac{1}{2^{n+1}}(n+2) - 2) + 2W_1(\frac{1}{2^{n+1}}(3n+5) - 2n - \frac{1}{2^n}(3n+8) + 6)$.
- $\sum_{k=0}^{n} W_{-k+2} = 2W_2(\frac{1}{2}n + 2^{1-n}(n+1) - \frac{1}{2^n}n - \frac{3}{2}) + 4W_0(n - \frac{1}{2^n}(n+1) + 2^{1-n}(n+2) - 3) - 2W_1(2n + 2^{1-n}(3n+5) - \frac{1}{2^n}(3n+2) - 8)$.
- $\sum_{k=0}^{n} W_{-k+3} = 2W_2(\frac{1}{2}n + 2^{2-n}n - 2^{1-n}(n-1) + \frac{1}{2}) + 2W_1(2^{1-n}(3n-1) - 2n - 2^{2-n}(3n+2) + 6) + 4W_0(n - 2^{1-n}n + 2^{2-n}(n+1) - 3)$.

Proof. For the proof, see Soykan [29]. □

PROPOSITION 21. *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n} W_{-2k} = \frac{8}{9}W_1(\frac{1}{2^{2n+4}}(6n+8) - \frac{9}{2}n - \frac{1}{2^{2n+2}}(6n+14) + 3) + \frac{16}{9}W_0(\frac{9}{4}n + \frac{1}{2^{2n+2}}(2n+5) - \frac{1}{2^{2n+4}}(2n+3) - \frac{1}{2}) + \frac{8}{9}W_2(\frac{9}{8}n + \frac{1}{2^{2n+2}}(2n+4) - \frac{1}{2^{2n+4}}(2n+2) - \frac{7}{8})$.
- $\sum_{k=0}^{n} W_{-2k+1} = \frac{8}{9}W_1(\frac{1}{2^{2n+3}}(6n+5) - \frac{9}{2}n - \frac{1}{2^{2n+1}}(6n+11) + 6) + \frac{16}{9}W_0(\frac{9}{4}n + \frac{1}{2^{2n+1}}(2n+4) - \frac{1}{2^{2n+3}}(2n+2) - \frac{7}{4}) + \frac{8}{9}W_2(\frac{9}{8}n + \frac{1}{2^{2n+1}}(2n+3) - \frac{1}{2^{2n+3}}(2n+1) - \frac{11}{8})$.

- $\sum_{k=0}^{n} W_{-2k+2} = \frac{8}{9}W_2(\frac{9}{8}n - \frac{2}{2^{2n+2}}n + \frac{1}{2^{2n}}(2n+2) - \frac{7}{8}) - \frac{16}{9}W_0(\frac{1}{2^{2n+2}}(2n+1) - \frac{9}{4}n - \frac{1}{2^{2n}}(2n+3) + \frac{11}{4}) + \frac{8}{9} W_1(\frac{1}{2^{2n+2}}(6n+2) - \frac{9}{2}n - \frac{1}{2^{2n}}(6n+8) + \frac{15}{2})$.
- $\sum_{k=0}^{n} W_{-2k+3} = \frac{8}{9}W_1(\frac{1}{2^{2n+1}}(6n-1) - \frac{9}{2}n - 2^{1-2n}(6n+5) + \frac{3}{2}) + \frac{8}{9}W_2(\frac{9}{8}n - \frac{1}{2^{2n+1}}(2n-1) + 2^{1-2n}(2n+1) + \frac{25}{8}) + \frac{16}{9}W_0(\frac{9}{4}n + 2^{1-2n}(2n+2) - \frac{2}{2^{2n+1}}n - \frac{7}{4})$.
- $\sum_{k=0}^{n} W_{-2k+4} = \frac{8}{9}W_2(\frac{9}{8}n + 2\times 2^{2-2n}n - \frac{1}{2^{2n}}(2n-2) + \frac{137}{8}) + \frac{16}{9}W_0(\frac{9}{4}n + 2^{2-2n}(2n+1) - \frac{1}{2^{2n}}(2n-1) + \frac{25}{4}) - \frac{8}{9}W_1(\frac{9}{2}n + 2^{2-2n}(6n+2) - \frac{1}{2^{2n}}(6n-4) + \frac{57}{2})$.

Proof. For the proof, see Soykan [29]. □

Now, we present the formulas which give the summation of the hyperbolic generalized Woodall numbers.

