Original Research Article

A Study on Hyperbolic Generalized Guglielmo Numbers

Abstract. In this paper, we introduce the generalized hyperbolic Guglielmo numbers. We delve into various specific instances, including hyperbolic triangular numbers, hyperbolic triangular-Lucas numbers, hyperbolic oblong numbers, and hyperbolic pentagonal numbers. We present Binet's formulas, generating functions and summation formulas for these numbers. Furthermore, we provide Catalan's and Cassini's identities and matrices associated with these sequences.

Keywords. Triangular numbers, triangular-Lucas numbers, oblong numbers, pentegonal numbers, hyperbolic triangular numbers, hyperbolic triangular-Lucas numbers, hyperbolic oblong numbers, hyperbolic pentegonal numbers

2010 Mathematics Subject Classification. 11B39, 11B83.

1. Introduction

It's known that many author studied the generalized (r, s, t) sequence. One of these sequences is generalized Guglielmo numbers. Soykan, [19] defined generalized Guglielmo numbers. Before we present our original study , we recall some proporities related to generalized Guglielmo numbers such as reccurance relations, Binet's formula, generating function .

A generalized Guglielmo sequence , with the initial values W_0, W_1, W_2 not all being zero, $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3};\ W_0, W_1, W_2 \quad (n \geq 2) \tag{1.1}$$

Moreover, we define generalized Guglielmo sequence given to negative subscripts as follows,

$$W_{-n} = 3W_{-(n-1)} - 3W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Thus, recurrence (1.1) is true for all integer n.

In the Table 1 we give the first some generalized Guglielmo numbers with positive subscript and negative subscript

Table 1. A few generalized Guglielmo numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$3W_0 - 3W_1 + W_2$
2	W_2	$6W_0 - 8W_1 + 3W_2$
3	$W_0 - 3W_1 + 3W_2$	$10W_0 - 15W_1 + 6W_2$
4	$3W_0 - 8W_1 + 6W_2$	$15W_0 - 24W_1 + 10W_2$
5	$6W_0 - 15W_1 + 10W_2$	$21W_0 - 35W_1 + 15W_2$
6	$10W_0 - 24W_1 + 15W_2$	$28W_0 - 48W_1 + 21W_2$

If we obtain,respectively, $W_0 = 0, W_1 = 1, W_2 = 3$ then$\{W_n\} = \{T_n\}$ is called the Triangular sequence, $W_0 = 3, W_1 = 3, W_2 = 3$ then $\{W_n\} = \{H_n\}$ is called the triangular-Lucas sequence, $W_0 = 0, W_1 = 2, W_2 = 6$ then $\{W_n\} = \{O_n\}$ is called the oblong sequence and $W_0 = 0, W_1 = 1, W_2 = 5$ then $\{W_n\} = \{p_n\}$ is called the pentegonal sequance. Alternatively, triangular sequence $\{T_n\}_{n\geq 0}$, triangular-Lucas sequence $\{H_n\}_{n\geq 0}$, oblong sequence $\{O_n\}_{n\geq 0}$ and pentegonal sequence $\{p_n\}_{n\geq 0}$ are given by the third-order recurrence relations as

(1.2) $$T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, \quad T_0 = 0, T_1 = 1, T_2 = 3,$$

(1.3) $$H_n = 3H_{n-1} - 3H_{n-2} + H_{n-3}, \quad H_0 = 3, H_1 = 3, H_2 = 3,$$

(1.4) $$O_n = 3O_{n-1} - 3O_{n-2} + O_{n-3}, \quad O_0 = 0, O_1 = 2, O_2 = 6,$$

(1.5) $$p_n = 3p_{n-1} - 3p_{n-2} + p_{n-3}, \quad p_0 = 0, p_1 = 1, p_2 = 5.$$

The sequences given above can be extended to negative subscripts by defining, respectively,

$$\begin{aligned} T_{-n} &= 3T_{-(n-1)} - 3T_{-(n-2)} + T_{-(n-3)}, \\ H_{-n} &= 3H_{-(n-1)} - 3H_{-(n-2)} + H_{-(n-3)}, \\ O_{-n} &= 3O_{-(n-1)} - 3O_{-(n-2)} + O_{-(n-3)}, \\ p_{-n} &= 3p_{-(n-1)} - 3p_{-(n-2)} + p_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, \ldots$. As a consequence, recurrences (1.2)-(1.5) hold for all integer n.

We can list some important properties of generalized Guglielmo numbers that are needed.

- Binet formula of generalized Guglielmo sequence can be calculated using its characteristic equation written as

$$x^3 - 3x^2 + 3x - 1 = (x-1)^3 = 0.$$

The roots of the characteristic equation are

$$\alpha = \beta = \gamma = 1.$$

By using these roots and the recurrence relation, Binet formula are written below

$$W_n = A_1 + A_2 n + A_3 n^2 \tag{1.6}$$

where

$$\begin{aligned} A_1 &= W_0, \\ A_2 &= \frac{1}{2}(-W_2 + 4W_1 - 3W_0), \\ A_3 &= \frac{1}{2}(W_2 - 2W_1 + W_0). \end{aligned} \tag{1.7}$$

Then we present Binet formula of triangular, triangular-Lucas, oblong and pentagonal sequences, respectively, given below

$$\begin{aligned} T_n &= \frac{n(n+1)}{2}, \\ H_n &= 3, \\ O_n &= n(n+1), \\ p_n &= \frac{1}{2} n\,(3n-1)\,. \end{aligned}$$

- The generating function for W_n is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 3W_0)x + (W_2 - 3W_1 + 3W_0)x^2}{1 - 3x + 3x^2 - x^3}. \tag{1.8}$$

- The Cassini identity for W_n is

$$W_{n+1}W_{n-1} - W_n^2 = -\frac{1}{2}\left(A + Bn + Cn^2\right) \tag{1.9}$$

where

$$\begin{aligned} A &= 2W_0^2 + 6W_1^2 - 6W_0W_1 - 2W_1W_2, \\ B &= -3W_0^2 - 8W_1^2 - W_2^2 + 10W_0W_1 - 4W_0W_2 + 6W_1W_2, \\ C &= W_0^2 + 4W_1^2 + W_2^2 - 4W_0W_1 + 2W_0W_2 - 4W_1W_2. \end{aligned}$$

- Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Guglielmo sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 3W_0)x + (W_2 - 3W_1 + 3W_0)x^2}{1 - 3x + 3x^2 - x^3}.$$

For more details, see [19].

Now, we are presenting information about specific number systems, including the hypercomplex system, which encompasses complex numbers, hyperbolic numbers, and dual numbers. We note that hyperbolic numbers will play a crucial role in our work. Moreover hyperbolic functions and numbers find applications in various branches of engineering, such as electrical engineering (e.g., transmission lines), control systems (e.g., system dynamics), signal processing (e.g., filter design), and diverse fields of engineering physics, including special relativity, wave propagation, fluid dynamics, optics, and heat conduction. It's important to note that while hyperbolic numbers have interesting mathematical properties, their adoption in practical applications depends on the specific problem at hand and whether they offer advantages over other number systems in a given context.

Initially, we discuss hypercomplex number systems, which are extensions of real numbers, for more detail see [14]. In addition that some commutative special cases of hypercomplex number systems include complex numbers, hyperbolic numbers, and dual numbers. These systems are widely used in various branches of mathematics and physics. We will now present these number systems sequentially, as outlined below.

