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Didactic Approaches to Proofs in Mathematics: Difficulties of Proofs

ABSTRACT

The theme of proof in mathematics is often the subject of debate in the educational world. Several different approaches have addressed the problem of learning a proof. In this work we studied the learning of proof in the teaching of mathematics, then we gave analyzes of the difficulties of the pupils to understand its nature and its usefulness. Analyzes of approaches to proof from the teacher's side are provided.

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Keywords: Didactic of mathematics, proofs and obstacles in mathematics.

AMS Class (2020): Primary 03Fxx, 03F03, 03F99; Secondary 97E50, 03F07, 03F20.

1. INTRODUCTION

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The proof, in general according to (Arsac [1], [2]; Balacheff [3]; Daili [4]; Glaeser [5]), is a rigorous and coherent reasoning by which we conclude the need for a proposition from one or more propositions already known. It seems to be the best way to lead us to the truth. We can find the proofs in various fields such as: mathematics, physics, economics, sociology ... etc.

In mathematics, a proof is a reasoning which allows, from certain axioms, to establish that an assertion is true. The proofs use constructions based on logic but usually include elements of natural language while avoiding as much as possible the introduction of ambiguities.

So a proof is a chain and each link is nested in others, and the failure of one link causes the whole chain to break.

To build a proof it is not at all sufficient to choose a language and appropriate rules; its construction requires reasoning which can be explained by a particular language and rules.

2. MAIN RESULTS

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What is a difficulty? Are there any difficulties with mathematics proofs? If so, is there a link between reading and proof in mathematics?

Are the difficulties in mathematical proofs of the same order for all age groups? Is there a report of difficulty in mathematical proofs for different genders? How do students and teachers behave when faced with a mathematics proof?

This work represents a complement to the work (Daili [4]) of the author. It gives answers to these questions.

The quests for truth and proof have been part of humanity's history for centuries. Having the right knowledge and proof help us solve current problems, as well as anticipate and solve complex and potential future problems. Most educational programs adhere to the constructivist approach, which emphasizes the importance of proof and true knowledge. According to this approach, new knowledge is constructed on the basis of existing knowledge (Kardag et al [6]; Aksoy and Narli [7]).

47 Proof is a notoriously difficult mathematical concept for students. Empirical studies have
48 shown that many students emerge from proof-oriented courses such real analysis (Bills and
49 Tall [8]), introduction to proof (Weber [9], [10]; Moore [11]; Knapp [12]; Morali and Ahsen
50 [13]; Sevgi and Kartalci [14]), geometry (Senk [15]), and abstract algebra (Weber [9]) unable
51 to construct anything beyond very trivial proofs. Furthermore, most university students do not
52 know what constitutes a proof (Recio and Godino [16]) and cannot determine whether a
53 purported proof is valid (Selden and Selden [17]).

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55 **2.1 THE DIFFICULTIES OF PROOFS IN MATHEMATICS**

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57 Proofs come in all kinds of styles and sizes (Almeida [18]; Velleman [19]; Kiresten and
58 Greefrath [20]). Some are short and effective, like those found in textbooks. Others, which
59 present the very latest research in great detail, are the subject of specific treatment in
60 journals which devote thousands of pages to them (by way of example let us quote two
61 examples: Fermat's theorem and its proof by Wiles; zeros of the zeta function and the
62 results obtained in this direction). Very few people can then get a sense of the full reasoning.
63 In addition, certain foundations sometimes cause problems. For example, a small number of
64 mathematicians are not satisfied with the method of reasoning by the absurd when it comes
65 to proving the existence. If the proposition that there is no solution to an equation leads to a
66 contradiction, is that sufficient to prove that a solution exists? Critics of this method would
67 argue that this logic is just a sleight of hand and does not tell us how to really build a
68 concrete solution. The Constructivists, who belong to several currents, are the
69 mathematicians who stipulate that the method used in the proof does not give a numerical
70 meaning. They cover with contempt the classical mathematicians who regard the method of
71 recurrence as an essential weapon in the mathematical arsenal. On the other hand, the most
72 traditional mathematicians would say that to proscribe this type of reasoning is to deprive
73 oneself of a very useful tool, and, moreover, that to reject so many results proven by means
74 of this indirect method would leave only the weft of the mathematical tapestry.