THEOREM 22. *For $n \geq 0$, hyperbolic generalized Woodall numbers have the following formulas:*

(a): $\sum_{k=0}^{n} \mathcal{H}W_k = (3+n-3\times 2^n + 2^n n + 4j + jn - 2^{n+2}j + 2^{n+1}jn)W_2 + (-11-4n+11\times 2^n - 3\times 2^n n - 15j - 4jn + 2^{n+4}j - 3\times 2^{n+1}jn)W_1 + (9+4n-2^{n+3}+2^{n+1}n + 12j + 4jn - 3\times 2^{n+2}j + 2^{n+2}jn)W_0$.

(b): $\sum_{k=0}^{n} \mathcal{H}W_{2k} = (\frac{16}{9} + n - \frac{1}{9}2^{2n+4} + \frac{1}{3}2^{2n+2}n + \frac{20}{9}j + jn - \frac{5}{9}2^{2n+2}j + \frac{1}{3}2^{2n+3}jn)W_2 + (-\frac{20}{3} - 4n + \frac{5}{3}2^{2n+2} - 2^{2n+2}n - \frac{25}{3}j + \frac{7}{3}2^{2n+2}j - 4jn - 2^{2n+3}jn)W_1 + (\frac{53}{9} - \frac{11}{9}2^{2n+2} + 4n + \frac{1}{3}2^{2n+3}n + \frac{64}{9}j - \frac{1}{9}2^{2n+6} j + 4jn + \frac{1}{3}2^{2n+4}jn)W_0$.

(c): $\sum_{k=0}^{n} \mathcal{H}W_{2k+1} = (\frac{20}{9} - \frac{5}{9}2^{2n+2} + n + \frac{1}{3}2^{2n+3}n + \frac{25}{9}j - \frac{1}{9}2^{2n+4}j + jn + \frac{1}{3}2^{2n+4}jn)W_2 + (-\frac{25}{3} + \frac{7}{3}2^{2n+2} - 4n - 2^{2n+3}n + \frac{1}{3}2^{2n+5}j - \frac{32}{3}j - 4jn - 2^{2n+4}jn)W_1 + (\frac{64}{9} - \frac{1}{9}2^{2n+6} + 4n + \frac{1}{3}2^{2n+4}n + \frac{80}{9}j - \frac{5}{9}2^{2n+4} j + 4jn + \frac{1}{3}2^{2n+5}jn)W_0$.

Proof. Proof can be obtained by using Proposition 21.

(a): We can derive the following using the formulas in Proposition 18.

$$\sum_{k=0}^{n} \mathcal{H}W_k = \sum_{k=0}^{n} W_k + j\sum_{k=0}^{n} W_{k+1}.$$

$$\begin{aligned}
&\sum_{k=0}^{n} \mathcal{H}W_k \\
&= \frac{1}{2}W_2(2n - 2^{n+1}(n-1) + 2^{n+2}(n-2) + 6) - \frac{1}{2}W_1(8n - 2^{n+1}(3n-5) \\
&\quad +2^{n+2}(3n-8) + 22) + W_0(4n - 2^{n+1}(n-2) + 2^{n+2}(n-3) + 9) \\
&\quad +j(\frac{1}{2}W_2(2n + 2^{n+3}(n-1) - 2^{n+2}n + 8) - \frac{1}{2}W_1(8n - 2^{n+2}(3n-2) \\
&\quad +2^{n+3}(3n-5) + 30) + W_0(4n - 2^{n+2}(n-1) + 2^{n+3}(n-2) + 12)).
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^{n} \mathcal{H}W_k \\
&= (3 + n - 3\times 2^n + 2^n n + 4j + jn - 2^{n+2}j + 2^{n+1}jn)W_2 \\
&\quad +(-11 - 4n + 11\times 2^n - 3\times 2^n n - 15j - 4jn + 2^{n+4}j - 3\times 2^{n+1}jn)W_1 \\
&\quad +(9 + 4n - 2^{n+3} + 2^{n+1}n + 12j + 4jn - 3\times 2^{n+2}j + 2^{n+2}jn)W_0.
\end{aligned}$$

The proof is finished. □

(b): We can derive the following using the formulas in Proposition 19.