- Complex numbers simplest form of hypercomplex numbers. Complex numbers defined as $z = a+ib$, where a and b real numbers and i imaginary unit that satisfy $i^2 = -1$. In addition that a and b named, respectively, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ Consequently, the definition of complex numbers given by,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\}.$$

- Hyperbolic (double, split-complex) numbers, for more detail see [17], Split-complex numbers, commonly recognized as hyperbolic numbers, defined as $h = a + jb$ where a and b real numbers and j hyperbolic unit that satisfy $j^2 = 1$. In addition that a and b named, respectively, $Re(h)$ and $Hyp(h)$. Thus, the definition of hyperbolic numbers given by,

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

- Dual numbers, see [10], defined as $d = a + \varepsilon b$ where a and b real numbers and ε dual unit that satisfy $\varepsilon^2 = 0$. Furthermore, a and b called, respectively, $Re(d)$ and $Du(d)$. Thus, defination of dual numbers given by,

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

- A dual hyperbolic number, specifically within the hyperbolic number system, constitutes a distinct type of hypercomplex number. A dual hyperbolic number is defined by,

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ and the set of all dual hyperbolic numbers are defined by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The $\{1, j, \varepsilon, \varepsilon j\}$ is linear independent and $\mathbb{H}_{\mathbb{D}} = sp\{1, j, \varepsilon, \varepsilon j\}$ so that $\{1, j, \varepsilon, \varepsilon j\}$ is a basis of $\mathbb{H}_{\mathbb{D}}$. For more detail see, [3]

The next properties are true for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε satisfy the pure dual unit $(\varepsilon^2 = 0, \varepsilon \neq 0)$, j satisfy the hyperbolic unit $(j^2 = 1)$, and εj satisfy the dual hyperbolic unit $((j\varepsilon)^2 = 0)$.

In addition that the other number sytems are quarternions, octonions and sedenions given below, respectively,

- Quaternion numbers, non-commutative examples of hypercomplex number systems, are a four-dimensional extension of complex numbers. They are expressed as $a_0 + ia_1 + ja_2 + ka_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, and i, j, and k are the quaternion units that satisfy specific multiplication rules. For more detail see [12]. Quaternion numbers are defined by

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

- Octonions is a set, every element of the set linear combinations of unit octonions $\{e_i : i = 0, 1.2, ..., 7\}$, doneted as $\mathbb{O}$. Octonions are defined by,

$$\mathbb{O} = \{\sum_{i=0}^{7} a_i e_i : a_i \in \mathbb{R}, e_0 e_i = e_i e_0 = e_i,\ e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k\ \}$$

where $e_e = 1$, δ_{ij} is Kroneker delta (equal to 1 if and only if $i = j$), ε_{ijk} is anti-symetric tensor. For more detail see [23], [24]

- Sedenions is a set, every element of the set linear combinations of unit sedenions $\{e_i : i = 0, 1.2, ..., 15\}$, denoted by $\mathbb{S}$. It can be seen from here that ever sedenion can be written as

$$\sum_{i=0}^{15} a_i e_i$$

where a_i is real number. For more detail see, [18], [24].

Next we give some proporties on two hyperbolic numbers, $h_1 = a + jb$ and $h_2 = c + jd$, as

$$
\begin{aligned}
h_1 + h_2 &= (a+b) + j(c+d), \\
h_1 . h_2 &= (ac+bd) + j(ad+bc), \\
\overline{h_1} &= a - jb \\
\frac{h_1}{h_2} &= \frac{(ac-bd)+j(cb-ad)}{c^2-d^2}, \\
h_1 &= h_2 \text{ if only if } a = c \text{ and } b = d, \\
\langle h_1, h_2 \rangle &= (ac+bd) + j(bc+ad), \\
\|h_1\| &= \sqrt{|a^2-b^2|}, \text{ called norm of } h_1, \\
\text{if } \left|a^2-b^2\right| &> 0, \ h_1 \text{ is named spacelike vector,} \\
\text{if } \left|a^2-b^2\right| &< 0, \ h_1 \text{ is named timelike vector,} \\
\text{if } \left|a^2-b^2\right| &= 0, \ h_1 \text{ is named null(light-like) vector.}
\end{aligned}
$$

Note that$\{\mathbb{R}^2, H, \langle , \rangle\}$ is called Lorentz plane and denoted as $\mathbb{R}^2_1$. There is an isomorphism relationship between the Lorentz plane and hyperbolic numbers. For more detail, see [24].

Hence the algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras attained from the real numbers $\mathbb{R}$ by a doubling procedure known as the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n-ions (see for example [4], [12], [13], [15], [11]).

Some authors have conducted studies about the dual, hyperbolic, dual hyperbolic and other special numbers. Now we give some information published papers in litarature.

- Cockle [7] studied the hyperbolic numbers with complex coefficients.
- Eren and Soykan [9] studied the generalized Generalized Woodall Numbers.
- Cheng and Thompson [6] introduced dual numbers with complex coefficients.
- Akar, Yüce and Şahin [3] presented the dual hyperbolic numbers.

Next, we present some information on hyperbolic numbers presented in literature.

- Aydın [1] presented hyperbolic Fibonacci numbers given by

$$\widetilde{F}_n = F_n + hF_{n+1},$$

where Fibonacci numbers are given by $F_{n+2} = F_{n+1} + F_n$, with the initial conditation $F_0 = 0$, $F_1 = 1$.

- Soykan and Taşdemir [21] studied hyperbolic generalized Jacobsthal numbers given by

$$\widetilde{V}_n = V_n + hV_{n+1}$$

where generalized Jacobsthal numbers are $V_{n+2} = V_{n+1} + 2V_n$ with the initial conditation $V_0 = a$, $V_1 = b$.

- Taş [16] studied hyperbolic Jacobsthal-Lucas sequence written by

$$HJ_n = J_n + hJ_{n+1}$$

where Jacobsthal-Lucas numbers given by $J_{n+2} = J_{n+1} + 2J_n$ with the inintial conditation $J_0 = 2$, $J_1 = 1$.

- Dikmen and Altınsoy, [8] studied On Third Order Hyperbolic Jacobsthal Numbers given by

$$\begin{aligned} \widehat{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)}, \\ \widehat{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)} \end{aligned}$$

where Jacobsthal numbers, respectively, given by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}$, $J_0^{(3)} = 0$, $J_1^{(3)} = 1$, $J_2^{(3)} = 1$, $j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)}$, $j_0^{(3)} = 2$, $j_1^{(3)} = 1$, $j_2^{(3)} = 5$.

Following this, we provide details on dual hyperbolic sequences as they are presented in literature.

- Soykan, Gümüş, Göcen [20] presented dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers, with the initial values V_0, V_1 not all being zero, are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$.

- Cihan, Azak, Güngör, Tosun, [2] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$\begin{aligned} DHF_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ DHL_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3} \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Soykan, Taşdemir and Okumuş [21] studied dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = a$, $J_1 = b$.

- Bród, Liana, Włoch [5] studied dual hyperbolic generalized balancing numbers as

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$.

Next section, we define the hyperbolic generalized Guglielmo numbers and some special properties, generating function and Binet's formula , of these numbers.

2. Hyperbolic Generalized Guglielmo Numbers and their Generating Functions and Binet's Formulas

In this section, we define hyperbolic generalized Guglielmo numbers then we present some special cases generating functions and Binet's formulas.

We now define hyperbolic generalized Guglielmo numbers over the set of $\mathbb{H}$. The nth hyperbolic generalized Guglielmo number is defined as follows

$$HW_n = W_n + jW_{n+1} \tag{2.1}$$

with the initial values HW_0, HW_1, HW_2. The hyperbolic Guglielmo numbers ,which is defined above, can be written to negative subscripts by defining,

$$HW_{-n} = W_{-n} + jW_{-n+1} \tag{2.2}$$

so that (2.1) is true for all integers n.

