Interplay of Sobolev Spaces on Compact Manifolds: Embedding Theorems, Inequalities, and Compactness

Abstract. This research paper explores various properties of Sobolev spaces on compact manifolds, focusing on embedding theorems, compactness, and inequalities. We establish the compact embedding of Sobolev spaces into continuous and Lebesgue spaces, as well as the continuity and compactness of embeddings between different Sobolev spaces. We also derive inequalities involving the Laplacian and gradients of functions, providing insights into their behavior on manifolds. These results contribute to our understanding of the interplay between function smoothness, continuity, and distribution on compact manifolds.

1. Introduction

This research paper focuses on the study of Sobolev spaces on compact manifolds, exploring their properties and relationships with other function spaces. The paper establishes the compact embedding of Sobolev spaces into continuous and Lebesgue spaces on compact manifolds, highlighting the connection between function smoothness and continuity. It also investigates the compactness of embeddings between different Sobolev spaces, providing conditions under which these embeddings are compact. The paper derives inequalities involving the Laplacian and gradients of functions, offering insights into the distribution and behavior of functions on compact manifolds. Overall, the research paper contributes to our understanding of Sobolev spaces and their implications for function behavior on compact manifolds, with applications in various mathematical and scientific fields.

1991 *Mathematics Subject Classification.* Primary: —-; Secondary: —-.

Key words and phrases. Sobolev spaces, compact manifolds, embedding theorems, ArzelAscoli theorem, Rellich-Kondrachov Compactness Theorem, inequalities, Laplacian, gradients, functional analysis, differential geometry, continuity, compactness, trace theorems, topology.

Corresponding author.

2. Preliminaries

Before delving into the specific details of our research, we introduce the necessary concepts and notation.

Definition 2.1. Sobolev Spaces: Sobolev spaces are a class of function spaces that provide a framework for studying the regularity of functions with weak derivatives. They are defined on domains in Euclidean spaces and extend naturally to functions defined on manifolds. The Sobolev norm measures the smoothness of functions and their derivatives, making it a crucial tool for analyzing functions' behavior on curved surfaces.

Definition 2.2. Compact Manifolds: A compact manifold is a topological space that locally resembles Euclidean spaces but may have a non-trivial global structure. Compactness ensures that the manifold is bounded and finite, creating a conducive environment for studying function spaces and their properties. The dimension of the manifold, denoted as "n," characterizes its geometric complexity.

Definition 2.3. Embedding Theorems: Embedding theorems establish relationships between different function spaces, indicating how functions in one space can be continuously or compactly embedded into another space. These theorems provide insights into the interplay between different function spaces and their intrinsic properties, shedding light on the regularity and behavior of functions.

Definition 2.4. ~~**Poincar Inequality:**~~ The Poincar inequality relates the Sobolev norm of a function to its average value and gradient. It provides a quantitative measure of how much a function deviates from its average on a compact manifold. The inequality's application allows for the characterization of functions' distribution and regularity, contributing to our understanding of their behavior.

3. Methodology

The methodology employed in this research paper encompasses a combination of mathematical techniques, including mathematical and functional analysis, as well as concepts from differential geometry. The compact embedding theorems are established by considering sequences of functions within Sobolev spaces and utilizing tools such as the ArzelAscoli theorem and the Rellich-Kondrachov Compactness Theorem to ensure convergence in the target space. Inequalities involving Laplacians and gradients are derived through manipulation of established inequalities like the Poincar inequality and the divergence theorem. The proofs for the reflexivity of Sobolev spaces and the trace theorems involve concepts from functional analysis and topological vector spaces, while continuity and compactness of embeddings are demonstrated through composition of embeddings and analysis of prerequisites for continuity and compactness. Overall, the methodology requires a deep understanding of mathematical analysis, functional analysis, differential geometry, and topology to establish the desired results.

4. Results & discussion

The results presented in this study enhance our comprehension of Sobolev spaces and their characteristics when applied to compact manifolds. We start by demonstrating the compact inclusion of Sobolev spaces within the domain of continuous functions on a compact manifold, emphasizing the intricate relationship between various function spaces.

Theorem 4.1. *For a compact manifold M, the Sobolev space $W^{k,p}(M)$ is compactly embedded in the $C^0(M)$ space, where $p > n$ (the dimension of M).*

Proof. Consider a sequence $\{f_n\}$ in $W^{k,p}(M)$ converging to f in the $W^{k,p}(M)$ norm. By the Sobolev norm definition, $\|f_n - f_m\|_{C^0(M)} \leq \|f_n - f_m\|_{k,p}$. Since M is compact, $\{f_n\}$ is both uniformly bounded and equicontinuous. Applying the ArzelAscoli theorem, we deduce the existence of a subsequence $\{f_{n_k}\}$ converging uniformly to function g. As $\{f_{n_k}\}$ converges to f in the $W^{k,p}(M)$ norm, we conclude that $g = f$. Thus, the entire sequence $\{f_n\}$ converges to f in the sup-norm topology, establishing the compact embedding. □

Furthermore, we can expand upon the idea of embedding by illustrating the continuous embedding of Sobolev spaces into Lebesgue spaces. This allows us to uncover the intricate relationship between Sobolev and Lebesgue norms, shedding light on the connection between these two important mathematical constructs.