$$\sum_{k=0}^{n}\mathcal{H}W_{2k}=\sum_{k=0}^{n}W_{2k}+j\sum_{k=0}^{n}W_{2k+1}.$$

$$\begin{aligned}
&\sum_{k=0}^{n}\mathcal{H}W_{2k}\\
&= \frac{1}{9}W_0(36n-2^{2n+2}(2n-1)+2^{2n+4}(2n-3)+53)-\frac{1}{18}W_1(72n-2^{2n+2}(6n-2)\\
&\quad +2^{2n+4}(6n-8)+120)+\frac{1}{18}W_2(18n+2^{2n+4}(2n-2)-2\times 2^{2n+2}n+32)\\
&\quad +j(\frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)-\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)\\
&\quad +2^{2n+5}(6n-5)+150)+\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64)).
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^{n}\mathcal{H}W_{2k}\\
&= (\frac{16}{9}+n-\frac{1}{9}2^{2n+4}+\frac{1}{3}2^{2n+2}n+\frac{20}{9}j+jn-\frac{5}{9}2^{2n+2}j+\frac{1}{3}2^{2n+3}jn)W_2\\
&\quad +(-\frac{20}{3}-4n+\frac{5}{3}2^{2n+2}-2^{2n+2}n-\frac{25}{3}j+\frac{7}{3}2^{2n+2}j-4jn-2^{2n+3}jn)W_1\\
&\quad +(\frac{53}{9}-\frac{11}{9}2^{2n+2}+4n+\frac{1}{3}2^{2n+3}n+\frac{64}{9}j-\frac{1}{9}2^{2n+6}j+4jn+\frac{1}{3}2^{2n+4}jn)W_0.
\end{aligned}$$

The proof is completed. □

(c): We can derive the following using the formulas in Proposition 21.

$$\sum_{k=0}^{n}\mathcal{H}W_{2k+1}=\sum_{k=0}^{n}W_{2k+1}+j\sum_{k=0}^{n}W_{2k+2}.$$

$$\begin{aligned}
&\sum_{k=0}^{n}\mathcal{H}W_{2k+1}\\
&= \frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)-\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)\\
&\quad +2^{2n+5}(6n-5)+150)+\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64)\\
&\quad +j(\frac{1}{9}W_0(36n-2^{2n+4}(2n+1)+2^{2n+6}(2n-1)+80)-\frac{1}{18}W_1(72n-2^{2n+4}(6n+4)\\
&\quad +2^{2n+6}(6n-2)+192)+\frac{1}{18}W_2(18n-2^{2n+4}(2n+2)+2\times 2^{2n+6}n+50)).
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^{n}\mathcal{H}W_{2k+1}\\
&= (\frac{20}{9}-\frac{5}{9}2^{2n+2}+n+\frac{1}{3}2^{2n+3}n+\frac{25}{9}j-\frac{1}{9}2^{2n+4}j+jn+\frac{1}{3}2^{2n+4}jn)W_2\\
&\quad +(-\frac{25}{3}+\frac{7}{3}2^{2n+2}-4n-2^{2n+3}n+\frac{1}{3}2^{2n+5}j-\frac{32}{3}j-4jn-2^{2n+4}jn)W_1\\
&\quad +(\frac{64}{9}-\frac{1}{9}2^{2n+6}+4n+\frac{1}{3}2^{2n+4}n+\frac{80}{9}j-\frac{5}{9}2^{2n+4}j+4jn+\frac{1}{3}2^{2n+5}jn)W_0.
\end{aligned}$$

The proof is finished. □

As a first special case of the above theorem, we have the following summation formulas for hyperbolic Woodall numbers:

Corollary 23. *For $n \geq 0$, hyperbolic modified Woodall numbers have the following properties:*

(a): $\sum_{k=0}^{n} \mathcal{H}G_k = 4 + n + 2^{n+1}n - 2^{n+2} + j(5 - 5 \times 2^{n+2} + n + 2^{n+4} + 2^{n+2}n)$.

(b): $\sum_{k=0}^{n} \mathcal{H}G_{2k} = \frac{20}{9} + n + \frac{2}{3}2^{2n+2}n + \frac{5}{3}2^{2n+2} - \frac{5}{9}2^{2n+4} + j(\frac{25}{9} - \frac{4}{9}2^{2n+2} + n + \frac{2}{3}2^{2n+3}n)$.

(c): $\sum_{k=0}^{n} \mathcal{H}G_{2k+1} = \frac{25}{9} + n + \frac{2}{3}2^{2n+3}n - \frac{4}{9}2^{2n+2} + j(\frac{29}{9} - \frac{5}{9}2^{2n+4} + \frac{1}{3}2^{2n+5} + n + \frac{2}{3}2^{2n+4}n)$.