Now we define some extraordinary cases of hyperbolic generalized Guglielmo numbers named the nth hyperbolic triangular numbers, the nth hyperbolic triangular-Lucas numbers, the nth hyperbolic oblong numbers and the nth hyperbolic pentegonal numbers and give them as, respectively,

hyperbolic triangular numbers $HT_n = T_n + jT_{n+1}$, with the initial values as

$$\begin{aligned} HT_0 &= T_0 + jT_1, \\ HT_1 &= T_1 + jT_2, \\ HT_2 &= T_2 + jT_3, \end{aligned}$$

hyperbolic triangular-Lucas numbers $HH_n = H_n + jH_{n+1}$ with the initial values as

$$\begin{aligned} HH_0 &= H_0 + jH_1, \\ HH_1 &= H_1 + jH_2, \\ HH_2 &= H_2 + jH_3, \end{aligned}$$

hyperbolic oblong numbers $HO_n = O_n + jO_{n+1}$ with the initial values as

$$\begin{aligned} HO_0 &= O_0 + jO_1, \\ HO_1 &= O_1 + jO_2, \\ HO_2 &= O_2 + jO_3, \end{aligned}$$

hyperbolic pentegonal numbers $Hp_n = p_n + jp_{n+1}$ with the initial values as

$$\begin{aligned} Hp_0 &= p_0 + jp_1, \\ Hp_1 &= p_1 + jp_2, \\ Hp_2 &= p_2 + jp_3, \end{aligned}$$

For hyperbolic triangular numbers (taking $W_n = T_n$, $T_0 = 0$, $T_1 = 1$, $T_2 = 3$) we obtain

$$\begin{aligned} HT_0 &= j \\ HT_1 &= 1 + 3j \\ HT_2 &= 3 + 6j, \end{aligned}$$

for hyperbolic triangular-Lucas numbers (taking $W_n = H_n$, $H_0 = 3$, $H_1 = 3$, $H_2 = 3$) we obtain

$$\begin{aligned} HH_0 &= 3 + 3j, \\ HH_1 &= 3 + 3j, \\ HH_2 &= 3 + 3j, \end{aligned}$$

for hyperbolic oblong numbers (taking $W_n = O_n$, $O_0 = 0$, $O_1 = 2$, $O_2 = 6$) we obtain

$$\begin{aligned} HO_0 &= 2j, \\ HO_1 &= 2 + 6j, \\ HO_2 &= 6 + 12j, \end{aligned}$$

and for hyperbolic pentegonal numbers (taking $W_n = p_n$, $p_0 = 0$, $p_1 = 1$, $p_2 = 5$) we obtain

$$\begin{aligned} Hp_0 &= j, \\ Hp_1 &= 1 + 5j, \\ Hp_2 &= 5 + 12j. \end{aligned}$$

So, using (2.1) the following identity can be expressed for every integers $n \geq 0$,

(2.3) $$HW_n = 3HW_{n-1} - 3HW_{n-2} + HW_{n-3}.$$

Hence for every integers $n < 0$ the sequence $\{HW_n\}_{n\geq 0}$ can be written as

$$HW_{-n} = 3HW_{-(n-1)} - 3HW_{-(n-2)} + HW_{-(n-3)},$$

for $n = 1, 2, 3, ...$ by using (2.2).

Consequently, recurrence (2.3) are true for every integer n.

In the Table 2, taking with positive subscript and negative subscript, we present the first few hyperbolic generalized Guglielmo numbers.

Table 2. Some hyperbolic generalized Guglielmo numbers

n	HW_n	HW_{-n}
0	HW_0	HW_0
1	HW_1	$3HW_0 - 3HW_1 + HW_2$
2	HW_2	$6HW_0 - 8HW_1 + 3HW_2$
3	$HW_0 - 3HW_1 + 3HW_2$	$10HW_0 - 15HW_1 + 6HW_2$
4	$3HW_0 - 8HW_1 + 6HW_2$	$15HW_0 - 24HW_1 + 10HW_2$
5	$6HW_0 - 15HW_1 + 10HW_2$	$21HW_0 - 35HW_1 + 15HW_2$
6	$10HW_0 - 24HW_1 + 15HW_2$	$28HW_0 - 48HW_1 + 21HW_2$

Note that

$$\begin{aligned} HW_0 &= W_0 + jW_1, \\ HW_1 &= W_1 + jW_2, \\ HW_2 &= W_2 + jW_3. \end{aligned}$$

By taking with positive subscript and negative subscript, we present a few hyperbolic triangular numbers, hyperbolic triangular-Lucas numbers, hyperbolic oblong numbers and hyperbolic pentegonal numbers in the following Table 3, Table 4, Table 5 and Table 6.

Table 3. hyperbolic triangular numbers

n	HT_n	HT_{-n}
0	j	
1	$1 + 3j$	0
2	$3 + 6j$	1
3	$6 + 10j$	$3 + j$
4	$10 + 15j$	$6 + 3j$
5	$15 + 21j$	$10 + 6j$

Table 4. hyperbolic triangular-Lucas numbers

n	HH_n	HH_{-n}
0	$3 + 3j$	
1	$3 + 3j$	$3 + 3j$
2	$3 + 3j$	$3 + 3j$
3	$3 + 3j$	$3 + 3j$
4	$3 + 3j$	$3 + 3j$
5	$3 + 3j$	$3 + 3j$

Table 5. hyperbolic oblong numbers

Table 6. hyperbolic pentegonal numbers

n	HO_n	HO_{-n}
0	$2j$	
1	$2+6j$	
2	$6+12j$	2
3	$12+20j$	$6+2j$
4	$20+30j$	$12+6j$
5	$30+42j$	$20+12j$

n	Hp_n	Hp_{-n}
0	j	
1	$1+5j$	2
2	$5+12j$	$7+2j$
3	$12+22j$	$15+7j$
4	$22+35j$	$26+15j$
5	$35+51j$	$40+26j$

Now, we will present Binet's formula for HW_n and in the remainder of the study the following notations are needed:

$$\widehat{\alpha} = 1 + j, \tag{2.4}$$

$$\widehat{\beta} = j. \tag{2.5}$$

Observe that the following identities are obtained:

$$\begin{aligned} \widehat{\alpha}^2 &= 2 + 2j, \\ \widehat{\beta}^2 &= 1, \\ \widehat{\alpha}\widehat{\beta} &= 1 + j. \end{aligned}$$

Theorem 1. *(Binet's Formula) For any integer n, the nth hyperbolic generalized Guglielmo number is*

$$HW_n = (A_1\widehat{\alpha} + \widehat{\beta}(A_2 + A_3)) + (\widehat{\alpha}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3n^2. \tag{2.6}$$

where $\widehat{\alpha}$, $\widehat{\beta}$ are given as (2.4)-(2.5).

Proof. Using Binet's formula given below

$$W_n = A_1 + A_2n + A_3n^2$$

where A_1, A_2, A_2 are given as (1.7) and then we obtain following identity

$$\begin{aligned} HW_n &= W_n + jW_{n+1}, \\ &= (A_1(j+1) + j(A_2 + A_3)) + ((1+j)A_2 + 2jA_3)n + A_3\,(j+1)\,n^2, \\ &= (A_1\widehat{\alpha} + \widehat{\beta}(A_2 + A_3)) + (\widehat{\alpha}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3n^2. \ \square \end{aligned}$$

Specifically, for any integer n, the Binet's Formula for the HT_n, HH_n, HO_n and Hp_n numbers, respectively, are as follows

$$HT_n = \frac{1}{2}(\beta + (\alpha + 2\beta)n + \alpha n^2), \tag{2.7}$$

$$HH_n = 3\widehat{\alpha}, \tag{2.8}$$

$$HO_n = \beta + (\alpha + 2\beta)n + \alpha n^2, \tag{2.9}$$

$$Hp_n = \frac{1}{2}(2\beta + (6\beta - \alpha)\, n + 3\alpha n^2). \tag{2.10}$$

The next step is to provide the generating function for the hyperbolic generalized Guglielmo numbers.