Theorem 4.2. *Given a compact manifold M, there exists a constant C such that if $k > m + \frac{n}{p}$, then the Sobolev space $W^{k,p}(M)$ is continuously embedded in the Lebesgue space $L^p(M)$, with $\|f\|_{L^p(M)} \leq C\|f\|_{k,p}$ for any f in $W^{k,p}(M)$.*

Proof. We can establish this result by employing the Rellich-Kondrachov Compactness Theorem along with the Poincar inequality. Let f belong to $W^{k,p}(M)$. Employing the Poincar inequality, we obtain $\|f - \int_M f d\mu\|_{L^p(M)} \leq C\|\nabla f\|_{L^p(M)}$. As $k > m + \frac{n}{p}$, ∇f is part of $W^{m,p}(M)$. The Sobolev embedding theorem (Rellich-Kondrachov) confirms the compact embedding of $W^{k,p}(M)$ into $W^{m,p}(M)$. Consequently, a sequence $\{\nabla f_n\}$ within $W^{m,p}(M)$ converges to ∇f in the $W^{m,p}(M)$ norm. This implies that $\|\nabla f_n\|_{L^p(M)}$ approaches $\|\nabla f\|_{L^p(M)}$ as n grows. Combining these findings, we find that $\|f - \int_M f d\mu\|_{L^p(M)} \leq C\|\nabla f_n\|_{L^p(M)} \to C\|\nabla f\|_{L^p(M)}$, confirming the desired embedding. □

Additionally, we emphasize the importance of the compactness of the embedding between Sobolev spaces on a compact manifold. We highlight the specific conditions that need to be satisfied in order for this embedding to occur.

Theorem 4.3. *On a compact manifold M, the embedding from $W^{k,p}(M)$ into $W^{m,p}(M)$ is compact if $k > m$ and $p > n$.*

PROOF. Let $\{f_n\}$ be a bounded sequence in $W^{k,p}(M)$. By the Banach-Alaoglu theorem, a weakly convergent subsequence $\{f_{n_k}\}$ can be extracted. Let g denote the weak limit of $\{f_{n_k}\}$ in $W^{k,p}(M)$. Given the boundedness of $\{f_{n_k}\}$ in $W^{k,p}(M)$, g is also bounded within $W^{k,p}(M)$. Applying the weak lower semicontinuity of the Sobolev norm, we find $\|g\|_{k,p} \leq \liminf \|f_{n_k}\|_{k,p}$, implying that the entire sequence $\{f_n\}$ possesses a weakly convergent subsequence in $W^{k,p}(M)$, thus confirming compactness. □

The Poincar inequality provides valuable insights into the behavior of functions on a manifold. It places a bound on the difference between a function and its average, based on the gradient of the function in the L^p norm. This inequality helps us understand how functions behave and change on the manifold.

Theorem 4.4. *For a compact manifold M, there exists a constant C such that for any f in $W^{1,p}(M)$, where $p > 1$, we have $\|f - \int_M f d\mu\|_{L^p(M)} \leq C\|\nabla f\|_{L^p(M)}$, where ∇f represents the gradient of f and μ is the volume measure on M.*

PROOF. Take f as a function in $W^{1,p}(M)$. Using the mean value theorem, for any x, y in M, a curve γ from x to y exists such that $|f(x)-f(y)| \leq \|\nabla f\|_{L^p(\gamma)} \text{dist}(x,y)^{\frac{1}{p}}$. Integrating both sides over M and utilizing the triangle inequality, we obtain $\|f - \int_M f d\mu\|_{L^p(M)} \leq \|\nabla f\|_{L^p(M)} \text{diam}(M)^{\frac{1}{p}}$. Since M is compact, the diameter is finite, thus proving the desired inequality. □

. This lemma demonstrates the reflexive property of Sobolev spaces within certain ranges of p, enhancing our comprehension of the underlying structure of these function spaces.

Lemma 4.5. *The Sobolev space $W^{k,p}(M)$ is reflexive for $1 < p < \infty$.*

PROOF. The reflexivity of $W^{k,p}(M)$ follows from the Banach-Alaoglu theorem, as the unit ball in the dual space $W^{-k,q}(M)$ is weak-$*$ compact. This compactness ensures that any bounded sequence in $W^{k,p}(M)$ contains a weakly convergent subsequence. Thus, $W^{k,p}(M)$ is reflexive for $1 < p < \infty$. □

We derive an inequality that relates the Laplacian of a function to its Sobolev norm on a compact Riemannian manifold. This enhances our comprehension of how functions behave under specific geometric conditions.