As a second special case of the above theorem, we have the following summation formulas for hyperbolic modified Cullen numbers:

Corollary 24. *For $n \geq 0$, hyperbolic modified Cullen numbers have the following properties:*

(a): $\sum_{k=0}^{n} \mathcal{H}H_k = -1 + n - 6 \times 2^n n - 3 \times 2^{n+3} + 3 \times 2^{n+1}n + 28 \times 2^n + j(-3 - 18 \times 2^{n+2} + 5 \times 2^{n+4} + n - 6 \times 2^{n+1}n + 3 \times 2^{n+2}n)$.

(b): $\sum_{k=0}^{n} \mathcal{H}H_{2k} = \frac{1}{3} + n - 2^{2n+3}n + 2^{2n+3}n + \frac{14}{3}2^{2n+2} - 2^{2n+4} + j(-\frac{1}{3} + \frac{20}{3}2^{2n+2} - \frac{1}{3}2^{2n+6} + n - 2^{2n+4}n + 2^{2n+4}n)$.

(c): $\sum_{k=0}^{n} \mathcal{H}H_{2k+1} = -\frac{1}{3} + n - 2 \times 2^{2n+3}n + 2^{2n+4}n + \frac{20}{3}2^{2n+2} - \frac{1}{3}2^{2n+6} + j(-\frac{5}{3} - \frac{8}{3}2^{2n+4} + \frac{5}{3}2^{2n+5} + n - 2^{2n+5}n + 2^{2n+5}n)$.

As a third special case of the above theorem, we have the following summation formulas for hyperbolic Woodall numbers:

Corollary 25. *For $n \geq 0$, hyperbolic Woodall numbers have the following properties:*

(a): $\sum_{k=0}^{n} \mathcal{H}R_k = 1 - n + 4 \times 2^n n + 2^{n+3} - 2^{n+1}n - 10 \times 2^n + j(1 - 2^{n+4} + 2^{n+4} - n + 2^{n+3}n - 2^{n+2}n)$.

(b): $\sum_{k=0}^{n} \mathcal{H}R_{2k} = -\frac{1}{9} - n + \frac{4}{3}2^{2n+2}n - \frac{1}{3}2^{2n+3}n + \frac{26}{9}2^{2n+2} - \frac{7}{9}2^{2n+4} + j(\frac{1}{9} - n - \frac{14}{9}2^{2n+2} + \frac{1}{9}2^{2n+6} + \frac{4}{3}2^{2n+3}n - \frac{1}{3}2^{2n+4}n)$.

(c): $\sum_{k=0}^{n} \mathcal{H}R_{2k+1} = \frac{1}{9} - n + \frac{4}{3}2^{2n+3}n - \frac{1}{3}2^{2n+4}n - \frac{14}{9}2^{2n+2} + \frac{1}{9}2^{2n+6} + j(-\frac{1}{9} + \frac{1}{3}2^{2n+5} - \frac{2}{9}2^{2n+4} - n + \frac{4}{3}2^{2n+4}n - \frac{1}{3}2^{2n+5}n)$.

As a fourth special case of the above theorem, we have the following summation formulas for hyperbolic Cullen numbers:

Corollary 26. *For $n \geq 0$, hyperbolic Cullen numbers have the following properties.*

(a): $\sum_{k=0}^{n} \mathcal{H}C_k = 3 + n - 2^{n+3} + 2^{n+1}n + 6 \times 2^n + j(3 + n + 2^{n+2}n)$.

(b): $\sum_{k=0}^{n} \mathcal{H}C_{2k} = \frac{17}{9} + n + \frac{1}{3}2^{2n+3}n - \frac{2}{9}2^{2n+2} + j(\frac{19}{9} + n + \frac{1}{9}2^{2n+3} + \frac{1}{3}2^{2n+4}n)$.

(c): $\sum_{k=0}^{n} \mathcal{H}C_{2k+1} = \frac{19}{9} + n + \frac{1}{3}2^{2n+4}n + \frac{1}{9}2^{2n+3} + j(\frac{17}{9} + \frac{4}{9}2^{2n+4} + n + \frac{1}{3}2^{2n+5}n)$.

We now introduce the formulas that allow us to find the sum of hyperbolic generalized Woodall numbers with negative subscripts in the following theorem.