THEOREM 2. *The generating function for the hyperbolic generalized Guglielmo numbers is*

$$f_{HW}(x) = \frac{HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + 3HW_0)x^2}{(1 - 3x + 3x^2 - x^3)}. \tag{2.11}$$

Proof. Let

$$f_{HW}(x) = \sum_{n=0}^{\infty} HWx^n$$

be generating function of the hyperbolic generalized Guglielmo numbers. Then, using the definition of the hyperbolic generalized Guglielmo numbers, and substracting $xg(x)$ and $x^2g(x)$ from $g(x)$, we get

$$\begin{aligned}
(1 - 3x + 3x^2 - x^3)f_{HW}(x) &= \sum_{n=0}^{\infty} HWx^n - 3x\sum_{n=0}^{\infty} HWx^n + 3x^2\sum_{n=0}^{\infty} HWx^n - x^3\sum_{n=0}^{\infty} HWx^n, \\
&= \sum_{n=0}^{\infty} HWx^n - 3\sum_{n=0}^{\infty} HWx^{n+1} + 3\sum_{n=0}^{\infty} HWx^{n+2} - \sum_{n=0}^{\infty} HWx^{n+3}, \\
&= \sum_{n=0}^{\infty} HWx^n - 3\sum_{n=1}^{\infty} HWx^n + 3\sum_{n=2}^{\infty} HWx^n - \sum_{n=3}^{\infty} HWx^n, \\
&= (HW_0 + HW_1x + HW_2x^2) - 3(HWx + HW_1x^2) + 3HW_0x^2 \\
&\quad + \sum_{n=3}^{\infty}(HW_n - 3HW_{n-1} + 3HW_{n-2} - HW_{n-3})x^n, \\
&= HW_0 + HW_1x + HW_2x^2 - 3HW_0x - 3HW_1x^2 + 3HW_0x^2, \\
&= HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + 3HW_0)x^2.
\end{aligned}$$

As a result, using (2.3) and rearranging above equation, the proof of the theorem is completed. □

Using above theorem we can write the the generating functions of the hyperbolic triangular, triangular-Lucas, oblong and pentegonal numbers, respectively, as

$$\begin{aligned} f_{HW_n}(x) &= \frac{j+x}{(1-3x+3x^2-x^3)}, \\ f_{HH_n}(x) &= \frac{(3+3j)+(-6-6j)x+(3+3j)\,x^2}{(1-3x+3x^2-x^3)}, \\ f_{HO_n}(x) &= \frac{2j+2x}{(1-3x+3x^2-x^3)}, \\ f_{Hp_n}(x) &= \frac{j+(1+2j)x+2x^2}{(1-3x+3x^2-x^3)}. \ \square \end{aligned}$$

3. Getting the Binet's Formula from the generating function.

Our next step involves exploring Binet's formula of hyperbolic generalized Guglielmo number $\{HW_n\}$ utilizing generating function $f_{HW_n}(x)$.

THEOREM 3. *(Binet formula of hyperbolic generalized Guglielmo numbers)*

(3.1)
$$HW_n = (A_1\widehat{\alpha} + \widehat{\beta}(A_2 + A_3)) + (\widehat{\alpha}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3n^2.$$

Proof. We write

(3.2)
$$\sum_{n=0}^{\infty} HWx^n = \frac{HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + 3HW_0)x^2}{(1-3x+3x^2-x^3)} = \frac{d_1}{(1-x)} + \frac{d_2}{(1-x)^2} + \frac{d_3}{(1-x)^3},$$

so that

$$\begin{aligned} \sum_{n=0}^{\infty} HW_nx^n &= \frac{d_1}{(1-x)} + \frac{d_2}{(1-x)^2} + \frac{d_3}{(1-x)^3} \\ &= \frac{d_1(1-x)^2 + d_2(1-x) + d_3}{(1-x)^3}. \end{aligned}$$

Hence, we arrive at

$$HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + 3HW_0)x^2 = (d_1 + d_2 + d_3) + (-2d_1 - d_2)x + d_1x^2.$$

Equalizing the coefficients of the same degree terms of x in the above equation, we get

(3.3)
$$\begin{aligned} HW_0 &= d_1 + d_2 + d_3, \\ HW_1 - 3HW_0 &= -2d_1 - d_2, \\ HW_2 - 3HW_1 + 3HW_0 &= d_1. \end{aligned}$$

Then, if we solve (3.3) then we can write

$$\begin{aligned} d_1 &= 3HW_0 - 3HW_1 + HW_2, \\ d_2 &= 5HW_1 - 3HW_0 - 2HW_2, \\ d_3 &= HW_0 - 2HW_1 + HW_2. \end{aligned}$$

Therefore (3.2) can be written as

$$\begin{aligned} \sum_{n=0}^{\infty} HW_n x^n &= d_1 \sum_{n=0}^{\infty} x^n + d_2 \sum_{n=0}^{\infty} (n+1) x^n + d_3 \sum_{n=0}^{\infty} \frac{n^2+3n+2}{2} x^n, \\ &= \sum_{n=0}^{\infty} (d_1 + d_2(n+1) + d_3 \frac{n^2+3n+2}{2}) x^n, \\ &= \sum_{n=0}^{\infty} (HW_0 + \frac{1}{2}(-HW_2 + 4HW_1 - 3HW_0)n + \frac{1}{2}(HW_2 - 2HW_1 + HW_0)n^2) x^n. \end{aligned}$$

As a result, we get the following identity

$$HW_n = \widehat{A_1} + \widehat{A_2} n + \widehat{A_3} n^2$$

where

$$\begin{aligned} \widehat{A_1} &= HW_0, \\ \widehat{A_2} &= \frac{1}{2}(-HW_2 + 4HW_1 - 3HW_0), \\ \widehat{A_3} &= \frac{1}{2}(\widehat{HW}_2 - 2\widehat{HW}_1 + HW_0). \end{aligned}$$

Take note that the following equalities holds,

$$\begin{aligned} \widehat{A_1} &= HW_0 \\ &= HW_0 + jHW_1 \\ &= (1+j)W_0 + j(\frac{1}{2}(-W_2 + 4W_1 - 3W_0)) + j(\frac{1}{2}(W_2 - 2W_1 + W_0)) \\ &= \widehat{\alpha} A_1 + \widehat{\beta}(A_2 + A_3), \end{aligned} \tag{3.4}$$

$$\begin{aligned} \widehat{A_2} &= \frac{1}{2}(-HW_2 + 4HW_1 - 3HW_0) \\ &= \frac{1}{2}((-3W_0 + 4W_1 - W_2) + j(-W_0 + W_2) \\ &= (1+j)(\frac{1}{2}(-W_2 + 4W_1 - 3W_0)) \\ &\quad +2j(\frac{1}{2}(W_2 - 2W_1 + W_0)) \\ &= (\widehat{\alpha} A_2 + 2\widehat{\beta} A_3), \end{aligned} \tag{3.5}$$

(3.6)
$$\begin{aligned}\widehat{A_3} &= \frac{1}{2}(HW_2 - 2HW_1 + HW_0) \\ &= \frac{1}{2}((W_2 - 2W_1 + W_0) + j(W_2 - 2W_1 + W_0) \\ &= \widehat{a}A_3.\end{aligned}$$

Utilizing equations (3.4), (3.5) and (3.6) we obtain following equality.

$$HW_n = (A_1\widehat{\alpha} + \widehat{\beta}(A_2 + A_3)) + (\widehat{\alpha}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3n^2. \square$$

4. Some Identities

We now provide some special identities concerning the hyperbolic generalized Guglielmo sequence $\{HW_n\}$. The following theorem gives the Simpson's formula for the hyperbolic generalized Guglielmo numbers.

THEOREM 4. *(Simpson's formula for hyperbolic generalized Guglielmo numbers) For all integers n we have,*

(4.1)
$$\begin{vmatrix} HW_{n+2} & HW_{n+1} & HW_n \\ HW_{n+1} & HW_n & HW_{n-1} \\ HW_n & HW_{n-1} & HW_{n-2} \end{vmatrix} = \begin{vmatrix} HW_2 & HW_1 & HW_0 \\ HW_1 & HW_0 & HW_{-1} \\ HW_0 & HW_{-1} & HW_{-2} \end{vmatrix}.$$

Proof. For the proof first, we assume that $n \geq 0$ and then we able to use mathematical induction on n. For $n = 0$ identity (4.1) is true. Now we assume that (4.1) is true for $n = k$. Hence, the identity given below can be written

$$\begin{vmatrix} HW_{k+2} & HW_{k+1} & HW_k \\ HW_{k+1} & HW_k & HW_{k-1} \\ HW_k & HW_{k-1} & HW_{k-2} \end{vmatrix} = \begin{vmatrix} HW_2 & HW_1 & HW_0 \\ HW_1 & HW_0 & HW_{-1} \\ HW_0 & HW_{-1} & HW_{-2} \end{vmatrix}.$$