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76 **2.2 THE PROOF IN EXPERIMENTAL DIDACTICS OF MATHEMATICS**

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78 According to (Glaeser [5]; Pedemonte [21]; Sarr [22]), the theme of proof is often a subject of
79 debate in the educational world. Several different approaches have addressed the problem
80 of learning a proof:

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- 82 ■ epistemological studies to characterize a proof in mathematics;
- 83 ■ analyzes of students' difficulties to understand its nature and usefulness;
- 84 ■ analyzes of approaches to proof on the part of the teacher.

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86 **2.3 LEARNING FROM THE PROOF IN MATHEMATICS EDUCATION**

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88 There are two opposing views on this subject. The debate is vast and important. We cannot
89 summarize it in a few lines as each author has his conception of this learning. We just want
90 to distinguish two major currents of thought. Some didacticians, like Balacheff ([3], [23]) or
91 Lehman ([24]), refuse to separate learning from proof from work solving problems that give it
92 meaning. The proof should only intervene as a tool and should not be the subject of a
93 specific teaching. Other researchers refuse to equate the proof with a path, natural for some,
94 inconceivable for others. They support the thesis that proof is a notion that can be taught like
95 so many others. The methods vary but some points seem fundamental in this teaching of

96 proof as a technique. The development of the ternary structure (hypotheses, theorem,
97 conclusion) of the basic link of the proof; the status of each statement, to be linked to its
98 language context and not to the content; learning different proof techniques such as proof by
99 absurdity, by contraposition, by disjunction of cases, ... etc.

100 A teaching of proof is not only possible, but also necessary, without however turning to
101 extremist practices which would make the work of the pupils lose all meaning. And work
102 exercises of the technique of proof based on the deductive step.

103 **2.4 BEHAVIOR OF STUDENTS FACING TO THE PROOF**

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105 The opposition between the real objective of a proof and the misinterpretation of this
106 objective by pupils and teachers is the basis of a didactic problem.

107 Students look for an explanation in a proof. They strive to read a proof as a tool to convince
108 oneself and others. This effort is often requested by teachers without giving students the
109 tools to achieve it. Often students need to do experiments, empirical verifications, after proof
110 because the proof does not convince them. Faced with constructed proof, students often
111 remain skeptical. They fail to understand the guarantee that a given proof in relation to an
112 argument. They prefer narrative arguments that use diagrams because they are closer to
113 their way of expressing justification.

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115 **2.5 DIFFICULTIES IMPOSED ON STUDENTS**

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117 According to Sarr ([22]), among students, we encounter difficulties related to:

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119 **2.5.1 The start-up**

120 We situate the problems of starting a proof at the level of:

121 1) lack of knowledge of the framework and procedures used in the proof;

122 2) how to use the material at their disposal in the statements, figures and their own
123 knowledge.

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125 **2.5.2 In search**

126 In the search for proof, learners are usually not sure where to start. They don't have a
127 research methodology. They do not know how to exploit the clues offered by a statement or
128 a figure.

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130 **2.5.3 To the editorial staff**

131 After having looked for a proof, the learners have problems to write it up and to make
132 coherence the ideas which they have brought out. They have problems following the
133 framework of deductive reasoning (hypothesis, theorem, conclusion). They confuse the role
134 and the value of the words or groups of words (conjunctions, adverbs ... etc) used in a proof.
135 To this, we must add the problems related to the ordering of the clauses composing the text
136 of the proof.

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138 **2.6 DIFFICULTIES IMPOSED ON TEACHERS**

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140 According to Sarr ([22]), among teachers, the difficulties are of a didactic nature and
141 pedagogical management of large numbers.

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143 **2.6.1 Didactic Difficulties**

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145 **2.6.1.1 The lack of benchmarks to give to students**

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147 Most teachers provide proofs without explaining the framework, procedures and other
148 elements that make up the proofs. These elements, implicit in the speeches or the answers
149 of the teachers, are sometimes difficult to decipher by the pupil. The latter manage blindly or
150 from vague reminiscences.

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152 **2.6.1.2 Lack of relevant activities to offer**

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154 Most of the time, the proof is taught in one step without differentiating between the four types
155 of students' difficulties mentioned above. Appropriate activities are not given to the pupils to
156 make them aware of the difficulties and the skills to be developed on each of them.