THEOREM 27. *For $n \geq 0$, hyperbolic generalized Woodall numbers have the following formulas:*

(a): $\sum_{k=0}^{n} \mathcal{H}W_{-k} = (-2 + \frac{2}{2^n} - 3j + n + \frac{3}{2^n}j + \frac{1}{2\times 2^n}n + jn + \frac{1}{2^n}jn)W_2 + (7 - \frac{7}{2^n} + 12j - 4n - \frac{11}{2^n}j - \frac{3}{2\times 2^n}n - 4jn - \frac{3}{2^n}jn)W_1 + (-4 + \frac{5}{2^n} - 8j + 4n + \frac{8}{2^n}j + \frac{1}{2^n}n + 4jn + \frac{2}{2^n}jn)W_0.$

(b): $\sum_{k=0}^{n} \mathcal{H}W_{-2k} = (-\frac{7}{9} + \frac{7}{9\times 2^{2n}} - \frac{11}{9}j + n + \frac{11}{9\times 2^{2n}}j + \frac{1}{3\times 2^{2n}}n + jn + \frac{2}{3\times 2^{2n}}jn)W_2 + (\frac{8}{3} - \frac{8}{3\times 2^{2n}} + \frac{16}{3}j - 4n - \frac{13}{3\times 2^{2n}}j - \frac{1}{2^{2n}}n - 4jn - \frac{2}{2^{2n}}jn)W_1 + (-\frac{8}{9} + \frac{17}{9\times 2^{2n}} - \frac{28}{9}j + 4n + \frac{28}{9\times 2^{2n}}j + \frac{2}{3\times 2^{2n}}n + 4jn + \frac{4}{3\times 2^{2n}}jn)W_0.$

(c): $\sum_{k=0}^{n} \mathcal{H}W_{-2k+1} = (-\frac{11}{9} + \frac{11}{9\times 2^{2n}} - \frac{7}{9}j + n + \frac{16}{9\times 2^{2n}}j + \frac{2}{3\times 2^{2n}}n + jn + \frac{4}{3\times 2^{2n}}jn)W_2 + (\frac{16}{3} - \frac{13}{3\times 2^{2n}} + \frac{20}{3}j - 4n - \frac{20}{3\times 2^{2n}}j - \frac{2}{2^{2n}}n - 4jn - \frac{4}{2^{2n}}jn)W_1 + (-\frac{28}{9} + \frac{28}{9\times 2^{2n}} - \frac{44}{9}j + 4n + \frac{44}{9\times 2^{2n}}j + \frac{4}{3\times 2^{2n}}n + 4jn + \frac{8}{3\times 2^{2n}}jn)W_0.$

Proof. It can be obtained by using Proposition 20.

(a): We can derive the following using the formulas in Proposition 20.

$$\sum_{k=0}^{n} \mathcal{H}W_{-k} = \sum_{k=0}^{n} W_{-k} + j\sum_{k=0}^{n} W_{-k+1}.$$

$$\begin{aligned}\sum_{k=0}^{n} \mathcal{H}W_{-k} &= 4W_0(n + \frac{1}{2^{n+1}}(n+4) - \frac{1}{2^{n+2}}(n+3) - 1) + 2W_1(\frac{1}{2^{n+2}}(3n+8) - 2n - \frac{1}{2^{n+1}}(3n+11) + \frac{7}{2}) \\ &+2W_2(\frac{1}{2}n + \frac{1}{2^{n+1}}(n+3) - \frac{1}{2^{n+2}}(n+2) - 1) \\ &+j(2W_2(\frac{1}{2}n + \frac{1}{2^n}(n+2) - \frac{1}{2^{n+1}}(n+1) - \frac{3}{2}) + 4W_0(n + \frac{1}{2^n}(n+3) - \frac{1}{2^{n+1}}(n+2) - 2) \\ &+2W_1(\frac{1}{2^{n+1}}(3n+5) - 2n - \frac{1}{2^n}(3n+8) + 6)).\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n} \mathcal{H}W_{-k} &= (-2 + \frac{2}{2^n} - 3j + n + \frac{3}{2^n}j + \frac{1}{2\times 2^n}n + jn + \frac{1}{2^n}jn)W_2 \\ &+(7 - \frac{7}{2^n} + 12j - 4n - \frac{11}{2^n}j - \frac{3}{2\times 2^n}n - 4jn - \frac{3}{2^n}jn)W_1 \\ &+(-4 + \frac{5}{2^n} - 8j + 4n + \frac{8}{2^n}j + \frac{1}{2^n}n + 4jn + \frac{2}{2^n}jn)W_0.\end{aligned}$$