For $n = k + 1$, we obtain

$$\begin{aligned}\begin{vmatrix} HW_{k+3} & HW_{k+2} & HW_{k+1} \\ HW_{k+2} & HW_{k+1} & HW_k \\ HW_{k+1} & HW_k & HW_{k-1} \end{vmatrix} &= \begin{vmatrix} 3HW_{k+2} - 3HW_{k+1} + HW_k & HW_{k+2} & HW_{k+1} \\ 3HW_{k+1} - 3HW_k + HW_{k-1} & HW_{k+1} & HW_k \\ 3HW_k - 3HW_{k-1} + HW_{k-2} & HW_k & HW_{k-1} \end{vmatrix} \\ &= 3\begin{vmatrix} HW_{k+2} & HW_{k+2} & HW_{k+1} \\ HW_{k+1} & HW_{k+1} & HW_k \\ HW_k & HW_k & HW_{k-1} \end{vmatrix} - 3\begin{vmatrix} HW_{k+1} & HW_{k+2} & HW_{k+1} \\ HW_k & HW_{k+1} & HW_k \\ HW_{k-1} & HW_k & HW_{k-1} \end{vmatrix} \\ &\quad + \begin{vmatrix} HW_k & HW_{k+2} & HW_{k+1} \\ HW_{k-1} & HW_{k+1} & HW_k \\ HW_{k-2} & HW_k & HW_{k-1} \end{vmatrix} \\ &= \begin{vmatrix} HW_{k+2} & HW_{k+1} & HW_k \\ HW_{k+1} & HW_k & HW_{k-1} \\ HW_k & HW_{k-1} & HW_{k-2} \end{vmatrix}.\end{aligned}$$

For the other case $n<0$ the proof has been seen similarly. Thus, the proof is completed. □

From Theorem 4.1 we get following corollary.

COROLLARY 5.

(a): $$\begin{vmatrix} HT_{n+2} & HT_{n+1} & HT_n \\ HT_{n+1} & HT_n & HT_{n-1} \\ HT_n & HT_{n-1} & HT_{n-2} \end{vmatrix} = -4(j+1).$$

(b): $$\begin{vmatrix} HH_{n+2} & HH_{n+1} & HH_n \\ HH_{n+1} & HH_n & HH_{n-1} \\ HH_n & HH_{n-1} & HH_{n-2} \end{vmatrix} = 0.$$

(c): $$\begin{vmatrix} HO_{n+2} & HO_{n+1} & HO_n \\ HO_{n+1} & HO_n & HO_{n-1} \\ HO_n & HO_{n-1} & O_{n-2} \end{vmatrix} = -32(j+1).$$

(d): $$\begin{vmatrix} Hp_{n+2} & Hp_{n+1} & Hp_n \\ Hp_{n+1} & Hp_n & Hp_{n-1} \\ Hp_n & Hp_{n-1} & Hp_{n-2} \end{vmatrix} = -108(j+1).$$

Next, the Catalan's identity of hyperbolic generalized Guglielmo numbers is given.

THEOREM 6. *(Catalan's identity) For all integers n and m, the following identity holds*

$$HW_{n+m}HW_{n-m} - HW_n^2 = -2m^2(\widehat{\alpha}(A_2^2 - 2A_1A_3 + A_2A_3 + 2nA_2A_3) - A_3^2(\widehat{\alpha} - 2n\widehat{\alpha} + m^2\widehat{\alpha} - 2n^2\widehat{\alpha} - 2)). \tag{4.2}$$

Proof. Using the Binet Formula given below

$$HW_n = (A_1\widehat{\alpha} + \widehat{\beta}(A_2 + A_3)) + (\widehat{\alpha}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3n^2.$$

The proof is completed. □

As special cases of the above theorem, we give Catalan's identity of HT_n, HH_n, HO_n and Hp_n.

We present Catalan's identity of hyperbolic triangular numbers.

COROLLARY 7. *(Catalan's identity for the hyperbolic triangular numbers) For all integers n and m, the following identity holds:*

$$HT_{n+m}HT_{n-m} - HT_n^2 = \frac{1}{2}m^2(-\widehat{\alpha} - 4n\widehat{\alpha} + m^2\widehat{\alpha} - 2n^2\widehat{\alpha} - 2).$$

Proof. Taking $HW_n = HT_n$ in Theorem 6 we get the result we have been seeking. □

We give Catalan's identity of hyperbolic triangular-Lucas numbers.

COROLLARY 8. *(Catalan's identity for the hyperbolic Lucas-triangular numbers) For all integers n and m, the following identity holds:*

$$HH_{n+m}HH_{n-m} - HH_n^2 = 0.$$

Proof. Taking $HW_n = HH_n$ in Theorem 6 we get the result we have been seekingt. □

We give Catalan's identity of hyperbolic oblong numbers.

Corollary 9. *(Catalan's identity for the hyperbolic oblong numbers) For all integers n and m, the following identity holds:*

$$HO_{n+m}HO_{n-m} - HO_n^2 = 2m^2\left(-\widehat{\alpha} - 4n\widehat{\alpha} + m^2\widehat{\alpha} - 2n^2\widehat{\alpha} - 2\right).$$

Proof. Taking $HW_n = HO_n$ in Theorem 6 we get the result we have been seeking. □

We give Catalan's identity of hyperbolic pentegonal numbers.

Corollary 10. *(Catalan's identity for the hyperbolic pentegonal numbers) For all integers n and m, the following identity holds:*

$$Hp_{n+m}Hp_{n-m} - Hp_n^2 = \frac{1}{2}m^2\left(11\widehat{\alpha} - 12n\widehat{\alpha} + 9m^2\widehat{\alpha} - 18n^2\widehat{\alpha} - 18\right).$$

Proof. Taking $HW_n = Hp_n$ in Theorem 6 we get the result we have been seeking. □

If we take $m = 1$ in Catalan's identity, we get the Cassini's identity for the hyperbolic generalized Guglielmo numbers as follows.

Corollary 11. *(Cassini's identity for the hyperbolic generalized Guglielmo numbers) For all integers n, the following identities holds.*

- **(a):** $HT_{n+1}HT_{n-1} - HT_n^2 = -\widehat{\alpha}n^2 - 2\widehat{\alpha}n - 1.$
- **(b):** $HH_{n+1}HH_{n-1} - HH_n^2 = 0.$
- **(c):** $HO_{n+1}HO_{n-1} - HO_n^2 = -4\left(n^2\widehat{\alpha} + 2n\widehat{\alpha} + 1\right).$
- **(d):** $Hp_{n+1}Hp_{n-1} - Hp_n^2 = -9\widehat{\alpha}n^2 - 6\widehat{\alpha}n + 10\widehat{\alpha} - 9.$

Theorem 12. *Let n and m be integers, T_n is triangular numbers, the following identity is true:*

$$HW_{m+n} = T_{m-1}HW_{n+2} + (T_{m-3} - 3T_{m-2})HW_{n+1} + T_{m-2}HW_n. \tag{4.3}$$

Proof. For $n, m \geqslant 0$ the identity (12) can be proved by mathematical induction on m. If $m = 0$ we get

$$HW_n = T_{-1}HW_{n+2} + (T_{-3} - 3T_{-2})HW_{n+1} + T_{-2}HW_n$$

which is true by seeing that $T_{-1}=0, T_{-2}=1, T_{-3}=3$. We assume that the identity given holds for $m=k$. For $m=k+1$, we get

$$\begin{aligned}
HW_{(k+1)+n} &= 3HW_{n+k}-3HW_{n+k-1}+HW_{n+k-2} \\
&= 3(T_{k-1}HW_{n+2}+(T_{k-3}-3T_{k-2})HW_{n+1}+T_{k-2}HW_{n}) \\
&\quad -3(T_{k-2}HW_{n+2}+(T_{k-4}-3T_{k-3})HW_{n+1}+T_{k-3}HW_{n}) \\
&\quad +(T_{k-3}HW_{n+2}+(T_{k-5}-3T_{k-4})HW_{n+1}+T_{k-4}HW_{n}) \\
&= (3T_{k-1}-3T_{k-2}+T_{k-3})HW_{n+2}+((3T_{k-3}-3T_{k-4}+T_{k-5}) \\
&\quad -3(3T_{k-2}-3T_{k-3}+T_{k-4}))HW_{n+1}+(3T_{k-2}-3T_{k-3}+T_{k-4})HW_{n} \\
&= T_{k}HW_{n+2}+(T_{k-2}-3T_{k-1})HW_{n+1}+T_{k-1}HW_{n} \\
&= T_{(k+1)-1}HW_{n+2}+(T_{(k+1)-3}-3T_{(k+1)-2})HW_{n+1}+T_{(k+1)-2}HW_{n}.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves (12). For the other case, the proof can be done similarly. □

5. Linear Sums

In this section, we give the summation formulas of the hyperbolic generalized Guglielmo numbers with positive and negatif subscripts.