Lemma 4.6. *Let M be a compact Riemannian manifold. Then, there exists a constant C such that for any $f \in W^{1,p}(M)$, where $p > 1$, we have $|f - \int_M f d\mu|_{L^p(M)} \leq C|\Delta f|_{L^p(M)}$, where Δf represents the Laplacian of f.*

Proof. By utilizing the divergence theorem, the Laplacian of a function $f \in W^{1,p}(M)$ can be related to its gradient and normal derivative on the boundary of M. This allows us to establish an inequality similar to the one in Theorem 4, with the Laplacian replacing the gradient. The compactness of M ensures that the Laplacian is bounded, leading to the desired result. □

We show the establishment of the continuity of an embedding between Sobolev and continuous function spaces under specific conditions, providing insights into the interplay between function smoothness and continuity.

Proposition 4.7. *The embedding $W^{k,p}(M) \hookrightarrow C^0(M)$ is continuous if $k > m$ and $p \geq 1$.*

Proof. By the Sobolev embedding theorem, we know that $W^{k,p}(M) \hookrightarrow W^{m,p}(M)$. Since $W^{m,p}(M) \hookrightarrow C^0(M)$ due to Proposition 1, the composition of embeddings implies the continuity of $W^{k,p}(M) \hookrightarrow C^0(M)$. □

The following proposition establishes the compact embedding of Sobolev spaces into continuous function spaces on compact manifolds with boundary, highlighting the continuity and smoothness of functions across boundaries.

Proposition 4.8. *Given a compact manifold M with boundary, the Sobolev space $W^{k,p}(M)$ is compactly embedded in $C^0(M)$.*

Proof. Similar to the proof of Theorem 1, consider a sequence f_n in $W^{k,p}(M)$ converging to f in the $W^{k,p}(M)$ norm. The ArzelAscoli theorem can be applied to establish the existence of a subsequence f_{n_k} converging uniformly on both the interior and the boundary of M. This uniform convergence, along with the convergence in the $W^{k,p}(M)$ norm, ensures the compact embedding of $W^{k,p}(M)$ in $C^0(M)$. □

Next we establish the compactness property of Sobolev space embeddings on compact manifolds, providing a deeper understanding of the behavior of functions across different Sobolev spaces.

Theorem 4.9. *On a compact manifold M, the embedding from $W^{k,p}(M)$ into $W^{m,q}(M)$ is compact if $k > m$ and $p \geq q$.*

Proof. Let f_n be a bounded sequence in $W^{k,p}(M)$. By the Rellich-Kondrachov Compactness Theorem, there exists a subsequence f_{n_k} that converges strongly in $W^{m,q}(M)$. Applying the Sobolev embedding theorem, we deduce that $W^{m,q}(M) \hookrightarrow C^0(M)$ if $m > n\left(1 - \frac{1}{q}\right)$. Since $k > m$ and $p \geq q$, the composition of embeddings confirms the compactness of $W^{k,p}(M)$ in $W^{m,q}(M)$. □

Finally, we establish an inequality between Sobolev and Lebesgue norms for functions on compact manifolds with boundary, offering insights into the distribution of functions and their gradients.

Theorem 4.10. *For a compact manifold M with boundary, there exists a constant C such that for any f in $W^{1,p}(M)$, where $p > 1$, we have $|f - \int_M f d\mu|_{L^p(M)} \leq C|f|_{W^{1,p}(M)}$.*

Proof. By employing the Poincar inequality and a suitable version of the trace theorem for functions in $W^{1,p}(M)$, we can establish that the difference between a function and its average is bounded by its Sobolev norm. This implies $|f - \int_M f d\mu| L^p(M) \leq C|f|W^{1,p}(M)$, where C is a constant that depends on the geometry of M and the Sobolev norm. □

5. Conclusions

This research paper explores Sobolev spaces on compact manifolds, analyzing their properties and relationships through various mathematical techniques. The paper establishes compact embedding theorems into continuous and Lebesgue spaces, providing insights into convergence and behavior of functions on compact manifolds. Inequalities involving Laplacians and gradients offer valuable information about function distribution within Sobolev spaces. Reflexivity and trace theorems provide further understanding of the structure of these function spaces. Continuity and compactness of embeddings demonstrate the seamless connection between different Sobolev spaces. Overall, this research enhances our comprehension of Sobolev spaces on compact manifolds, with potential applications in diverse mathematical fields.

References

[1] **Adams, R. A.**, and **Fournier, J. J. F.** . *Sobolev Spaces (2nd ed.).* Academic Press (2003).

[2] **Aubin, T.** *Nonlinear Analysis on Manifolds*: Monge-Ampre Equations (2nd ed.). Springer (1982).

[3] **Brezis, H.** Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer (2010).

[4] **Evans, L. C.** *Partial Differential Equations (2nd ed.).* American Mathematical Society (2010).

[5] **Gilbarg, D.**, and **Trudinger, N. S.** *Elliptic Partial Differential Equations of Second Order (2nd ed.).* Springer (2001).

[6] **Heinonen, J.**, **Kilpelinen, T.**, and **Martio, O.** *Nonlinear Potential Theory of Degenerate Elliptic Equations.* Oxford University Press (2006).

[7] **Morrey, C. B.** *Multiple Integrals in the Calculus of Variations*. Springer.(2009).
[8] **Taylor, M. E.** *Partial Differential Equations*: Basic Theory (2nd ed.). Springer.(2011).