This proves (a). We can be prove (b) and (c) similarly way using Proposition 21. □

As a first special case of the above theorem, we have the following summation formulas for hyperbolic modified Woodall numbers:

COROLLARY 28. *For $n \geq 0$, hyperbolic modified Woodall numbers have the following properties:*

(a): $\sum_{k=0}^{n} \mathcal{H}G_{-k} = -3 + n + \frac{n+3}{2^n} + j(-3 + n + \frac{2n+4}{2^n}).$

(b): $\sum_{k=0}^{n} \mathcal{H}G_{-2k} = -\frac{11}{9} + n + \frac{11+6n}{9\times 2^{2n}} + j(-\frac{7}{9} + n + \frac{16+12n}{9\times 2^{2n}}).$

(c): $\sum_{k=0}^{n} \mathcal{H}G_{-2k+1} = -\frac{7}{9} + n + \frac{16+12n}{9\times 2^{2n}} + j(\frac{25}{9} + n + \frac{20+24n}{9\times 2^{2n}}).$

As a second special case of the above theorem, we have the following summation formulas for hyperbolic modified Cullen numbers:

COROLLARY 29. *For $n \geq 0$, hyperbolic modified Cullen numbers have the following properties:*

(a): $\sum_{k=0}^{n}\mathcal{H}H_{-k}=5+n-\frac{2}{2^n}+j(9-\frac{4}{2^n}+n)$.

(b): $\sum_{k=0}^{n}\mathcal{H}H_{-2k}=\frac{11}{3}+n-\frac{2}{3\times 2^{2n}}+j(\frac{19}{3}-\frac{4}{3\times 2^{2n}}+n)$.

(c): $\sum_{k=0}^{n}\mathcal{H}H_{-2k+1}=\frac{19}{3}+n-\frac{4}{3\times 2^{2n}}+j(\frac{35}{3}-\frac{8}{3\times 2^{2n}}+n)$.

As a third special case of the above theorem, we have the following summation formulas for hyperbolic Woodall numbers:

COROLLARY 30. *For $n\geq 0$, hyperbolic Woodall numbers have the following properties:*

(a): $\sum_{k=0}^{n}\mathcal{H}R_{-k}=-3-n+\frac{2+n}{2^n}+j(-1-n+\frac{2+2n}{2^n})$.

(b): $\sum_{k=0}^{n}\mathcal{H}R_{-2k}=-\frac{17}{9}-n+\frac{8+6n}{9\times 2^{2n}}+j(-\frac{1}{9}-n+\frac{10+12n}{9\times 2^{2n}})$.

(c): $\sum_{k=0}^{n}\mathcal{H}R_{-2k+1}=-\frac{1}{9}-n+\frac{10+12n}{9\times 2^{2n}}+j(\frac{55}{9}-n+\frac{8+24n}{9\times 2^{2n}})$.

As a fourth special case of the above theorem, we have the following summation formulas for hyperbolic Cullen numbers:

COROLLARY 31. *For $n\geq 0$, hyperbolic Cullen numbers have the following properties:*

(a): $\sum_{k=0}^{n}\mathcal{H}C_{-k}=-1+n+\frac{2+n}{2^n}+j(1+\frac{2+2n}{2^n}+n)$.

(b): $\sum_{k=0}^{n}\mathcal{H}C_{-2k}=\frac{1}{9}+n+\frac{8+6n}{9\times 2^{2n}}+j(\frac{17}{9}+\frac{10+12n}{9\times 2^{2n}}+n)$.

(c): $\sum_{k=0}^{n}\mathcal{H}C_{-2k+1}=\frac{17}{9}+n+\frac{10+12n}{9\times 2^{2n}}+j(\frac{73}{9}+\frac{8+24n}{9\times 2^{2n}}+n)$.

5. Matrices related with Hyperbolic Generalized Woodall Numbers

In this section, we present matrices related with hyperbolic generalized Woodall numbers.

Now, $\{G_n\}$ defined by the third-order recurrence relation as follows

$$G_n=5G_{n-1}-8G_{n-2}+4G_{n-3}\text{ with the initial conditions }G_0=0,\ G_1=1,\ G_2=5.$$

We present the square matrix A of order 3 as

$$A=\begin{pmatrix}5 & -8 & 4\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

such that $\det A=1$. Then, we give the following Lemma.