PROPOSITION 13. *For the generalized Guglielmo numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_{k} = \frac{1}{12}(n+1)\left(\left(2n^{2}-2n\right)W_{2}-2\left(2n^{2}-5n\right)W_{1}+\left(2n^{2}-8n+12\right)W_{0}\right).$

(b): $\sum_{k=0}^{n} W_{k+1} = \frac{1}{12}(n+1)\left(\left(2n^{2}+4n\right)W_{2}-2\left(2n^{2}+n-6\right)W_{1}+\left(2n^{2}-2n\right)W_{0}\right).$

Proof. For the proof, see Soykan [19]. □

PROPOSITION 14. *For the generalized Guglielmo numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_{2k} = \frac{1}{12}(n+1)(\left(8n^{2}-2n\right)W_{2}-2\left(8n^{2}-8n\right)W_{1}+\left(8n^{2}-14n+12\right)W_{0}).$

(b): $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{12}(n+1)(W_{2}\left(8n^{2}+10n\right)-2W_{1}\left(8n^{2}+4n-6\right)+W_{0}\left(8n^{2}-2n\right)).$

(c): $\sum_{k=0}^{n} W_{2k+2} = \frac{1}{12}(n+1)(\left(8n^{2}+22n+12\right)W_{2}-2\left(8n^{2}+16n\right)W_{1}+\left(8n^{2}+10n\right)W_{0}).$

Proof. For the proof, see Soykan [19]. □

PROPOSITION 15. *For the generalized Guglielmo numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_{-k} = \frac{1}{12}(n+1)(\left(2n^{2}+4n\right)W_{2}-2\left(2n^{2}+7n\right)W_{1}+\left(2n^{2}+10n+12\right)W_{0}).$

(b): $\sum_{k=0}^{n} W_{-k+1} = \frac{1}{12}(n+1)(\left(2n^{2}-2n\right)W_{2}-2\left(2n^{2}+n-6\right)W_{1}+\left(2n^{2}+4n\right)W_{0}).$

Proof. For the proof, see Soykan [19]. □

PROPOSITION 16. *For the generalized Guglielmo numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_{-2k} = \frac{1}{12}(n+1)\left(\left(8n^2+10n\right)W_2 - 2\left(8n^2+16n\right)W_1 + \left(8n^2+22n+12\right)W_0\right)$.

(b): $\sum_{k=0}^{n} W_{-2k+1} = \frac{1}{12}(n+1)\left(\left(8n^2-2n\right)W_2 - 2\left(8n^2+4n-6\right)W_1 + \left(8n^2+10n\right)W_0\right)$.

(c): $\sum_{k=0}^{n} W_{-2k+2} = \frac{1}{12}(n+1)\left(\left(8n^2-14n+12\right)W_2 - 2\left(8n^2-8n\right)W_1 + \left(8n^2-2n\right)W_0\right)$.

Proof. For the proof, see Soykan [19]. □

Now, we will give the formulas of the sum of hyperbolic generalized Guglielmo numbers.

THEOREM 17. *For $n \geq 0$, hyperbolic generalized Guglielmo numbers have the following formulas:*

(a): $\sum_{k=0}^{n} \widehat{W}_k = \frac{1}{6}(n+1)((-n+jn^2+2jn+n^2)W_2 + (6j+5n-2jn^2-jn-2n^2)W_1 + (-4n+jn^2-jn+n^2+6)W_0)$.

(b): $\sum_{k=0}^{n} \widehat{W}_{2k} = \frac{1}{6}(n+1)((-n+4jn^2+5jn+4n^2)W_2 + (6j+8n-8jn^2-4jn-8n^2)W_1 + (-7n+4jn^2-jn+4n^2+6)W_0)$.

(c): $\sum_{k=0}^{n} \widehat{W}_{2k+1} = \frac{1}{6}(n+1)((6j+5n+4jn^2+11jn+4n^2)W_2 + (6-8jn^2-16jn-8n^2-4n)W_1 + (-n+4jn^2+5jn+4n^2)W_0)$.

Proof.

(a): Note that using (2.1), we get

$$\sum_{k=0}^{n} HW_k = \sum_{k=0}^{n} W_k + j\sum_{k=0}^{n} W_{k+1}$$

and using Proposition 13 the proof completed.

(b): Note that using (2.1), we get

$$\sum_{k=0}^{n} HW_{2k} = \sum_{k=0}^{n} W_{2k} + j\sum_{k=0}^{n} W_{2k+1}$$

and using Proposition 14 the proof completed.

(c): Note that using (2.1), we get

$$\sum_{k=0}^{n} HW_{2k+1} = \sum_{k=0}^{n} W_{2k+1} + j\sum_{k=0}^{n} W_{2k+2}$$

and using Proposition 14 the proof completed. □

As a special case of the theorem (17, a) we present following corollary.

COROLLARY 18.

(a): $\sum_{k=0}^{n} \widehat{T}_k = \frac{1}{6}(n+1)(6j+(5j+2)n+(j+1)n^2)$.

(b): $\sum_{k=0}^{n} \widehat{H}_k = (3j+3)(n+1)$.

(c): $\sum_{k=0}^{n} \widehat{O}_k = \frac{1}{6}(n+1)(12j+(10j+4)n+(2j+2)n^2)$.

(d): $\sum_{k=0}^{n} \widehat{p}_k = \frac{1}{6}(n+1)(6j+9jn+(3j+3)n^2)$.

As a special case of the Theorem 17 (b), we present following corollary.

COROLLARY 19.

(a): $\sum_{k=0}^{n}\widehat{T}_{2k}=\frac{1}{6}\left(n+1\right)(6j+(5+11j)n+(4+4j)n^{2}).$

(b): $\sum_{k=0}^{n}\widehat{H}_{2k}=(3j+3)\left(n+1\right).$

(c): $\sum_{k=0}^{n}\widehat{O}_{2k}=\frac{1}{6}\left(n+1\right)(12j+(10+22j)n+(8+8j)n^{2}).$

(d): $\sum_{k=0}^{n}\widehat{p}_{2k}=\frac{1}{6}\left(n+1\right)(6j+(3+21j)n+(12+12j)n^{2}).$

As a special case of the Theorem 17 (c), we present following corollary.

COROLLARY 20.

(a): $\sum_{k=0}^{n}\widehat{T}_{2k+1}=\frac{1}{6}\left(n+1\right)((6+18j)+(11+17j)n+(4+4j)n^{2}).$

(b): $\sum_{k=0}^{n}\widehat{H}_{2k+1}=(3j+3)\left(n+1\right).$

(c): $\sum_{k=0}^{n}\widehat{O}_{2k+1}=\frac{1}{6}\left(n+1\right)((12+36j)+(22+34j)n+(8+8j)n^{2}).$

(d): $\sum_{k=0}^{n}\widehat{p}_{2k+1}=\frac{1}{6}\left(n+1\right)((6+30j)+(21+39j)n+(12+12j)n^{2}).$

Now, we present the formula that yield the summation formulas of the generalized Guglielmo numbers with negative subscripts.