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158 **2.6.2 Difficulties related to class sizes**

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160 The problems related to staffing are of the order of:

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162 **2.6.2.1 Student-Teacher Communication**

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164 The excess number of students increases the distance between the students and the
165 teacher. It makes movement and proximity difficult which allow a more direct and frequent
166 dialogue between the student and the teacher. It promotes the isolation of some students
167 who need to be "shaken" to invest and encourages them to drown in the mass of strangers.

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169 **2.6.2.2 Student-Student Communication**

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171 The excess number of students does not encourage the teacher to develop a continuous
172 debate between the students so as not, as some say, to waste time and promote anarchy.
173 Students do not frequently share their ideas and experiences.

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175 **2.6.2.3 Lack of Sub-Grouping and Communication**

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177 In general, teachers do not have sub-group and communication techniques that allow
178 students to search together, to share their experiences in a framework full of enthusiasm and
179 fruitfulness.

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181 **2.7 DIFFICULTIES RELATING TO THE CONSTRUCTION OF PROOF**

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183 According to (Arsac [2]; Barbin [25]; Daili [4]; Pluvinage [26]), the difficulties of proof are
184 numerous and of a very various. Here are some of the most commonly encountered:

185 ■ lexical difficulties: imperfect knowledge of the meaning of terms; this can lead to difficulty
186 in understanding what is required to be proven;

187 ■ difficulties related to the discernment of mathematical objects: not knowing which
188 hypothesis or which proposition to use to prove. This often leads to confusion in concepts;

189 ■ memory-related difficulties: summary or partial recollection of acquired knowledge. This
190 may be due to an imperfection in the acquisition process or a failure in the process of storing
191 mathematical information;

192 ■ difficulties related to reasoning: the construction of a proof calls for innumerable mental
193 operations, complex, which often consist in connecting pieces of knowledge between them,
194 and to adapt them using logical operations to the problem of which one find the solution;

195 ■ the difficulty of a proof depends on at least two variables the number of elementary
196 deductions and the number of statements involved (definitions, axioms, or theorem). It is
197 therefore important to know the value of these variables.

198

199 **3. CONCLUSIONS**

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201 ■ The proof, in general, is central to any activity of a human being who runs behind the truth.

202 In particular, proof in mathematics occupies a special place since it represents a
203 fundamental tool which especially characterizes this discipline. It is therefore understandable
204 that it plays an important role in the school curriculum.

205 It is, therefore, an object of study a priori privileged for the mathematician and all the more
206 so since its introduction is a source of difficulties for many human beings.

207 ■ Any research on its teaching poses the problem of transmission of its construction, just
208 like any mathematical concept, although the proof is not strictly speaking a concept, but
209 rather a technique.

210 The importance given to proof is partly attributable to the fact that it separates mathematics
211 from experimental sciences, it also allows to explain, to discover new mathematics, to
212 systematize, to communicate, in addition to offering intellectual challenges its importance in
213 the life of professional mathematicians as well as the various roles and goals for which it is
214 intended confirm and justify that it is indeed an essential characteristic of the mathematical
215 universe.

216 ■ The proof also has its importance in university education. Indeed, it occupies a prominent
217 place in university-level mathematics courses. In these advanced mathematics courses,
218 taking proof is often the first goal, in addition to being one of the only ways to assess student
219 performance. Proof, at the university level, is a central subject both for its contribution to the
220 mathematical discipline and for the connections it maintains with other fields such as the
221 natural sciences. Proof is at the heart of the mathematician's activity, so it goes without
222 saying that university mathematics devotes a capital place to it.

223 Its learning requires a change of vision of mathematics, progressing beyond experimental
224 verification and intuition to adopt an approach consistent with a mathematical theory. It is not
225 a question of learning techniques but of knowing how to situate them in the theory because
226 proving it can be learned: it is not the result of a eureka for big head it is the result of a good
227 conduct method.

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ACKNOWLEDGEMENTS

The author takes pleasure in thanking all the referees of this article and their comments which contributed to the improvement of the writing of this final version.

COMPETING INTERESTS

No conflict of interest is declared by author.

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