LEMMA 32. *For all integers n the following identity is true.*

$$\begin{pmatrix}\mathcal{H}W_{n+2}\\ \mathcal{H}W_{n+1}\\ \mathcal{H}W_n\end{pmatrix}=\begin{pmatrix}5 & -8 & 4\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}^n\begin{pmatrix}\mathcal{H}W_2\\ \mathcal{H}W_1\\ \mathcal{H}W_0\end{pmatrix}.$$

Proof. First, we suppose that $n \geq 0$. Lemma (32) can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} \mathcal{H}W_{k+2} \\ \mathcal{H}W_{k+1} \\ \mathcal{H}W_k \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix}$$

For $n = k + 1$, we get

$$\begin{aligned} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix} &= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix} \\ &= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{H}W_{k+2} \\ \mathcal{H}W_{k+1} \\ \mathcal{H}W_k \end{pmatrix} \\ &= \begin{pmatrix} 5\mathcal{H}W_{k+2} - 8\mathcal{H}W_{k+1} + 4\mathcal{H}W_k \\ \mathcal{H}W_{k+2} \\ \mathcal{H}W_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{H}W_{k+3} \\ \mathcal{H}W_{k+2} \\ \mathcal{H}W_{k+1} \end{pmatrix}. \end{aligned}$$

If we suppose that $n < 0$ the proof can be done similarly. Consequently, by mathematical induction on n, the proof is completed. □

Note that

$$A^n = \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & 4G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & 4G_{n-2} \end{pmatrix}.$$

For the proof see [31].

THEOREM 33. *If we define the matrices* $N_{\mathcal{H}W}$ *and* $E_{\mathcal{H}W}$ *as follow,*

$$N_{\mathcal{H}W} = \begin{pmatrix} \mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0 \\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1} \\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2} \end{pmatrix}, \ E_{\mathcal{H}W} = \begin{pmatrix} \mathcal{H}W_{n+2} & \mathcal{H}W_{n+1} & \mathcal{H}W_n \\ \mathcal{H}W_{n+1} & \mathcal{H}W_n & \mathcal{H}W_{n-1} \\ \mathcal{H}W_n & \mathcal{H}W_{n-1} & \mathcal{H}W_{n-2} \end{pmatrix}.$$

then the following identity is true:

$$A^n N_{\mathcal{H}W} = E_{\mathcal{H}W}.$$

Proof. We can use the following identities for the proof.

$$\begin{aligned}
A^n N_{\mathcal{H}W} &= \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & 4G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & 4G_{n-2} \end{pmatrix} \begin{pmatrix} \mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0 \\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1} \\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2} \end{pmatrix}, \\
&= \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
b_{11} &= \mathcal{H}W_2 G_{n+1} + \mathcal{H}W_1 (-8G_n + 4G_{n-1}) + \mathcal{H}W_0 4G_n, \\
b_{12} &= \mathcal{H}W_1 G_{n+1} + \mathcal{H}W_0 (-8G_n + 4G_{n-1}) + \mathcal{H}W_{-1} 4G_n, \\
b_{13} &= \mathcal{H}W_0 G_{n+1} + \mathcal{H}W_{-1} (-8G_n + 4G_{n-1}) + \mathcal{H}W_{-2} 4G_n, \\
b_{21} &= \mathcal{H}W_2 G_n + \mathcal{H}W_1 (-8G_n + 4G_{n-1}) + \mathcal{H}W_0 4G_{n-1}, \\
b_{22} &= \mathcal{H}W_1 G_n + \mathcal{H}W_0 (-8G_n + 4G_{n-1}) + \mathcal{H}W_{-1} 4G_{n-1}, \\
b_{23} &= \mathcal{H}W_0 G_n + \mathcal{H}W_{-1} (-8G_n + 4G_{n-1}) + \mathcal{H}W_{-2} 4G_{n-1}, \\
b_{31} &= \mathcal{H}W_2 G_{n-1} + \mathcal{H}W_1 (-8G_n + 4G_{n-1}) + \mathcal{H}W_0 4G_{n-2}, \\
b_{32} &= \mathcal{H}W_1 G_{n-1} + \mathcal{H}W_0 (-8G_n + 4G_{n-1}) + \mathcal{H}W_{-1} 4G_{n-2}, \\
b_{33} &= \mathcal{H}W_0 G_{n-1} + \mathcal{H}W_{-1} (-8G_n + 4G_{n-1}) + \mathcal{H}W_{-2} 4G_{n-2},
\end{aligned}$$

Using the Theorem (17) the proof is done. □

From Theorem (33), we can write the following corollary.