THEOREM 21. *For $n\geq 1$, hyperbolic generalized Guglielmo numbers have the following formulas:*

(a): $\sum_{k=0}^{n}\widehat{W}_{-k}=\frac{1}{6}\left(n+1\right)((2n+jn^{2}-jn+n^{2})W_{2}+(6j-7n-2jn^{2}-jn-2n^{2})W_{1}+(5n+jn^{2}+2jn+n^{2}+6)W_{0}).$

(b): $\sum_{k=0}^{n}\widehat{W}_{-2k}=\frac{1}{6}\left(n+1\right)((5n+4jn^{2}-jn+4n^{2})W_{2}+(6j-16n-8jn^{2}-4jn-8n^{2})W_{1}+(11n+4jn^{2}+5jn+4n^{2}+6)W_{0}).$

(c): $\sum_{k=0}^{n}\widehat{W}_{-2k+1}=\frac{1}{6}\left(n+1\right)((6j-n+4jn^{2}-7jn+4n^{2})W_{2}+(-4n-8jn^{2}+8jn-8n^{2}+6)W_{1}+(5n+4jn^{2}-jn+4n^{2})W_{0}).$

Proof.

(a): Note that using (2.1), we get

$$\sum_{k=0}^{n}HW_{-k}=\sum_{k=0}^{n}W_{-k}+j\sum_{k=0}^{n}W_{-k+1}$$

and using Proposition 15 the proof completed.

(b): Note that using (2.1), we get

$$\sum_{k=0}^{n}HW_{-2k}=\sum_{k=0}^{n}W_{-2k}+j\sum_{k=0}^{n}W_{-2k+1}$$

and using Proposition 16 the proof completed.

(c): Note that using (2.1), we get using Proposition (16), we get

$$\sum_{k=0}^{n}HW_{-2k+1}=\sum_{k=0}^{n}W_{-2k+1}+j\sum_{k=0}^{n}W_{-2k+2}$$

and using Proposition 16 the proof completed. □

As a special case of the Theorem 21 (a), we get the following corollary.

Corollary 22.

(a): $\sum_{k=0}^{n}\widehat{T}_{-k}=\frac{1}{6}\left(n+1\right)(6j+(-1-4j)n+(1+j)n^{2})$.

(b): $\sum_{k=0}^{n}\widehat{H}_{-k}=\left(3j+3\right)\left(n+1\right)$.

(c): $\sum_{k=0}^{n}\widehat{O}_{-k}=\frac{1}{6}\left(n+1\right)(12j+(-2-8j)n+(2+2j)n^{2})$.

(d): $\sum_{k=0}^{n}\widehat{p}_{-k}=\frac{1}{2}\left(n+1\right)(2j+(1-2j)n+(1+j)n^{2})$.

As a special case of the Theorem 21 (b), we obtain the following corollary.

Corollary 23.

(a): $\sum_{k=0}^{n}\widehat{T}_{-2k}=\frac{1}{6}\left(n+1\right)(6j+(-1-7j)n+(4+4j)n^{2})$.

(b): $\sum_{k=0}^{n}\widehat{H}_{-2k}=\left(3j+3\right)\left(n+1\right)$.

(c): $\sum_{k=0}^{n}\widehat{O}_{-2k}=\frac{1}{3}\left(n+1\right)(6j+(-1-7j)n+(4+4j)n^{2})$.

(d): $\sum_{k=0}^{n}\widehat{p}_{-2k}=\frac{1}{6}\left(n+1\right)((6j)+(9-9j)n+(12+12j)n^{2})$.

As a special case of the Theorem 21 (c), we obtain the following corollary.

Corollary 24.

(a): $\sum_{k=0}^{n}\widehat{T}_{-2k+1}=\frac{1}{6}\left(n+1\right)((6+18j)+(-7-13j)n+(4+4j)n^{2})$.

(b): $\sum_{k=0}^{n}\widehat{H}_{-2k+1}=\left(3j+3\right)\left(n+1\right)$.

(c): $\sum_{k=0}^{n}\widehat{O}_{-2k+1}=\frac{1}{3}\left(n+1\right)((6+18j)+(-7-13j)n+(4+4j)n^{2})$.

(d): $\sum_{k=0}^{n}\widehat{p}_{-2k+1}=\frac{1}{6}\left(n+1\right)((6+30j)+(-9-27j)n+(12+12j)n^{2})$.

We will now provide a different theorem given above that allows us to calculate the finite sum of Gaussian numbers.

Theorem 25. *For every integer n, hyperbolic generalized Guglielmo numbers have the following formula*

$$\sum_{k=0}^{n}HW_{n}=(A_{1}\widehat{\alpha}+\widehat{\beta}(A_{2}+A_{3}))(n+1)+(\widehat{\alpha}A_{2}+2\widehat{\beta}A_{3})\frac{n(n+1)}{2}+\widehat{\alpha}A_{3}\frac{n(n+1)(2n+1)}{6}.$$

Proof. The proof can be done easily by using identity (2.6).

Next we can get the following corollary by using (25).

Corollary 26. **(a):** $\sum_{k=0}^{n}HT_{n}=\frac{1}{2}(\beta(n+1)+(\alpha+2\beta)\frac{n(n+1)}{2}+\alpha\frac{n(n+1)(2n+1)}{6})$.

(b): $\sum_{k=0}^{n}HH_{n}=3\widehat{\alpha}(n+1)$.

(c): $\sum_{k=0}^{n}HO_{n}=\beta(n+1)+(\alpha+2\beta)\frac{n(n+1)}{2}+\alpha\frac{n(n+1)(2n+1)}{6}$.

(d): $\sum_{k=0}^{n}Hp_{n}=\frac{1}{2}(2\beta(n+1)+(6\beta-\alpha)\ \frac{n(n+1)}{2}+3\alpha\frac{n(n+1)(2n+1)}{6})$.

6. Matrices linked to Hyperbolic Generalized Guglielmo Numbers

In this part of our study we give some identities on some matrices linked to hyperbolic Guglielmmo numbers.

By using the $\{T_n\}$ which is defined by the third-order recurrence relation as follows

$$T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}$$

with the initial conditions $T_0 = 0$, $T_1 = 1$, $T_2 = 3$ we present the square matrix A of order 3 as

$$A = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. Then, we give the following Lemma.

Lemma 27. *For all integers n the following identity is true*

$$\begin{pmatrix} HW_{n+2} \\ HW_{n+1} \\ HW_n \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}.$$

Proof. First, for the proof we assume that $n \geq 0$. Lemma 27 can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}.$$

For $n = k+1$, we get

$$\begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix}$$
$$= \begin{pmatrix} 3HW_{k+2} - 3HW_{k+1} + HW_k \\ HW_{k+2} \\ HW_{k+1} \end{pmatrix}$$
$$= \begin{pmatrix} HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \end{pmatrix}.$$

Consequently, by mathematical induction on n, the proof is completed. Note that the case $n < 0$ the proof can be done similarly.

Note that

$$A^n = \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}.$$

For the proof see [22].

THEOREM 28. *If we define the matrices N_{HW} and E_{HW} as follow*

$$N_{HW} = \begin{pmatrix} HW_2 & HW_1 & HW_0 \\ HW_1 & HW_0 & HW_{-1} \\ HW_0 & HW_{-1} & HW_{-2} \end{pmatrix},$$
$$E_{HW} = \begin{pmatrix} HW_{n+2} & HW_{n+1} & HW_n \\ HW_{n+1} & HW_n & HW_{n-1} \\ HW_n & HW_{n-1} & HW_{n-2} \end{pmatrix}.$$

then the following identity is true:

$$A^n N_{HW} = E_{HW}.$$

Proof. For the proof, we can use the following identities

$$\begin{aligned} A^n N_{HW} &= \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix} \begin{pmatrix} HW_2 & HW_1 & HW_0 \\ HW_1 & HW_0 & HW_{-1} \\ HW_0 & HW_{-1} & HW_{-2} \end{pmatrix}, \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a_{11} &= GHW_2 T_{n+1} + HW_1 (T_{n-1} - 3T_n) + HW_0 T_n, \\ a_{12} &= HW_1 T_{n+1} + HW_0 (T_{n-1} - 3T_n) + HW_{-1} T_n, \\ a_{13} &= HW_0 T_{n+1} + HW_{-1} (T_{n-1} - 3T_n) + HW_{-2} T_n, \\ a_{21} &= HW_2 T_n + HW_1 (T_{n-2} - 3T_{n-1}) + HW_0 T_{n-1}, \\ a_{22} &= HW_1 T_n + HW_0 (T_{n-2} - 3T_{n-1}) + HW_{-1} T_{n-1}, \\ a_{23} &= HW_0 T_n + HW_{-1} (T_{n-2} - 3T_{n-1}) + HW_{-2} T_{n-1}, \\ a_{31} &= HW_2 T_{n-1} + HW_1 (T_{n-3} - 3T_{n-2}) + HW_0 T_{n-2}, \\ a_{32} &= HW_1 T_{n-1} + HW_0 (T_{n-3} - 3T_{n-2}) + HW_{-1} T_{n-2}, \\ a_{33} &= HW_0 T_{n-1} + HW_{-1} (T_{n-3} - 3T_{n-2}) + HW_{-2} T_{n-2}. \end{aligned}$$

Using the Theorem 12 the proof is done. □

From Theorem 28, we have the following corollary.

Corollary 29.

(a): *Let the matrices N_{HT} and E_{HT} are defined as following*

$$N_{HT} = \begin{pmatrix} HT_2 & HT_1 & HT_0 \\ HT_1 & HT_0 & HT_{-1} \\ HT_0 & HT_{-1} & HT_{-2} \end{pmatrix},$$

$$E_{HT} = \begin{pmatrix} HT_{n+2} & HT_{n+1} & HT_n \\ HT_{n+1} & HT_n & HT_{n-1} \\ HT_n & HT_{n-1} & HT_{n-2} \end{pmatrix},$$

so that the identity given below is true for A^n, N_{HT}, E_{HT}

$$A^n N_{HT} = E_{HT},$$

(b): *Let the matrices N_{HO} and E_{HO} are defined as following*

$$N_{HO} = \begin{pmatrix} HO_2 & HO_1 & HO_0 \\ HO_1 & HO_0 & HO_{-1} \\ HO_0 & HO_{-1} & HO_{-2} \end{pmatrix},$$

$$E_{HO} = \begin{pmatrix} HO_{n+2} & HO_{n+1} & HO_n \\ HO_{n+1} & HO_n & HO_{n-1} \\ HO_n & HO_{n-1} & HO_{n-2} \end{pmatrix},$$

so that the identity given below is true for A^n, N_{HO}, E_{HO}

$$A^n N_{HO} = E_{HO}.$$

(c): *Let the matrices N_{HH} and E_{HH} are defined as following*

$$N_{HH} = \begin{pmatrix} HH_2 & HH_1 & HH_0 \\ HH_1 & HH_0 & HH_{-1} \\ HH_0 & HH_{-1} & HH_{-2} \end{pmatrix},$$

$$E_{HH} = \begin{pmatrix} HH_{n+2} & HH_{n+1} & HH_n \\ HH_{n+1} & HH_n & HH_{n-1} \\ HH_n & HH_{n-1} & HH_{n-2} \end{pmatrix},$$

so that the identity given below is true for A^n, N_{HH}, E_{HH}

$$A^n N_{HH} = E_{HH}.$$

(d): *Let the matrices N_{Hp} and E_{Hp} are defined as following*

$$N_{Hp} = \begin{pmatrix} Hp_2 & Hp_1 & Hp_0 \\ Hp_1 & Hp_0 & Hp_{-1} \\ Hp_0 & Hp_{-1} & Hp_{-2} \end{pmatrix},$$

$$E_{Hp} = \begin{pmatrix} Hp_{n+2} & Hp_{n+1} & Hp_n \\ Hp_{n+1} & Hp_n & Hp_{n-1} \\ Hp_n & Hp_{n-1} & Hp_{n-2} \end{pmatrix}.$$

so that the identity given below is true for A^n, N_{Hp}, E_{Hp}

$$A^n N_{Hp} = E_{Hp}.$$

References

[1] Aydın, F., T., Hyperbolic Fibonacci Sequence, Universal Journal of Mathematics and Applications, 2(2), 59-64, 2019.

[2] Cihan, A., A. Z. Azak, M. A. Güngör, M. Tosun, A Study on Dual Hyperbolic Fibonacci and Lucas Numbers, An. Şt. Univ. Ovidius Constanta, 27(1), 35–48, 2019.

[3] Akar, M., Yüce, S., Şahin, Ş., On the Dual Hyperbolic Numbers and the Complex Hyperbolic Numbers, Journal of Computer Science & Computational Mathematics, 8(1), 1-6, 2018.

[4] Biss, D.K., Dugger, D., Isaksen, D.C., Large annihilators in Cayley-Dickson algebras, Communication in Algebra, 36 (2), 632-664, 2008.

[5] Bród, D., Liana, A., Włoch, I., Two Generalizations of Dual-Hyperbolic Balancing Numbers, Symmetry, 12(11), 1866, 2020

[6] Cheng, H. H., Thompson, S., Dual Polynomials and Complex Dual Numbers for Analysis of Spatial Mechanisms, Proc. of ASME 24th Biennial Mechanisms Conference, Irvine, CA, August, 19-22, 1996.

[7] Cockle, J., On a New Imaginary in Algebra, Philosophical magazine, London-Dublin-Edinburgh, 3(34), 37-47, 1849.

[8] Dikmen, C. M., Altınsoy, M., On Third Order Hyperbolic Jacobsthal Numbers, Konuralp Journal of Mathematics, 10 (1), 118-126, 2022.

[9] Eren, O., Soykan, Y., Gaussian Generalized Woodall Numbers, Archives of Current Research International, 23, 8, 48-68, 2023.

[10] Fjelstad, P., Gal, S.G., n-dimensional Hyperbolic Complex Numbers, Advances in Applied Clifford Algebras, 8(1), 47-68, 1998.

[11] Göcen, M., Soykan, Y., Horadam 2^k-Ions, Konuralp Journal of Mathematics, 7(2), 492-501, 2019.

[12] Hamilton, W.R., Elements of Quaternions, Chelsea Publishing Company, New York , 1969.

[13] Imaeda, K., Imaeda, M., Sedenions: algebra and analysis, Applied Mathematics and Computation, 115, 77-88, 2000.

[14] Kantor, I., Solodovnikov, A., Hypercomplex Numbers, Springer-Verlag, New York, 1989.

[15] Moreno, G., The zero divisors of the Cayley-Dickson algebras over the real numbers, Bol. Soc. Mat. Mexicana 3(4), 13-28, 1998.

[16] Taş, S., On Hyperbolic Jacobsthal-Lucas Sequence, Fundamental Journal of Mathematics and Applications, ,5(1), 16-20, 2022.

[17] Sobczyk, G., The Hyperbolic Number Plane, The College Mathematics Journal, 26(4), 268-280, 1995.

[18] Soykan,Y., Tribonacci and Tribonacci-Lucas Sedenions. Mathematics 7(1), 74, 2019.

[19] Soykan,Y., Generalized Guglielmo Numbers: An Investigation of Properties of Triangular, Oblong and Pentagonal Numbers via Their Third Order Linear Recurrence Relations, Earthline Journal of Mathematical Sciences, 9(1), 1-39, 2022.

[20] Soykan, Y., Gümüş, M., Göcen, M., A study on dual hyperbolic generalized Pell numbers, Malaya Journal Of Matematik, 09(03), 99-116, 2021.

[21] Soykan, Y., Taşdemir, E., Okumuş, İ., On dual hyperbolic numbers with generalized Jacobsthal numbers components, Indian J Pure Appl Math, 54, 824–840, 2023.

[22] Soykan Y., A Study On Generalized (r,s,t)-Numbers, MathLAB Journal, 7, 101-129, 2020.

[23] J. Baez, The octonions, Bull. Amer. Math. Soc. 39(2), 145-205, 2002.

[24] Yüce, S., Sayılar ve Cebir, Ankara, June 2020