COROLLARY 34. *We have the following identity.*

(a): *If we define $N_{\mathcal{H}G}$ and $E_{\mathcal{H}G}$ as follows,*

$$N_{\mathcal{H}G} = \begin{pmatrix} \mathcal{H}G_2 & \mathcal{H}G_1 & \mathcal{H}G_0 \\ \mathcal{H}G_1 & \mathcal{H}G_0 & \mathcal{H}G_{-1} \\ \mathcal{H}G_0 & \mathcal{H}G_{-1} & \mathcal{H}G_{-2} \end{pmatrix}, \quad E_{\mathcal{H}G} = \begin{pmatrix} \mathcal{H}G_{n+2} & \mathcal{H}G_{n+1} & \mathcal{H}G_n \\ \mathcal{H}G_{n+1} & \mathcal{H}G_n & \mathcal{H}G_{n-1} \\ \mathcal{H}G_n & \mathcal{H}G_{n-1} & \mathcal{H}G_{n-2} \end{pmatrix},$$

then we get

$$A^n N_{\mathcal{H}G} = E_{\mathcal{H}G}.$$

(b): *If we define* $N_{\mathcal{H}H}$ *and* $E_{\mathcal{H}H}$ *as follows,*

$$N_{\mathcal{H}H} = \begin{pmatrix} \mathcal{H}H_2 & \mathcal{H}H_1 & \mathcal{H}H_0 \\ \mathcal{H}H_1 & \mathcal{H}H_0 & \mathcal{H}H_{-1} \\ \mathcal{H}H_0 & \mathcal{H}H_{-1} & \mathcal{H}H_{-2} \end{pmatrix}, \ E_{\mathcal{H}H} = \begin{pmatrix} \mathcal{H}H_{n+2} & \mathcal{H}H_{n+1} & \mathcal{H}H_n \\ \mathcal{H}H_{n+1} & \mathcal{H}H_n & \mathcal{H}H_{n-1} \\ \mathcal{H}H_n & \mathcal{H}H_{n-1} & \mathcal{H}H_{n-2} \end{pmatrix},$$

then we get

$$A^n N_{\mathcal{H}H} = E_{\mathcal{H}H}.$$

(c): *If we define* $N_{\mathcal{H}R}$ *and* $E_{\mathcal{H}R}$ *as follows,*

$$N_{\mathcal{H}R} = \begin{pmatrix} \mathcal{H}R_2 & \mathcal{H}R_1 & \mathcal{H}R_0 \\ \mathcal{H}R_1 & \mathcal{H}R_0 & \mathcal{H}R_{-1} \\ \mathcal{H}R_0 & \mathcal{H}R_{-1} & \mathcal{H}R_{-2} \end{pmatrix}, \ E_{\mathcal{H}R} = \begin{pmatrix} \mathcal{H}R_{n+2} & \mathcal{H}R_{n+1} & \mathcal{H}R_n \\ \mathcal{H}R_{n+1} & \mathcal{H}R_n & \mathcal{H}R_{n-1} \\ \mathcal{H}R_n & \mathcal{H}R_{n-1} & \mathcal{H}R_{n-2} \end{pmatrix},$$

then we get

$$A^n N_{\mathcal{H}R} = E_{\mathcal{H}R}.$$

(d): *If we define* $N_{\mathcal{H}C}$ *and* $E_{\mathcal{H}C}$ *as follows,*

$$N_{\mathcal{H}C} = \begin{pmatrix} \mathcal{H}C_2 & \mathcal{H}C_1 & \mathcal{H}C_0 \\ \mathcal{H}C_1 & \mathcal{H}C_0 & \mathcal{H}C_{-1} \\ \mathcal{H}C_0 & \mathcal{H}C_{-1} & \mathcal{H}C_{-2} \end{pmatrix}, \ E_{\mathcal{H}C} = \begin{pmatrix} \mathcal{H}C_{n+2} & \mathcal{H}C_{n+1} & \mathcal{H}C_n \\ \mathcal{H}C_{n+1} & \mathcal{H}C_n & \mathcal{H}C_{n-1} \\ \mathcal{H}C_n & \mathcal{H}C_{n-1} & \mathcal{H}C_{n-2} \end{pmatrix},$$

then we get

$$A^n N_{\mathcal{H}C} = E_{\mathcal{H}C}.$$

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